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A convergence property for quasisuperminimizers on metric measure spaces

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§1. Preliminaries

We assume that $X=(X, d, \mu)$ be a complete metric space with a metric d and a positive Borel regular measure μ which is finite on a bounded set.

Let u be a real valued function on X. A nonnegative Borel measurable function g on X is said to be an upper gradient of u if for every rectifiable path γ joining x and y in X,

$$|u(x)-u(y)| \leq \int_{\gamma} g\, ds. \tag{1.1}$$

The p-modulus of a family Γ of paths in X is defined by

$$\inf_{\rho} \int_{X} \rho^{p}\, d\mu,$$

where the infimum is taken over all nonnegative Borel measurable functions ρ such that for all rectifiable paths γ in Γ

$$\int_{\gamma} \rho\, ds \geq 1.$$

We say that a property holds for p-almost every path if the family of paths on which the property does not hold is of zero the p-modulus. If (1.1) holds for p-almost every path γ, then we say that g is a p-weak upper gradient of u.

Let $1<p<\infty$ and $L^{p}(X)$ be the space of functions f on X such that $|f|^{p}$ is integrable with respect to the measure μ. A function u belongs the space $\widetilde{N}^{1,p}(X)$ if $u \in L^{p}(X)$ and u has a p-weak upper gradient g such that $g \in L^{p}(X)$. For a function $u \in \widetilde{N}^{1,p}(X)$, we define

$$||u||_{\widetilde{N}^{1,p}(X)}=||u||_{L^{p}(X)}+\inf_{g}||g||_{L^{p}(X)},$$

where the infimum is taken over all p-weak upper gradients of u. For functions $u,\ v \in \widetilde{N}^{1,p}(X)$, we define the relation $u \sim v$ if and only if $||u-v||_{\widetilde{N}^{1,p}(X)} = 0$. We define the Newtonian space $N^{1,p}(X) = \widetilde{N}^{1,p}(X)/\sim$ equipped with the norm $||\cdot||_{N^{1,p}(X)}$.

Following properties of the Newtonian spaces are known (see **[S1]**):

(i) $N^{1,p}(X)$ is a Banach space.

(ii) Lipschitz functions are dense in $N^{1,p}(X)$.

(iii) Every $u \in N^{1,p}(X)$ has a unique minimal p-weak upper gradient $g_u \in L^p(X)$ in the sense that for every p-weak upper gradient g of u, $g_u \leq g$ μ-a.e in X.

For a set E in X, the p-capacity of E is defined by

$$C_p(E) = \inf_u ||u||_{N^{1,p}(X)},$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u = 1$ on E, and the Newtonian space with zero boundary values is defined by

$$N_0^{1,p}(E) = \{u \in N^{1,p}(X) \,|\, C_p(\{x \in X \setminus E \,|\, u(x) \neq 0\}) = 0\}.$$

Let Ω be an open subset in X. If $u \in N^{1,p}(E)$ for every measurable set $E \Subset \Omega$, we write $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$. For more various properties of Newtonian spaces, see **[S1]**.

In addition, we assume following two conditions:

(I) The measure μ is doubling, that is, there exists a constant $C > 0$ such that

$$0 < \mu(2B) \leq C\,\mu(B)$$

whenever $B = B(x_0, r) = \{x \in X \,|\, d(x, x_0) < r\}$ is a ball in X and $\lambda B = B(x_0, \lambda r)$ for $\lambda \in \mathbf{R}$.

(II) X supports a weak $(1,p)$-Poincaré inequality, that is, there exist constants $C > 0$ and $\lambda \geq 1$ such that for all balls $B \subset X$, all measurable functions f on X and all upper gradients g of f,

$$\frac{1}{\mu(B)} \int_B |f - f_B| \, d\mu \leq C(\text{dima B}) \left(\frac{1}{\mu(\lambda B)} \int_{\lambda B} g^p \, d\mu \right)^{1/p},$$

where $f_B = \frac{1}{\mu(B)} \int_B f \, d\mu$.

In [**B**] there are various examples of spaces equipped with a doubling measure and supporting Poincaré inequality.

§2. Quasisuperminimizers

Let a constant $Q \geq 1$. A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is said to be a (Q,p)-*quasiminimizer* in Ω if for all open $\Omega' \Subset \Omega$ and all $\varphi \in N^{1,p}_0(\Omega')$ we have

$$\int_{\Omega'} g_u^p \, d\mu \leq Q \int_{\Omega'} g_{u+\varphi}^p \, d\mu. \tag{2.1}$$

A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is said to be a (Q,p)-*quasisuperminimizer* in Ω if (2.1) holds for all nonnegative $\varphi \in N^{1,p}_0(\Omega')$. A function u is said to be a (Q,p)-*quasisubminimizer* if $-u$ is a (Q,p)-*quasisuperminimizer*. A function u is a (Q,p)-quasiminimizer if and only if u is a (Q,p)-quasisuperminimizer and a (Q,p)-quasisubminimizer.

A (Q,p)-quasiminimizer (respectively, (Q,p)-quasisuperminimizer) has a continuous (respectively, lower semicontinuous) representative (see [**KM1**; Theorem 5.1], [**KM2**; Lemma 5.3] and [**KS**; Proposition 3.3 and Theorem 5.2]). If u is a $(1,p)$-quasiminimizer (respectively, $(1,p)$-quasisuperminimizer), we say that u is a minimizer (respectively, superminimizer). A continuous minimizer is said to be p-harmonic. Potential theory for p-harmonic functions on metric measure spaces has been studied in [**C**], [**S2**], [**KM1**], [**BBS1**] and [**BBS2**] etc.

If u is a (Q,p)-quasisuperminimizer and $\lambda \geq 0$, τ are constants, then $\lambda u + \tau$ is a (Q,p)-quasisuperminimizer.

§3. A convergence property for quasisuperminimizers

In [**KM2**; Theorem 6.1] the following convergence result for quasisuperminimizers was established:

Proposition. *Let Ω be an open set in X and let $\{u_n\}$ be a nondecreasing sequence of (Q,p)-quaisuperminimizers in Ω and $u = \lim_{n\to\infty} u_n$. If either u is locally bounded above or $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$, then u is a (Q,p)-quaisuperminimizer in Ω.*

We can relax the condition in the above proposition as follows.

Theorem. *Let Ω be an open set in X and let $\{u_n\}$ be a nondecreasing sequence of (Q,p)-quaisuperminimizers in Ω. If there is a function $f \in$*

$N^{1,p}_{\mathrm{loc}}(\Omega)$ *such that* $u_n \le f$ μ*-a.e. for all* n*, then* $u = \lim_{n\to\infty} u_n$ *is a* (Q,p)*-quaisuperminimizer in* Ω*.*

Let Ω be an open subset of X. A function $u : \Omega \to \mathbf{R} \cup \{\infty\}$ is said to be (Q,p)-quaisuperharmonic in Ω in the sense of [**KM2**] if

(i) u is lower semicontinuous,
(ii) $u \not\equiv \infty$ in Ω, and
(ii) there exist an exhaustion $\{\Omega_n\}$ of Ω and a nondecreasing sequence $\{u_n\}$ of (Q,p)-quaisuperminimizers in Ω_n such that $u = \lim_{n\to\infty} u_n^*$, where $u_n^*(x) = \operatorname{ess\,lim\,inf}_{y\to x} u_n(y)$.

If u is a (Q,p)-quaisuperminimizers, then u has a (Q,p)-quaisuperharmonic representative (see [**KM2** ; Proposition 7.2]).

From the above theorem the next corollary follows immediately.

Corollary. *Let* Ω *be an open set in* X *and let* u *be a* (Q,p)*-quaisuperharmonic function in the sense of* [**KM2**] *in* Ω*. If there is a function* $f \in N^{1,p}_{\mathrm{loc}}(\Omega)$ *such that* $u \le f$ μ*-a.e., then* u *is a* (Q,p)*-quaisuperminimizers in* Ω*.*

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