数理解析研究所講究録
第 1555 巻 2007 年 57-60

CM-TRIVIALITY AND GEOMETRIC ELIMINATION OF IMAGINARIES

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1. Introduction

To show CM-triviality (of generic relational structures), first of all, we showed weak elimination of imaginaries, and then, working in the real sort, we could show CM-triviality. In this note, we show that *CM-triviality in the real sort*, defined in the second section, implies geometric elimination of imaginaries and CM-triviality (in the real and imaginary sorts). To show this, we give a characterization of geometric elimination of imaginaries in simple theories.

Our notation is standard. Let T be a complete L-theory, and let $\mathcal{M}$ be the big model of T. $\bar{a}, \bar{b}, \ldots (\subset_\omega \mathcal{M})$ denote finite sequences in $\mathcal{M}$. We work in $\mathcal{M}^{\mathrm{eq}}$, which consists of $\bar{a}_E$, the E-class of $\bar{a}$, for any 0-definable equivalence relation E and $\bar{a} \subset_\omega \mathcal{M}$. AB denotes $A \cup B$ for any $A, B \subset \mathcal{M}^{\mathrm{eq}}$.

For $a \in \mathcal{M}^{\mathrm{eq}}, A \subset \mathcal{M}^{\mathrm{eq}}$, we write $a \in \mathrm{dcl}^{\mathrm{eq}}(A)$, if a is fixed by any automorphism pointwise fixing A. And we write $a \in \mathrm{acl}^{\mathrm{eq}}(A)$, if the orbit of a by automorphisms pointwise fixing A, is finite. We write $\bar{a} \equiv_A \bar{b}$ for $\mathrm{tp}(\bar{a}/A) = \mathrm{tp}(\bar{b}/A)$ in T.

We said that T geometrically eliminates imaginaries (T has GEI), if for any $e \in \mathcal{M}^{\mathrm{eq}}$, there exists $\bar{b} \subset_\omega \mathcal{M}$ such that $e \in \mathrm{acl}^{\mathrm{eq}}(\bar{b})$ and $\bar{b} \in \mathrm{acl}^{\mathrm{eq}}(e)$.

2. A characterization of GEI in simple theories

Let T be a simple theory.

Definition 2.1. We say that T has *the independence over intersections* (T has IND/I), for any $\bar{a}, A, B \subset \mathcal{M}$ with $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_A B, \bar{a} \mathop{\smile\!\!\!\!\!\mid}_B A$, we have $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{acl}(A)\cap \mathrm{acl}(B)} AB$.

Proposition 2.2. *IND/I implies GEI.*

1991 *Mathematics Subject Classification.* 03C45.

Proof. Fix $e = \bar{a}_E \in \mathcal{M}^{\text{eq}}$. Take $\bar{b}, \bar{c} \models \text{tp}(\bar{a}/e)$ such that $\bar{b}, \bar{c}, \bar{a}$ are independent over e. Let $A = \text{acl}(\bar{b}) \cap \text{acl}(\bar{c})$. Then $\bar{a} \perp_A \bar{b}\bar{c}$ by IND/I. By $e \in \text{dcl}^{\text{eq}}(\bar{a}) \cap \text{dcl}^{\text{eq}}(\bar{b}\bar{c})$, $e \in \text{acl}^{\text{eq}}(A)$. On the other hand, $A \subset \text{acl}^{\text{eq}}(e)$ follows from $\bar{b} \perp_e \bar{c}$. □

Lemma 2.3. *Suppose that T has GEI. Then, for any* $\text{acl}(A) = A, \text{acl}(B) = B \subset \mathcal{M}$, *we have*

$$\text{acl}^{\text{eq}}(A) \cap \text{acl}^{\text{eq}}(B) = \text{acl}^{\text{eq}}(A \cap B).$$

Proof. Let $e \in \text{acl}^{\text{eq}}(A) \cap \text{acl}^{\text{eq}}(B)$. By GEI, there exists $\bar{a} \subset_\omega \mathcal{M}$ such that $e \in \text{acl}^{\text{eq}}(\bar{a})$ and $\bar{a} \in \text{acl}^{\text{eq}}(e)$. As $\bar{a} \in \text{acl}^{\text{eq}}(A)$ and $\bar{a} \in \text{acl}^{\text{eq}}(B)$, we see $\bar{a} \subset A \cap B$. Thus, $e \in \text{acl}^{\text{eq}}(A \cap B)$. □

From now on, we assume elimination of hyperimaginaries (EHI). Then the converse of Proposition 2.2 follows.

Proposition 2.4. *GEI* $\Leftrightarrow$ *IND/I*

Proof. ($\Leftarrow$) by Proposition 2.2. ($\Rightarrow$): Suppose that $\bar{a} \perp_A B, \bar{a} \perp_B A$ and $\text{acl}(A) = A, \text{acl}(B) = B$. By the above lemma and EHI, we see $\text{Cb}(a/AB) \subseteq \text{acl}^{\text{eq}}(A) \cap \text{acl}^{\text{eq}}(B) = \text{acl}^{\text{eq}}(A \cap B)$. □

3. Main theorem

Definition 3.1. We say that T is *CM-trivial in the real sort*, if, for any $\bar{a}, A = \text{acl}(A), B = \text{acl}(B) \subset \mathcal{M}$, $\bar{a} \perp_A B$ implies $\bar{a} \perp_{A \cap \text{acl}(\bar{a}, B)} B$.

Remark 3.2. The original definition of CM-triviality is as follows: For any $a, A = \text{acl}^{\text{eq}}(A), B = \text{acl}^{\text{eq}}(B) \subset \mathcal{M}^{\text{eq}}$, $a \perp_A B$ implies $a \perp_{A \cap \text{acl}^{\text{eq}}(a,B)} B$. Clearly, under assuming GEI, CM-triviality is equivalent to CM-triviality in the real sort. In the next remark, we lay out an example which shows the difference of the definitions.

Theorem 3.3. *If T is CM-trivial in the real sort, then T has GEI. So CM-triviality in the real sort implies (the original) CM-triviality.*

Proof. By Proposition 2.2, we will show that T has IND/I, i.e. if $\bar{a}, A = \text{acl}(A), B = \text{acl}(B) \subset \mathcal{M}$ and $\bar{a} \perp_A B, \bar{a} \perp_B A$, then $\bar{a} \perp_{A \cap B} AB$. By CM-triviality in the real sort, we have $\bar{a} \perp_{\text{acl}(\bar{a},B) \cap A} B$. By $\bar{a} \perp_B A$, we see $\text{acl}(\bar{a}, B) \cap AB = B$. As $A \cap B \subseteq A \cap \text{acl}(\bar{a}, B) \subseteq AB \cap \text{acl}(\bar{a}, B) = B$, we see

$$\text{acl}(\bar{a}, B) \cap A = A \cap B.$$

By $\bar{a} \perp_{\text{acl}(\bar{a},B) \cap A} B$ and $\bar{a} \perp_B A$, we see $\bar{a} \perp_{A \cap B} AB$. □

Remark 3.4. (1) Let T be the theory of a simple relational structure with a closure operator $\mathrm{cl}(*)$ such that

- $\mathrm{cl}(\mathrm{acl}(A)) = \mathrm{acl}(A)$ and $\mathrm{cl}(\mathrm{cl}(A) \cap \mathrm{cl}(B)) = \mathrm{cl}(A) \cap \mathrm{cl}(B)$,
- for any algebraically closed sets $A, B \subset \mathcal{M}$, $A \downfree_{A\cap B} B \Leftrightarrow$ "$AB = \mathrm{cl}(AB)$ and $R^{AB} = R^A \cup R^B$ for any predicate R".

Then T is CM-trivial in the real sort. (Suppose that $\bar{a} \downfree_A B$. Let $C = \mathrm{acl}(\bar{a}, A), D = \mathrm{acl}(AB)$. As $C \downfree_A B$ and $C \cap B = A$, $\mathrm{cl}(CB) = CB$ and $R^{CB} = R^C \cup R^B$ for any predicate R. Let $E = \mathrm{acl}(\bar{a}, B)$. Then $\mathrm{cl}(CB \cap E) = CB \cap E$ and $R^{CB\cap E} = R^{C\cap E} \cup R^{B\cap E}$ for any predicate R. So, we see $C\cap E \downfree_{A\cap E} B\cap E$. As $\bar{a} \subset C \cap E, B \subset B \cap E$, $\bar{a} \downfree_{A\cap \mathrm{acl}(\bar{a},B)} B$ follows.) So, by Theorem 3.3, CM-triviality of T follows.

(2) CM-triviality does not imply *CM-triviality in the real sort*: In [E], Evans gave an ω-categorical CM-trivial structure $\mathfrak{C}$, defined below, of SU-rank one without WEI.
Here, we check that $\mathfrak{C}$ does not have GEI.
Firstly, he constructed an ω-categorical generic structure M (coutable binary graph $R(x, y)$ with a predimension $\delta(A) = 2|A| - |R^A|$) of SU-rank two such that

- no triangles, no squres in M, and points and adjacent pairs of points are closed in M
- $\mathrm{cl}(*) = \mathrm{acl}(*)$ in M and M is of diameter 3.

Fix $a \in M$. Let C, D be the sets of vertices at distance 1, 2 from a. Then we have the canonical structure $\mathfrak{C}$ on C such that $\mathrm{Aut}(\mathfrak{C})$ is homeomorphic to $\mathrm{Aut}(M/a)$, so $\mathfrak{C}$ and (M, a) are biinterpretable. (See pp.136,139,348 in [H].) Then $\mathfrak{C}$ is of SU-rank one.
We see that $\mathfrak{C}$ does not have GEI as follows:
Let $c, c' \in C$ and $d, d' \in D$ be such that $M \models R(a, c) \wedge R(a, c') \wedge R(c, d) \wedge R(c', d')$. As no triangles and squares in M, we have $M \models \neg R(c, c') \wedge \neg R(c, d') \wedge \neg R(c', d)$. Note that $c \in \mathrm{dcl}(a, d)$ and $acd < acdc', acdd'$. So, $c', d' \notin \mathrm{cl}(a, d, c) = \mathrm{acl}(a, d, c)$. Therefore $\mathrm{cl}(a, d) = \mathrm{acl}(a, d) = \{a, c, d\}$ follows. On the other hand, $\mathrm{cl}(a, c) = \{a, c\}$. So, if $\mathfrak{C}$ has GEI, then, as $d \in \mathfrak{C}^{eq}$, there exist $\bar{c} \subset_\omega C$ such that $d \in \mathrm{acl}(a, \bar{c})$ and $\bar{c} \in \mathrm{acl}(a, d)$ in the sense of M. But such $\bar{c}$ must be a singleton $c \in C$ with $M \models R(a, c) \wedge R(c, d)$. Since $\mathrm{acl}(a, c) = \{a, c\}$ in M, so $d \notin \mathrm{acl}(a, c)$ in M.

Problem 3.5. *In stable theories, is CM-triviality equivalent to CM-triviality in the real sort?*

References

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