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Neighborhood Function for CA

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Abstract

In place of the traditional definition of a cellular automaton $CA = (S, Q, N, f)$, a new definition (S, Q, f_n, ν) is given by introducing an injection called *the neighborhood function* $\nu : \{0, 1, ..., n-1\} \rightarrow S$, which provides a connection between the variables of local function f_n of arity n and neighbors of CA: image(ν) is a neighborhood of size n. The new definition allows new analysis of cellular automata. We first show that from a single local function countably many CA are induced by changing ν and then prove that equivalence problem of such CA is decidable. Then we investigate what happens if we change the neighborhood. In particular we show that reversibility of CA is preserved by changing the neighborhood, if CA has 2 states but not if it has 3 states.

1 Introduction

The cellular automaton (CA for short) is a uniformly structured information processing system, which is traditionally defined by a 4-tuple (S, Q, N, f), where S is a discrete space consisting of (infinitely) many cells, Q is a finite set of states of each cell, N is a finite subset of S called *the neighborhood* of CA and f is the local state transition function $Q^N \rightarrow Q$. Among others, the neighborhood is a fundamental constituent of CA.

Most studies on CA assume a standard neighborhood (von Neumann, Moore) and then look for or investigate the local function that would meet a given problem. In 2003, however, H.Nishio and M.Margenstern began a general study of the neighborhood in its own right [4]. Following such a framework, we asked the question: How does the Neighborhood Affect the Global Behavior of Cellular Automata? It has been shown that there are some properties which depend on the choice of the neighborhood, while there are some neighborhood-independent properties [3].

Recently T. Worsch and H. Nishio (2006) designed CA which simulate arbitrarily many CA by changing the neighborhood [10]. During this joint work, we arrived at the definition of *the neighborhood function*, which gives rise to infinitely many CA from a single local function: CA is now defined by a 4-tuple (S, Q, f_n, ν), where $\nu : \{0, 1, ..., n-1\} \rightarrow \mathbb{Z}$ provides a connection between variables of f and neighbors of CA. ν is called a *neighborhood function*, since range(ν) corresponds to the ordinary neighborhood N of CA.

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2 Preliminaries

Though the theory applies to higher dimensional CA, we describe it principally for 1-dimensional CA with local functions in 3 variables. 1-dimensional CA is defined by a 4-tuple $(\mathbb{Z}, Q, f_n, \nu)$, where

1. $\mathbb{Z}$ is the set of integers,
2. Q is the set of states of a cell and assumed to be a finite field $GF(q)$ where $q = p^k$ with a prime p and a positive integer k,
3. $f_n : Q^n \to Q$ is the local function of arity n: $f_n(x_0, x_1, ..., x_{n-1})$ and
4. ν is an injection from $\{0, 1, ..., n-1\}$ to $\mathbb{Z}$ which we call *the neighborhood function*. The neighborhood function defines connection between variables of f_n and neighbors for CA: x_i is connected to $\nu(i)$ for $0 \le i \le n-1$. In other words range$(\nu) = (\nu(0), \nu(1), ..., \nu(n-1))$ is *the neighborhood* of CA in the ordinary sense. In this paper each neighborhood is expressed by an *ordered list* of integers (neighbors) such as $(-2, 0, 1)$.

The notion of the neighborhood function was first introduced by T. Worsch and H. Nishio (2006) for achieving universality of CA by changing neighborhoods [10]. Here, we redefine ν to be an injection from $\{0, 1, ..., n-1\}$ to $\mathbb{Z}$. The *degenerate neighborhood* where ν is not injective (many to one mapping) also will do, but we will not discuss such a case in this paper.

f_n is expressed by a polynomial over Q in n variables $(x_0, x_1, x_2, ..., x_{n-1})$, see [5]. In case of ternary function, it reads

$$f_3 = f(x, y, z) = u_0 + u_1x + u_2y + \cdots + u_ix^hy^jz^k + \cdots$$
$$+ u_{q^3-2}x^{q-1}y^{q-1}z^{q-2} + u_{q^3-1}x^{q-1}y^{q-1}z^{q-1},$$
$$\textit{where } u_i \in Q,\ 0 \le i \le q^3 - 1. \quad (1)$$

Furthermore, if $Q = GF(2) = \{0, 1\}$, we have

$$f_3 = f(x, y, z) = u_0 + u_1x + u_2y + u_3z + u_4xy + u_5xz + u_6yz + u_7xyz,$$
$$\textit{where } u_i \in \{0, 1\},\ 0 \le i \le 7. \quad (2)$$

A local function f_3 expressed by Equation (2) is called Elementary Local Function (ELF for short). By Equation (2) it is seen that there are $2^8 = 256$ ELF. The neighborhood function ν_E such that range$(\nu_E) = (-1, 0, 1)$ is called Elementary NeighBorhood (ENB for short). Then $(\mathbb{Z}, GF(2), f_3, \nu_E)$ is Elementary Cellular Automaton (ECA for short) as usually called.

Finally, the global map $F : C \to C$ where $C = Q^{\mathbb{Z}}$ is defined as usual: For any $c \in C$ and $j \in \mathbb{Z}$, $c(j)$ is the state of cell j in c. Then we have

$$F(c)(j) = f(c(j + \nu(0)), c(j + \nu(1)), ..., c(j + \nu(n-1))). \quad (3)$$

2.1 Illustrations

Traditional CA $(\mathbb{Z}, Q, N, f)$, with space $\mathbb{Z}$, cell states Q, neighborhood $N = (-1, 0, 1)$ and local function $f : Q^N \to Q$ is illustrated by Fig.1.

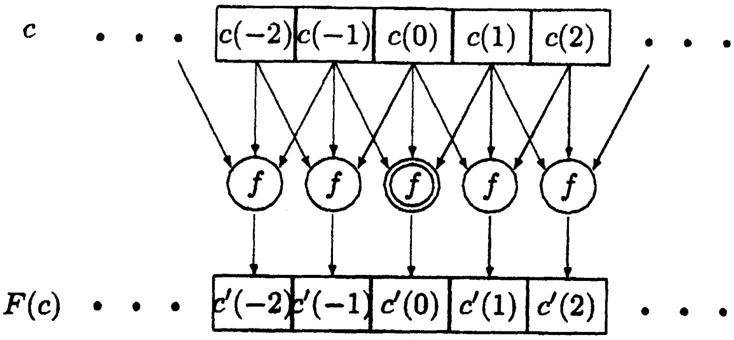

Figure 1: Traditional definition of CA

New CA $(\mathbb{Z}, Q, f_3, \nu)$, where ternary function $f_3(x_0, x_1, x_2)$ and neighborhood function $\nu : \{0,1,2\} \rightarrow \mathbb{Z}$ together define a CA, which has the local state transition rule $f : Q^{(\nu(0),\nu(1),\nu(2))} \rightarrow Q$. Fig. 2 illustrates the case image$(\nu) = (-2, 0, 1)$.

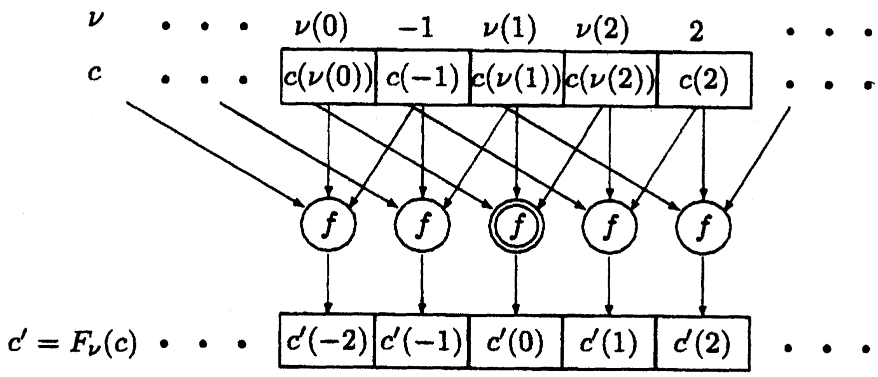

Figure 2: New definition of CA

From the above observation, it will be easily understood that different neighborhood functions give rise different CA for the same local function. The next section gives a formal proof for that.

3 Infinitely many CA made by changing neighborhoods

Theorem 1 *By changing the neighborhood function ν, infinitely many different global CA functions are induced by any single local function $f_3(x, y, z)$ which is not constant.*

Proof.
It is clear that to each non-constant function f at least one of the following three cases applies.

1) If $f_3(a,b,c) \neq f_3(a,b,c')$ for $a, b, c \neq c' \in Q$, consider CA and CA' which have the same local function $f_3(x,y,z)$ and different neighborhoods $(-1, 0, 1+k)$ and $(-1, 0, 1+k')$ where $0 \leq k < k'$. Then, for configuration $W = vab\delta c\delta' c' w$, where $W(0) = b$, δ and δ' are words of lengths $k-1$ and $k' - k - 1$ and v, w are semi-infinite words over Q, we have $F(W)(0) = f_3(a,b,c) \neq f_3(a,b,c') = F'(W)(0)$. That is $F(W) \neq F'(W)$.
In this way, countably many different CA $\{(\mathbb{Z}, Q, f_3, (-1, 0, 1+k)),\ k \geq 1\}$ are induced from a single

local function f_3.

2) If $f_3(a,b,c) \neq f_3(a,b',c)$ for $a, b \neq b', c \in Q$, consider CA and CA' which have the same local function $f_3(x,y,z)$ and different neighborhoods $(-1, 2+k, 1)$ and $(-1, 2+k', 1)$, where $0 \leq k < k'$. Then, for configuration $W = vadc\delta b\delta' b' w$, where $W(0) = d$, δ and δ' are words of lengths $k-1$ and $k'-k-1$ and v, w are semi-infinite words over Q, we have $F(W)(0) = f_3(a,b,c) \neq f_3(a,b',c) = F'(W)(0)$. That is $F(W) \neq F'(W)$.
In this way, countably many different CA $\{\{(\mathbb{Z}, Q, f_3, (-1, 2+k, 1)),\ k \geq 1\}$ are induced from a single local function f_3.

3) If $f_3(a,b,c) \neq f_3(a',b,c)$ for $a \neq a', b, c \in Q$, consider CA and CA' which have the same local function $f_3(x,y,z)$ and different neighborhoods $(-k-1, 0, 1)$ and $(-k'-1, 0, 1)$ where $0 \leq k < k'$. Then, for configuration $W = va'\delta' a\delta bcw$, where $W(0) = b$, δ and δ' are words of lengths $k-1$ and $k'-k-1$ and v, w are semi-infinite words over Q, we have $F(W)(0) = f_3(a,b,c) \neq f_3(a',b,c) = F'(W)(0)$. That is $F(W) \neq F'(W)$.
In this way, countably many different CA $\{(\mathbb{Z}, Q, f_3, (-1-k, 0, 1)),\ k \geq 1\}$ are induced from a single local function f_3. ■

Corollary 1 *There are infinitely many 2 states 3 neighbors CA different from any ECA.*

4 Equivalence Problem of CA

When $\mathbb{Z}$ and Q are understood, we denote $(\mathbb{Z}, Q, f_n, \nu)$ simply by (f_n, ν).

Definition 1 *Two CA (f_n, ν) and $(f'_{n'}, \nu')$ are called equivalent, denoted by $(f_n, \nu) \cong (f'_{n'}, \nu')$, if and only if their global maps are equal.*

Note that there is a local function which induces an equivalent CA for different neighborhood functions, while different local functions may induce an equivalent CA by changing the neighborhood function. For example, $(R85, (-1,0,1)) \cong (R51, (-1,1,0))$, where R85 and R51 are ELF in Wolfram number which give reversible ECA on ENB, see proof of Theorem 3.

We have here a decidability theorem whose proof is independent of dimensionality.

Theorem 2 *The equivalence problem of CA is decidable.*

Proof. Consider two CA (f_n, ν) and $(f'_{n'}, \nu')$ for the same set Q of states. Let $N = \text{range}(\nu) \cup \text{range}(\nu')$. We will consider finite "subconfigurations" $\ell : N \to Q$.

Changing in c the states of cells outside the finite part N has no influence in the computation of $F(c)(0)$ or $F'(c)(0)$. Thus any subconfiguration ℓ determines states $F(c)(0)$ or $F'(c)(0)$ which we denote $G(\ell)$ and $G'(\ell)$.

- Now assume, that the two CA are not equivalent: $(f_n, \nu) \not\cong (f'_{n'}, \nu')$, i.e. the corresponding global maps F and F' are not the same. Then there is a configuration c such that $F(c) \neq F'(c)$. Since global maps commute with the shift, it is without loss of generality to assume that $F(c)(0) \neq F'(c)(0)$. Hence in this case there is an $\ell = c|_N$ such that $G(\ell) \neq G'(\ell)$.
- On the other hand, when there exists an ℓ such that $G(\ell) \neq G'(\ell)$, then obviously F and F' will be different for any configuration c satisfying $c|_N = \ell$ and hence the CA are not equivalent.

For deciding the equivalence it is therefore sufficient to check whether for all $\ell : N \to Q$ holds: $G(\ell) = G'(\ell)$. If this is the case, the two CA are equivalent, if not they are not. ■

In the following, we generally discuss the case $n = n'$ (local functions are of the same arity). For an example of the case $n \neq n'$, see Proposition 1 below.

The following easily proved proposition shows that for CA defined by the neighborhood function ν, there is an equivalent CA' having the ordinary neighborhood of scope $2r + 1$.

Proposition 1 *For (f_n, ν), let $r = \max\{|\nu(i)| \mid 0 \leq i \leq n-1\}$. Then there is an equivalent (f'_{2r+1}, ν') such that* $\mathrm{range}(\nu') = (-r, -r+1, ..., 0, ..., r-1, r)$ *and f'_{2r+1} takes the same value as f_n on* $\mathrm{range}(\nu)$, *while variables x_i are don't care for i such that $\nu'(i) \notin$* $\mathrm{range}(\nu)$.

5 Neighborhood Family and Permutation Family

We define and analyze two families of CA which are obtained by changing and permuting the neighborhood.

Definition 2 *The neighborhood family $\mathcal{F}(f_n)$ of f_n is an infinite set of global functions defined by*

$$\mathcal{F}(f_n) = \bigcup_{\nu \in N_n} F(f_n, \nu), \tag{4}$$

where N_n is the set of all injections $\nu : \{0, \ldots, n-1\} \to \mathbb{Z}$ and $F(f, \nu)$ is the global function induced by f and ν.

Definition 3 *A permutation π of* $\mathrm{image}(\nu)$ *is denoted by $\pi(\nu)$ or simply π when ν is known. The permutation family $\mathcal{P}(f_n, \nu)$ of (f_n, ν) is a finite set of global functions defined by*

$$\mathcal{P}(f_n, \nu) = \bigcup_{i=0}^{n!-1} F(f_n, \pi_i(\nu)). \tag{5}$$

Example: In case of n=3 there are 6 permutations of ENB.

$$\pi_0 = (-1, 0, 1), \pi_1 = (-1, 1, 0), \pi_2 = (0, -1, 1),$$
$$\pi_3 = (0, 1, -1), \pi_4 = (1, -1, 0), \pi_5 = (1, 0, -1).$$

Lemma 1 *The set of CA $\bigcup_f (f, \nu) = \{(f_n, \nu) \mid f_n :$ n-ary function$\}$ is closed under permutation of the neighborhood. That is*

$$\bigcup_f \mathcal{P}(f, \nu) = \bigcup_f (f, \nu). \tag{6}$$

Proof: Since a permutation of the neighborhood amounts to a permutation of the variables of the local function with the neighborhood being fixed, for any f there is a function g and permutation π_i such that $(f, \nu) \cong (g, \pi_i(\nu))$ for some $1 \leq i \leq n! - 1$. ■

Some properties of CA are sensitive to changing the neighborhood but others are not. See also [3].

First, we give some propositions without proof for showing the properties which are not sensitive to changing the neighborhood.

Proposition 2 *$f_n(x_1, ..., x_n)$ is called totalistic if it is a function of $\sum_{i=1}^{n} x_i$. If f_n is totalistic, then any $(f_n, \nu) \in \mathcal{F}(f_n)$ is totalistic.*

Proposition 3 *An affine CA is defined by a local function*

$$f_n(x_1, x_2, ..., x_n) = u_0 + u_1x_1 + \cdots + u_nx_n, \text{ where } u_i \in Q,\ 0 \le i \le n.$$

If f_n is affine, then any $(f_n, \nu) \in \mathcal{F}(f_n)$ is affine.

Proposition 4 *A local function $f : Q^n \to Q$ is called balanced if $|f^{-1}(a)| = |Q|^{n-1}$, $\forall a \in Q$. A finite CA is called balanced if any global configuration has the same number of preimages.*
In case of finite CA, if (f_n, ν) is balanced then $(f_n, \pi(\nu))$ is balanced for any π.

In contrast, there is a property which is sensitive to permutations of the neighborhood.

Proposition 5 *The number-conserving ECA are sensitive to permutation.*

Proof. The only number-conserving ECA are $(R184, \pi_0)$ and its conjugate $(R226, \pi_0)$ [1]. It is seen that $(R184, \pi_2) \cong (R172, \pi_0)$ which is not number-conserving. A similar relation holds for $R226$. ■

6 Reversibility of CA

This section addresses the problem how the reversibility of 2 and 3 states 3 neighbors CA is affected by changing the neighborhood.

Theorem 3 *The set of 6 reversible ECA is closed under permutation.*

Proof. There are 6 reversible ECA; R15, R51, R85, R170, R204, R240 expressed by Wolfram numbers, see page 436 of [9]. Their local functions are listed in Table 1. In the sequel these 6 functions are called *elementary reversible functions*(ERF for short). Note that R204 is the conjugate of R51, R240 is the conjugate of R15 and R170 is the conjugate of R85.

Table 1. Reversible CA with 2 states 3 neighbors

local configuration	000	001	010	011	100	101	110	111
R15	1	1	1	1	0	0	0	0
R51	1	1	0	0	1	1	0	0
R85	1	0	1	0	1	0	1	0
R170	0	1	0	1	0	1	0	1
R204	0	0	1	1	0	0	1	1
R240	0	0	0	0	1	1	1	1

For instance, from R51, by permuting ENB, we obtain R15 and R85. Summing up, we see that

$$(R51, \pi_1) \cong (R85, \pi_0),\quad (R51, \pi_2) \cong (R15, \pi_0),\quad (R51, \pi_3) \cong (R15, \pi_0)$$
$$(R51, \pi_4) \cong (R15, \pi_0),\quad (R51, \pi_5) \cong (R51, \pi_0)\ \ x \leftrightarrow z \text{ symmetry.}$$

Similarly from R204 we obtain R170 and R240 by permutation. Note, however, that R170 can not be obtained by permutation of R51 but by taking conjugate. In other words, $\mathcal{P}(R51, \nu_E) \cap \mathcal{P}(R170, \nu_E) = \emptyset$.
■

Problem 1 It is not known if the set of reversible 3 states CA $(\mathbb{Z}, GF(3), f_3, \pi(\nu_E)$ is closed under permutation.

Theorem 4 *Any 2 states 3 neighbors local function* f_{ERF} *from Table 1 induces a reversible CA* (f_{ERF}, ν) *for any* ν*, particularly for* $\nu \neq ENB$.

Proof: $R15 = x + 1$, where variables y and z are don't care, and CA $(R15, ENB)$ is essentially *a right shift by 1 cell.* Now, it is seen that $(R15, (-k, l, m))$ is a right shift by k cells for any integers k, l, m, which is a reversible CA. Since $R51 = y + 1$ and $R85 = z + 1$, we have the same conclusion that they define reversible CAs for any neighborhood functions. As for $R170 = z, R204 = y$ and $R240 = x$, we have the same conclusion. ■

An analogous result holds for higher dimensional spaces.

Theorem 5 *n-dimensional CA* $(\mathbb{Z}^n, GF(2), f_{ERF}, \nu)$ *is reversible for any* $\nu : \{0, 1, 2\} \rightarrow \mathbb{Z}^n$.

Proof: As is seen from Table 1, any ERF is a *shift* in $\mathbb{Z}^n$ as well. ■

Problem 2 Are there irreversible ECA which become reversible by permuting or changing the neighborhood?

Theorem 6 *Reversible 3 states CA* $(\mathbb{Z}, GF(3), f_3, ENB)$ *is not always reversible, when the neighborhood is different from ENB.*

Proof:
We give a counter example for 3 stats 3 neighbors CA; Among 3^{3^3} 3 states CA on ENB, 1800 are reversible. A reversible CA R[270361043509] appearing in p.436 of [9] is proved not reversible when the neighborhood is changed to $(-1, 0, 2)$ as is shown below;

Injectivity: R[270361043509] on neighborhood $(-1, 0, 2)$ maps both global configurations $\bar{0}1\bar{0}$ and $\bar{0}11\bar{0}$ to $\bar{1}0\bar{1}$. So, it is not injective.

Surjectivity: David Sehnal [7], student of the University of Brno (CZ), used a Mathematica program to show that R[270361043509] is not surjective on $(-1, 0, 2)$. Naonori Tanimoto [8], graduate student of the Hiroshima University (Japan), also confirmed Sehnal's conclusion by his C-code computation.

Recently Clemens Lode [2], student of the University of Karlsruhe (Germany), wrote a Java program called *catest103* which checks injectivity and surjectivity of CA for arbitrary neighborhoods. The program classifies R[270361043509] as not injective and not surjective on $(-1, 0, 2)$. Moreover, catest103 can test injectivity and surjectivity of arbitrary local functions on all (6) permutations of ENB. Owing to the program, we see that R[270361043509] is reversible on ENB $= (-1, 0, 1)$ and $(1, 0, -1)$ but not on the other permutations of ENB. ■

Conjecture 1 *By use of the above mentioned program by C. Lode we see that the 3 states reversible CA R[277206003607] in [9] is reversible on all permutations of ENB and on permutations of many other neighborhoods such as* $(-1, 0, 2), (-1, 0, 3)$ *and* $(-2, 0, 1)$. *From this, we conjecture that R277206003607 is reversible for arbitrary neighborhoods of size 3 in* $\mathbb{Z}$.

7 Concluding Remarks

The new definition of CA using the neighborhood function creates new research of CA. The results established in this paper are fundamental ones and many interesting problems are being left unsolved. Apart from those already mentioned, we have the following problems: Is Rule 110 universal for a neighborhood different from ENB? Is there any other ELF which gives a universal CA for a cleverly chosen neighborhood?

The Java Applet simulator [6] of 1-dimensional CA coded by Christoph Scheben for the Institute of Informatics, University of Karlsruhe, works for arbitrary local functions, number of states, neighborhood and initial configuration (including random configurations) up to 1,000 cells with cyclic boundary and 1,000 time steps. The simulator is the first of this kind —arbitrary neighborhoods. It has been well finished and proved very useful for our research work.

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