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A High-Dimensional Keynesian Macrodynamic Model with and without Time Delay of Policy Response

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Abstract

In this paper, we study the macroeconomic effect of government's fiscal stabilization policy with and without time delay of policy response by using the analytical framework of 'nonlinear high-dimensional Keynesian macrodynamic model', which consists of a system of nonlinear differential equations with many variables. We can summarize the main conclusions of this paper as follows. (1) If the speed of the quantity adjustment of disequilibrium in the goods market is sufficiently high, the system becomes unstable under the lack of government's active stabilization policy. (2) If time delay of government's policy response is sufficiently short, the sufficiently active fiscal stabilization policy can stabilize the economy. (3) If time delay of policy response is sufficiently long, the economy becomes unstable irrespective of the value of the fiscal parameter. (4) Under some combinations of parameter values, endogenous cyclical fluctuations occur.

Keywords: high dimensional Keynesian macrodynamic model, stabilization policy, policy lag, cyclical fluctuation.

JEL classification: E31, E32, E52, E62

1. Introduction[1]

Recently, an international research group of theoretical economists including the author of this paper has developed a series of mathematical economic models called 'nonlinear high-dimensional Keynesian Macrodynamic models'. [2] 'Nonlinear

[1] Thanks are due to the financial support of this research(Chuo University grant for special research 2006) by Chuo University.
[2] See, for example, Asada, Chiarella, Flaschel and Franke(2003), Chiarella, Flaschel and Franke(2005), and Asada, Chen, Chiarella and Flaschel(2006).

high-dimensional dynamic model' means the model that consists of a system of nonlinear differential or difference equations with many variables. These models are disequilibrium dynamic macromodels which are based on Keynes(1936)'s vision on the working of the modern capitalist economy, and in these models many important macroeconomic variables such as national income, employment, capital, private and public debts, money, price level, exchange rate etc. fluctuate endogenously. Utilizing adapted versions of these models, Asada (2006a, 2006b, 2007) investigated the effects of macroeconomic stabilization policies by the 'consolidated government' including central bank. These papers also purported to present theoretical interpretations to the performance of the Japanese economy in the 1990s and the early 2000s. Asada(2006a, 2006b) studied the macroeconomic impact of the inflation targeting by the central bank, while Asada(2007) studied the macroeconomic effect of government's fiscal stabilization policy. In this paper, we restate the essence of the analysis in Asada(2007) without committing to the mathematical details. In section 2, we formulate a macrodynamic model with debt effect without time delay of policy response by the government, which consists of a system of five dimensional nonlinear differential equations. In section 3, we present the outline of the mathematical analysis of the model in section 2. In section 4, we reformulate the model introducing the time delay of policy response and summarize the analytical results of the reformulated model. Section 5 is devoted to the economic interpretation of the analytical results of our model.

2. A Model without policy lag

A version of the high-dimensional Keynesian macrodynamic model without time delay of policy response, which was formulated by Asada(2007), consists of the following system of equations.

$$\dot{d} = \phi(g) - s_f(r - id) - (g + \pi)d \tag{1}$$

$$\dot{y} = \alpha[\phi(g) + v + (1 - s_r)(\rho b + id) - \{s_f + (1 - s_f)s_r\}r - t_w - (1 - s_r)t_r] \;;\; \alpha>0 \tag{2}$$

$$\dot{e}/e = \dot{y}/y + g - n \tag{3}$$

$$\dot{m}/m = \mu - \pi - g \tag{4}$$

$$\dot{b}/b = \mu_B - \pi - g \tag{5}$$

$$i = \rho + \xi(d) = i(\rho, d) \;;\; \xi(d) \geqq 0,\ i_d = \xi'(d)>0 \text{ for } d>0,\ i_d<0 \text{ for } d<0 \tag{6}$$

$$g = g(r, \rho - \pi^e, d) \;;\; g_r = \partial g/\partial r>0,\ g_{\rho-\pi} = \partial g/\partial(\rho - \pi^e)<0,\ g_d = \partial g/\partial d<0 \tag{7}$$

$$r=\beta y \;;\; 0<\beta<1 \tag{8}$$

$$\pi = f(e)+\pi^{e} \;;\; f'(e)>0,\;\; f(\bar{e})=0,\;\; 0<\bar{e}<1 \tag{9}$$

$$\rho=\rho(y,m)=\begin{cases}\rho_0+(h_1y-m)/h_2 & \text{if} \quad h_1y-m\geqq 0\\ \rho_0 & \text{if} \quad h_1y-m<0\end{cases} \tag{10}$$

$$\mu m+\mu_B b = v+\rho b-(t_w+t_r) \tag{11}$$

$$\mu=\bar{\mu}>n \tag{12}$$

$$v=v_0+\delta(\bar{e}-e) \;;\; v_0>0,\;\; \delta\geqq 0 \tag{13}$$

$$\pi^{e}=\bar{\mu}-n \tag{14}$$

The meanings of the symbols are as follows. D = nominal stock of firms' private debt. K = real capital stock. p = price level. $d=D/pK$ = private debt-capital ratio. Y = real output(real national income). $y=Y/K$ = output-capital ratio, which is supposed to be proportional to 'rate of capacity utilization' of the capital stock. $g=\dot{K}/K$ = rate of capital accumulation. $\phi(g)$ = Uzawa(1969)'s adjustment cost function of investment with the properties $\phi'(g)\geqq 1$, $\phi''(g)\geqq 0$. $I=\phi(g)K$ = real private investment expenditure. ρ = nominal rate of interest of public bond. i = nominal rate of interest that is applied to firms' private debt. $\pi=\dot{p}/p$ = rate of price inflation. π^{e} = expected rate of price inflation. $\rho-\pi^{e}$ = expected real rate of interest of public bond. G = real government expenditure. $v=G/K$. B = nominal stock of public debt(public bond). $b=B/pK$ = public debt-capital ratio. T_w = real income tax on workers. $t_w=T_w/K$ = constant. T_r = real income tax on capitalists. $t_r=T_r/K$ = constant. N = labor employment. N_s = labor supply. $e=N/N_s$ = rate of employment = 1 – rate of unemployment. n_1 = growth rate of labor supply >0. $a=Y/N$ = average labor productivity. $n_2=\dot{a}/a$ = growth rate of average labor productivity. $n=n_1+n_2$ = 'natural' rate of growth. M = nominal money supply. $m=M/pK$ = money-capital ratio. $\mu=\dot{M}/M$ = growth rate of nominal money supply. $\mu_B=\dot{B}/B$ = growth rate of nominal public debt. β = share of pre tax profit in national income. s_f = rate of internal retention of firms ($0<s_f\leqq 1$). s_r = capitalists' propensity to save ($0<s_r\leqq 1$). α = adjustment speed in the goods market. $\bar{e}$ = 'natural' rate of employment = 1 – 'natural' rate of unemployment. v_0 = constant part of v. δ = measure of the strength of counter-cyclical fiscal stabilization policy.

A detailed exposition of the derivation of these equations are presented in Asada(2007), so that in this paper we comment only briefly on the economic meanings of these equations.

Eq. (1) is the dynamic law of firms' debt accumulation. Eq. (2) is the Keynesian/Kaldorian quantity adjustment process in the goods market. Equations (3), (4), and (5) are dynamics of rate of employment, money-capital ratio, and public debt-capital ratio respectively. Eq. (6) implies that the private and public bonds are the imperfect substitutes, and the interest rate differentials reflect the difference of the 'degree of risk' of these assets. Eq. (7) is the Keynesian/Kaleckian investment function with debt effect, which can be derived from firms' optimizing behavior under some reasonable assumptions(cf. Asada 1999). Eq. (8) means that the share of pre tax profit in national income is constant, which is supposed to reflect the 'degree of monopoly'. Eq. (9) is the standard expectation-augmented price Phillips curve, which is derived from the expected-augmented wage Phillips curve and firms' mark up pricing rule. Eq. (10) is a standard Keynesian 'LM equation', which is noting but the equilibrium condition of money market.[3] Eq. (11) is in fact the budget constraint of the 'consolidated government' including the central bank.[4] Eq. (12) means that the monetary policy of the central bank follows the simple 'monetarist rule' to keep a constant growth rate of nominal money supply.[5] Eq. (13) specifies the government's fiscal stabilization policy rule *without policy lag*. If $\delta>0$, fiscal policy is said to be counter-cyclical or 'Keynesian'. We can consider that the policy parameter δ is a measure of the strength of the counter-cyclical fiscal policy. Eq. (14) is called the 'quasi rational' expectation hypothesis, which means that the inflation expectation by the public is correct in the long run.[6] We can rationalize this expectation hypothesis if we can assume that the central bank announces the right hand side of Eq.(14) as the target rate of price inflation, and this announcement by the central bank is sufficiently credible for the public. The above system of equations *without policy lag* can be reduced to the following system of five-dimensional nonlinear differential equations, which is called the 'system (S_1)'.

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[3] Following Asada, Chiarella, Flaschel and Franke(2003), we specify the equilibrium condition of money market as $M = h_1 pY + (\rho_0 - \rho)h_2 pK$, $h_1>0$, $h_2>0$, $\rho \geqq \rho_0 \geqq 0$, where ρ_0 is the nonnegative lower bound of nominal interest rate of the government bond. Solving this equation with respect to ρ, we have (Eq.) 10.

[4] Budget constraint of the consolidated government means that the government deficit is financed through the issue of new money and/or new bond, which can be written as $\dot{M} + \dot{B} = pG + \rho B - pT = pG + \rho B - p(T_w + T_r)$. From this relationship, we obtain Eq. (11).

[5] See Asada(2006a, 2006b) for the models with an alternative monetary policy rule, which is called the 'activist rule'.

[6] In fact, we can show that the long run equilibrium rate of price inflation is exactly equal to the right hand side of Eq. (14).

(i) $\dot{d} = \phi(g(\beta y, \rho(y,m) - \bar{\mu} + n, d)) - s_f\{\beta y - i(\rho(y,m), d)d\}$

$$- \{g(\beta y, \rho(y,m) - \bar{\mu} + n, d) + f(e) + \bar{\mu} - n\}d = F_1(d, y, e, m)$$

(ii) $\dot{y} = \alpha[\phi(g(\beta y, \rho(y,m) - \bar{\mu} + n, d) + v_0 + \delta(\bar{e} - e) + (1 - s_r)\{\rho(y,m)b$

$$+ i(\rho(y,m), d)d - \{s_f + (1 - s_f)s_r\}\beta y - t_w - (1 - s_r)t_r] = F_2(d, y, e, m, b; \alpha, \delta)$$

(iii) $\dot{e} = e[F_2(d, y, e, m, b; \alpha, \delta)/y + g(\beta y, \rho(y,m) - \bar{\mu} + n, d) - n]$

$$= F_3(d, y, e, m, b; \alpha, \delta)$$

(iv) $\dot{m} = m[n - f(e) - g(\beta y, \rho(y,m) - \bar{\mu} + n, d)] = F_4(d, y, e, m)$

(v) $\dot{b} = v_0 + \delta(\bar{e} - e) + \rho(y,m)b - \bar{\mu}m - (t_w + t_r) - b[f(e) + \bar{\mu} - n + g(\beta y, \rho(y,m)$

$$- \bar{\mu} + n, d)] = F_5(d, y, e, m, b; \delta) \qquad (S_1)$$

3. Outline of the analysis of the system (S_1)

The long run equilibrium solution of the system (S_1) with the property $\dot{d} = \dot{y} = \dot{e} = \dot{m} = \dot{b} = 0$ is determined by the following system of equations.

(i) $n - s_f\{\beta y - i(\beta y - i(\rho(y,m), d)d\} - \bar{\mu}d = 0$

(ii) $n + v_0 + (1 - s_r)\{\rho(y,m)b + i(\rho(y,m), d)d - \{s_f + (1 - s_f)s_r\}\beta y - t_w$

$$- (1 - s_r)t_r\} = 0$$

(iii) $g(\beta y, \rho(y,m) - \bar{\mu} + n, d) = n$

(iv) $e = \bar{e}$

(v) $v_0 + \rho(y,m)b - \bar{\mu}(m + b) - (t_w + t_r) = 0 \qquad (15)$

This long run solution has the 'classical' properties such that $g = n$, $e = \bar{e}$, and $\pi = \pi^e = \bar{\mu} - n$. By the way, the expected real rate of interest $\rho - \pi^e$ must satisfy the following inequality because the nominal rate of interest of government bond has the nonnegative lower bound ρ_0.

$$\rho - \pi^e = \rho - \bar{\mu} + n \geqq \rho_0 - \bar{\mu} + n \qquad (16)$$

This means that the long run equilibrium may not exist because the expected real rate of interest is too high to support the 'natural rate of growth' if the target rate of inflation announced by the central bank $\bar{\mu} - n$ is too low (i. e., rate of growth of nominal money supply $\bar{\mu}$ is too low). In other words, money is not neutral even in the long run in our model. Henceforth, we assume that $\bar{\mu}$ is sufficiently high to ensure the existence of the long run equilibrium such that $d>0$, $y>0$, $m>0$, $b>0$, and $\rho(y,m)>\rho_0$.

The Jacobian matrix of the system (S_1) *at the equilibrium point* becomes as follows.

$$J_1 = \begin{bmatrix} F_{11} & F_{12} & -f'(\bar{e})d & F_{14} & 0 \\ \alpha G_{21} & \alpha G_{22} & -\alpha\delta & \alpha G_{24} & \alpha G_{25} \\ \bar{e}[\alpha G_{21}/y + g_d] & \bar{e}[\alpha G_{22}/y + H_{22}] & -\bar{e}\alpha\delta / y & \bar{e}[\alpha G_{24}/y + H_{24}] & \alpha G_{25}/y \\ -mg_d & -mH_{22} & -mf'(\bar{e}) & -mH_{24} & 0 \\ -bg_d & F_{52} & -\{\delta + bf'(\bar{e})\} & F_{54} & F_{55} \end{bmatrix} \tag{17}$$

where $F_{11} = \partial F_1/\partial d = (\underset{(+)}{\phi'(n)} - d)\underset{(-)}{g_d} - \bar{\mu} + s_r(\underset{(+)}{i_d}\, d + i)$,

$F_{12} = \partial F_1/\partial y = \beta\{(\underset{(+)}{\phi'(n)} - d)\underset{(+)}{g_r} - s_r\} + (1-d)(\underset{(+)}{\phi'(n)} - d)\underset{(-)}{g_{\rho-\pi}}\,\underset{(+)}{\rho_y} + s_f\,\underset{(+)}{\rho_y}\, d$,

$F_{14} = \partial F_1/\partial m = (\underset{(+)}{\phi'(n)} - d + s_f d)\underset{(-)}{g_{\rho-\pi}}\,\underset{(-)}{\rho_m}$,

$G_{21} = \partial(\frac{F_2}{\alpha})/\partial d = \underset{(+)}{\phi'(n)}\,\underset{(-)}{g_d} + (1-s_f)(\underset{(+)}{i_d}\, d + i)$,

$G_{22} = \partial(\frac{F_2}{\alpha})/\partial y = \beta[\underset{(+)}{\phi'(n)}\,\underset{(+)}{g_r} - \{s_f + (1-s_f)s_r\}] + \{\underset{(+)}{\phi'(n)}\,\underset{(-)}{g_{\rho-\pi}} + (1-s_r)(b+d)\underset{(+)}{\rho_y}$,

$G_{24} = \partial(\frac{F_2}{\alpha})/\partial m = \{\underset{(+)}{\phi'(n)}\,\underset{(-)}{g_{\rho-\pi}} + (1-s_r)(b+d)\}\underset{(-)}{\rho_m}$,

$G_{25} = \partial(\frac{F_2}{\alpha})/\partial b = (1-s_r)\rho \geqq 0$, $H_{22} = \beta\underset{(+)}{g_r} + \underset{(-)}{g_{\rho-\pi}}\,\underset{(+)}{\rho_y}$, $H_{24} = \underset{(-)}{g_{\rho-\pi}}\,\underset{(-)}{\rho_m} > 0$,

$F_{52} = \partial F_5/\partial y = b\{\beta\underset{(+)}{g_r} + (1+\underset{(-)}{g_{\rho-\pi}})\underset{(+)}{\rho_y}\}$, $F_{54} = \partial F_5/\partial m = b(1+\underset{(-)}{g_{\rho-\pi}})\underset{(-)}{\rho_m}$, and

$F_{55} = \partial F_5/\partial b = \rho - \bar{\mu}$.

We can write the characteristic equation of this system at the equilibrium point as

$$\Gamma_1(\lambda) = |\lambda I - J_1| = \lambda^5 + a_1\lambda^4 + a_2\lambda^3 + a_3\lambda^2 + a_4\lambda + a_5 = 0 \tag{18}$$

where $a_1 = -traceJ_1$, $a_k = (-1)^k$(sum of all principal k'th order minors of J_1) $(k = 2,3,4)$, and $a_5 = -\det J_1$.

Asada(2007) investigated the local stability/instability of the equilibrium point of this system under the following assumption.

Assumption 1.

$F_{11}<0$, $F_{12}>0$, $F_{14}>0$, $G_{21}<0$, $G_{22}>0$, $H_{22}>0$, and $F_{55}<0$.

A set of inequalities in **Assumption 1** will in fact be satisfied if sensitivity of investment adjustment cost ($\phi'(n)$), sensitivities of investment activities with respect to the changes of some crucial variables (g_r and $|g_d|$), sensitivity of money demand with respect to the changes of nominal rate of interest (h_2), and growth rate of nominal money supply (μ) are sufficiently large. Asada(2007) proved the following proposition rigorously under **Assumption 1** and some additional technical assumptions.

Proposition 1.

(i) Suppose that $\delta<\underset{(+)}{G_{22}}\, y/\bar{e}$. Then, the equilibrium point of the system (S_1) is unstable for all sufficiently large values of $\alpha>0$.

(ii) Suppose that $s_r=1$ or s_r is sufficiently close to 1. Then, the equilibrium point of the system (S_1) is locally asymptotically stable for all sufficiently large values of the fiscal policy parameter $\delta>0$ irrespective of the value of the parameter $\alpha>0$.

(iii) Suppose that $s_r=1$ or s_r is sufficiently close to 1. Furthermore, suppose that $\alpha>0$ is so large that the system (S_1) is unstable at $\delta=0$. Then, there exist the endogenous cyclical fluctuations at some intermediate range of the fiscal policy parameter values $\alpha>0$.

(Sketch of proof.)

(i) Under the relevant assumptions, we have $a_1<0$ for all sufficiently large values of $\alpha>0$, which violates one of the Routh-Hurwitz conditions for stable roots.

(ii) Suppose that $s_r=1$. Then, we have $G_{25}=0$ so that the Jacobian matrix J_1 becomes decomposable. In this case, the characteristic equation (18) has one negative real root $\lambda_5=F_{55}$, and other four roots are determined by the four dimensional subsystem. Applying Routh-Hurwits conditions for stable roots to this four dimensional system, we obtain **Proposition** 1 (ii).[7] We can extend this proposition concerning local stability to the case of $s_r<1$ as long as s_r is sufficiently close to 1, because of the continuity of values of the characteristic roots with respect to the changes of the coefficients of characteristic equation.

(iii) In this case, it follows from **Propositions** (i) and (ii) that the equilibrium point of the system (S_1) is unstable for all sufficiently small values of $\delta>0$, and it is locally asymptotically stable for all sufficiently large values of $\delta>0$. Therefore, there exists at least one 'bifurcation point' $\delta_0 \in (0,+\infty)$, at which the real part of at least one root of the characteristic equation (18) becomes zero. Under the relevant assumptions, however, we have $\Gamma_1(0) = a_5>0$, which means that the characteristic equation (18) have no real root such as $\lambda = 0$, and it must have at least a pair of pure imaginary roots at the bifurcation point. If it has only a pair of pure imaginary roots, the point δ_0 is the Hopf bifurcation point, and in this case the existence of the non-constant closed orbits is ensured at some range of the parameter values δ sufficiently close to δ_0.[8] If it has two pairs of pure imaginary roots, the existence of the closed orbits is not necessarily ensured. Even in this case, however, the existence of the cyclical fluctuations is ensured at some range of the parameter values δ sufficiently close to δ_0 because of the existence of (two pairs of) complex roots. □

4. A model with policy lag

It is well known that Friedman(1948) asserted that Keynesian stabilization policy may destabilize rather than stabilize the economy because of the existence of the time delay of government's policy response. In this section, we introduce the time delay of policy response to our formal model to test the validity of Friedman(1948)'s assertion theoretically. Following the procedure by Yoshida and Asada(2007), we replace Eq. (13) in section 2 with the following equation, which formalizes the policy lag by means of the continuously distributed lag.

$$v(t) = v_0 + \delta \int_{-\infty}^{t} \{\bar{e} - e(s)\}\omega(s)ds \ ; \ v_0>0, \ \delta \geqq 0 \tag{19}$$

where the function $\omega(s)$ is a weighting function that satisfies the following properties.

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[7] The Routh-Hurwitz conditions for stable roots of the four dimensional system $\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0$ becomes $a_j>0$ for all $j \in \{1,2,3,4\}$ and $a_1a_2a_3 - a_1^2a_4 - a_3^2>0$ (cf. Asada and Yoshida2003 and Yoshida and Asada2007). As for the Routh-Hurwitz conditions for the general n dimensional system, see Appendix.

[8] As for the Hopf bifurcation theorem, see Gandolfo(1996) Chap. 25 and Lorenz(1993) Chap. 3.

$$\omega(s) \geqq 0, \quad \int_{-\infty}^{t} \omega(s)ds = 1 \tag{20}$$

In this paper, we adopt the following simplest type of the weighting function, which means that the policy delay in our model is described by means of the 'simple exponential distributed lag' (cf. Shinkai1970 Chap. 6 and Yoshida and Asada2007).[9]

$$\omega(s) = (1/\tau)\exp[-(1/\tau)(t-s)] \geqq 0 \;\; ; \;\; \tau>0 \tag{21}$$

Now, let us define the variable $e_E(t)$ as follows.

$$e_E(t) = \int_{-\infty}^{t} e(s)\omega(s)ds \tag{22}$$

Substituting equations (20) and (22) into Eq. (19), we have the following expression.

$$v(t) = v_0 + \delta\{\bar{e} - e_E(t)\} \tag{23}$$

Furthermore, substituting Eq. (21) into Eq. (22), we have

$$e_E(t) = (1/\tau)\exp[-(1/\tau)t]\int_{-\infty}^{t} e(s)\cdot\exp[(1/\tau)s]ds \tag{24}$$

or equivalently,

$$e_E(t)\cdot\exp[(1/\tau)t] = (1/\tau)\int_{-\infty}^{t} e(s)\cdot\exp[(1/\tau)s]ds. \tag{25}$$

Differentiating Eq. (25) with respect to t, we obtain the following expression.

$$\dot{e}_E(t) = (1/\tau)\{e(t) - e_E(t)\} \tag{26}$$

In short, we obtain a set of equations (23) and (26) to formalize the time lag of policy response. We can provide clear economic interpretation to these equations. We can interpret the variable $e_E(t)$ as the *expected* rate of employment. Eq. (23) means that the government's fiscal policy is determined by the expected rate of employment rather than actual rate of employment. Eq. (26) means that the expected rate of employment changes according to the formula of the adaptive expectation hypothesis, and τ can be interpreted as the average time lag of policy response.[10] If we replace Eq. (13) in section 2 with equations (23) and (26), we have the following six dimensional system of nonlinear differential equations instead of the five dimensional system.

$$\text{(i)}\;\; \dot{d} = \phi(g(\beta y, \rho(y,m) - \bar{\mu} + n, d)) - s_f\{\beta y - i(\rho(y,m), d)d\}$$
$$- \{g(\beta y, \rho(y,m) - \bar{\mu} + n, d) + f(e) + \bar{\mu} - n\}d = F_1(d,y,e,m)$$

$$\text{(ii)}\;\; \dot{y} = \alpha[\phi(g(\beta y, \rho(y,m) - \bar{\mu} + n, d) + v_0 + \delta(\bar{e} - e_E) + (1-s_r)\{\rho(y,m)b$$

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[9] We have $\int_{-\infty}^{t}(1/\tau)\exp[-(1/\tau)(t-s)]ds = (1/\tau)\exp[-(1/\tau)t]\int_{-\infty}^{t}\exp[(1/\tau)s]ds$
$= \exp[-(1/\tau)t][\exp[(1/\tau)s]]_{s=-\infty}^{s=t} = 1.$

[10] Speed of adaptation is *inversely* proportional to τ.

$$+ i(\rho(y,m),d)d\} - \{s_f + (1-s_f)s_r\}\beta y - t_w - (1-s_r)t_r]$$

$$= F_2(d,y,m,b,e_E;\alpha,\delta)$$

(iii) $\dot{e} = e[F_2(d,y,m,b,e_E;\alpha,\delta)/y + g(\beta y, \rho(y,m) - \bar{\mu} + n, d) - n]$
$= F_3(d,y,m,b,e_E;\alpha,\delta)$

(iv) $\dot{m} = m[n - f(e) - g(\beta y, \rho(y,m) - \bar{\mu} + n, d)] = F_4(d,y,e,m)$

(v) $\dot{b} = v_0 + \delta(\bar{e} - e_E) + \rho(y,m)b - \bar{\mu}m - (t_w + t_r) - b[f(e) + \bar{\mu} - n + g(\beta y, \rho(y,m)$
$- \bar{\mu} + n, d)] = F_5(d,y,e,m,b,e_E;\delta)$

(vi) $\dot{e}_E = (1/\tau)(e - e_E) = F_6(e,e_E;\tau)$ $\qquad (S_2)$

The long run equilibrium solution of this system is exactly same as that of the system (S_1), and we can write the Jacobian matrix of this system at the equilibrium point as follows.[11]

$$J_2 = \begin{bmatrix} F_{11} & F_{12} & -f'(\bar{e})d & F_{14} & 0 & 0 \\ \alpha G_{21} & \alpha G_{22} & 0 & \alpha G_{24} & \alpha G_{25} & -\alpha\delta \\ \bar{e}[\alpha G_{21}/y + g_d] & \bar{e}[\alpha G_{22}/y + H_{22}] & 0 & \bar{e}[\alpha G_{24}/y + H_{24}] & \alpha G_{25}/y & -\bar{e}\alpha\delta/y \\ -mg_d & -mH_{22} & -mf'(\bar{e}) & -mH_{24} & 0 & 0 \\ -bg_d & F_{52} & -bf'(\bar{e}) & F_{54} & F_{55} & -\delta \\ 0 & 0 & 1/\tau & 0 & 0 & -1/\tau \end{bmatrix} \tag{27}$$

The characteristic equation of this system at the equilibrium point can be written as

$$\Gamma_2(\lambda) = |\lambda I - J_2| = \lambda^6 + b_1\lambda^5 + b_2\lambda^4 + b_3\lambda^3 + b_4\lambda^2 + b_5\lambda + b_6 = 0 \tag{28}$$

where $b_1 = -traceJ_2$, $b_k = (-1)^k$(sum of all principal k'th order minors of J_2) $(k = 2,\cdots,5)$, and $b_6 = \det J_2$.

After somewhat tedious calculations, we obtain the following proposition under **Assumption** 1 in section 3 and some additional technical assumptions.[12]

Proposition 2.

[11] The meanings of the symbols in Eq. (27) are the same as those in Eq. (17) except a new symbol τ.
[12] The method of the proof is almost the same as that of the proof of **Proposition 1**. We omit the proof because of the lack of the space.

(i) Suppose that the average policy lag $\tau > 0$ is sufficiently small. Then, **Proposition 1** applies to the system (S_2).

(ii) The equilibrium point of the system (S_2) becomes unstable for all sufficiently large values of τ irrespective of the value of the policy parameter $\delta > 0$.

(iii) Suppose that the value of the policy parameter $\delta > 0$ is fixed at sufficiently large level. Then, the equilibrium point of the system (S_2) is locally asymptotically stable for all sufficiently small values of $\tau > 0$. In this case, at some intermediate values of $\tau > 0$, cyclical fluctuations occur.

5. Economic interpretation of the analytical results

We can summarize the main conclusions of our analysis, which are derived from two propositions in this paper, as follows.

(1) If the speed of the quantity adjustment of disequilibrium in the goods market (α) is sufficiently high, the long run equilibrium point of the system becomes unstable under the lack of the active stabilization policy by the government.

(2) Suppose that the delay of the policy response by the government (τ) is sufficiently short. Then, the sufficiently active stabilization policy, which is reflected by sufficiently large value of the fiscal parameter δ, can stabilize the economy. In this case, the endogenous cyclical fluctuations occur at the intermediate levels of the parameter value δ.

(3) Suppose that the delay of the policy response is sufficiently long. Then, the economy becomes unstable irrespective of the value of the fiscal parameter.

In this section, we shall present some economic interpretation of the above results by means of the schematic representation of some important stabilizing negative feedback and destabilizing positive feedback causal chains which are embedded in our model.

A famous stabilizing negative feedback mechanism caused by the price change is called 'Keynes effect', which works through the effect of the changes of the nominal rate of interest on investment expenditure. We can express this effect schematically as follows.

$$(e\downarrow) \Rightarrow \pi\downarrow \Rightarrow m = (M/pK)\uparrow \Rightarrow \rho\downarrow \Rightarrow (\rho - \pi^e)\downarrow \Rightarrow g\uparrow \Rightarrow y\uparrow \Rightarrow (e\uparrow) \qquad \text{(KE)}$$

However, stabilizing 'Keynes effect' will be quite weak in the situation when the nominal rate of interest already fell to the level that is close to its lower bound ρ_0, as the Japanese economy in the late 1990s and the early 2000s.

On the other hand, it is also well known that price change has the following destabilizing positive feedback effect through the changes of the expected real rate of interest via the changes of the expected rate of inflation, which is called 'Mundell effect', if the price expectation formation of the public is highly adaptive or 'backward looking'(cf. Asada, Chiarella, Flaschel and Franke 2003, and Asada 2006a, 2006b).

$$(e\downarrow) \Rightarrow \pi\downarrow \Rightarrow \pi^e\downarrow \Rightarrow (\rho - \pi^e)\uparrow \Rightarrow g\downarrow \Rightarrow y\downarrow \Rightarrow (e\downarrow) \quad \text{(ME)}$$

This destabilizing positive feedback chain disappears if the price expectation formation by the public becomes highly 'forward looking' because of the fact that the announcement of the target rate of inflation by the central bank is highly 'credible'. Asada(2006a, 2006b) formulated the heterogeneous expectation formation hypothesis (mixture of adaptive and forward looking expectations) by means of a differential equation such as

$$\dot{\pi}^e = \gamma\{\theta(\bar{\mu} - n - \pi^e) + (1-\theta)(\pi - \pi^e)\} \;;\; \gamma>0, \quad 0 \leqq \theta \leqq 1, \quad (29)$$

where the parameter θ is interpreted to reflect the credibility of the central bank's announcement on the target rate of inflation. The more close to 1 θ is, the more credible is the central bank's announcement. Asada(2006a, 2006b) showed that the increase of θ has a stabilizing effect. In case of $\theta = 1$, Eq. (29) is reduced to

$$\dot{\pi}^e = \gamma(\bar{\mu} - n - \pi^e), \quad (30)$$

and in this case the expected rate of inflation (π^e) will converge to

$$\pi^{e*} = \bar{\mu} - n, \quad (31)$$

which is nothing but Eq. (14) in this paper. Therefore, in the model in this paper, destabilizing 'Mundell effect' does not exist by assumption in spite of the fact that the stabilizing 'Keynes effect' may be very weak.

Even in this case, however, there exists another destabilizing positive feedback mechanism of price changes that is called 'Fisher debt effect', which is represented schematically as follows.

$$(e\downarrow) \Rightarrow \pi\downarrow \Rightarrow d = (D/pK)\uparrow \Rightarrow g\downarrow \Rightarrow y\downarrow \Rightarrow (e\downarrow) \quad \text{(FDE)}$$

In other words, the price deflation in the depression process causes the rise of value of firms' real debt, which causes further decrease of the effective demand through the decrease of firms' investment expenditure.[13] The increase of the speed of quantity adjustment in the goods market (α) will strengthen this destabilizing positive feedback effect by reinforcing the part $g\downarrow \Rightarrow y\downarrow$.

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[13] In our model, the increase of firms' debt causes the increase of the consumption expenditure by the capitalists, who are the creditors. Needless to say, this is the stabilizing negative feedback effect, which is called 'wealth effect'. In our model, however, it is implicitly assumed that the stabilizing wealth effect is relatively weak compared with the destabilizing Fisher debt effect. It may be said that this assumption in fact applies to the Japanese economy in the late 1990s and the early 2000s.

In our model, it is assumed that the parameter α is so large that the long run equilibrium point is unstable under the lack of active stabilization policy by the government even if the inflation targeting by the central bank is highly credible. If the delay of policy response is sufficiently short, however, the government can stabilize the unstable economy by means of the fiscal stabilization policy that is represented schematically by

$$(e\downarrow) \Rightarrow v\uparrow \Rightarrow y\uparrow \Rightarrow (e\uparrow), \tag{FSE}$$

which may be called 'Fiscal stabilization effect'. Obviously, fiscal stabilization policy can be destabilizing if the delay of policy response is sufficiently long because of the inadequate timing of the policy enforcement, as Friedman(1948) asserted. In section 4 of this paper, we formalized this assertion by using a simple distributed lag model following the procedure by Yoshida and Asada(2007).

Appendix : Routh-Hurwitz conditions for stable roots for the n-dimensional system

Let us consider the following characteristic equation.

$$\Gamma(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_r\lambda^{n-r} + \cdots + a_{n-1}\lambda + a_n = 0 \tag{A1}$$

All the roots of this characteristic equation have negative real parts if and only if the following set of inequalities is satisfied(cf. Gandolfo 1996 pp. 221 – 222).

$$\Delta_1 = a_1 > 0,\quad \Delta_2 = \begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix} > 0,\quad \Delta_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} > 0,\quad \cdots\cdots,$$

$$\Delta_n = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 & \cdots & 0 \\ 1 & a_2 & a_4 & a_6 & \cdots & 0 \\ 0 & a_1 & a_3 & a_5 & \cdots & 0 \\ 0 & 1 & a_2 & a_4 & \cdots & 0 \\ 0 & 0 & a_1 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{vmatrix} > 0 \tag{A2}$$

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