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Representation of Preference Orderings on L^p-spaces by Integral Functionals: Myopia, Continuity and TAS Utility*

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1 Introduction

Time additive separable (TAS) utility functions have been used in the analysis of intertemporal optimal behaviors and equilibria over time in various fields. The first axiomatization of TAS utility with an infinite horizon was provided by Koopmans (1972b) in a discrete time framework. Koopmans employed a truncation method to embed preference orderings with an infinite horizon into finite dimensional preference orderings with an additive separable representation by using the result of Debreu (1960), and then extended the preference orderings with a finite horizon to those with an infinite horizon by a kind of limiting argument.

While the result of Koopmans was restricted to bounded programs, Dolmas (1995) generalized it to unbounded programs. Epstein (1986) obtained the TAS representation under the hypothesis of constancy of marginal rates of

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intertemporal substitution. However, the above works require strong assumptions and it is difficult to apply these results to a continuous time framework. In particular, Epstein (1986) requires that preference ordering is represented by differentiable utility functions and the truncation method of Koopmans (1972b) and Dolmas (1995) do not work because program spaces in continuous time are infinite dimensional even if time horizons are fixed to be finite.

The purpose of this paper is to present an axiomatic approach in a continuous time framework for representing preference orderings on L^p-spaces in terms of integral functionals. We show that if preference orderings on L^p-spaces satisfy continuity, separability, sensitivity, substitutability, additivity and lower boundedness, then there exists a utility function representing the preference orderings such that the utility function is an integral functional with an upper semicontinuous integrand satisfying the growth condition. Moreover, if the preference orderings satisfy the continuity with respect to the weak topology of L^p-spaces, then the integrand is a concave integrand. As a result, TAS utility functions with constant discount rates are obtained.

2 Finitely Additive Representation

Let $(\Omega, \mathscr{F}, \mu)$ be a measure space with $\mathscr{F}$ a countably generated σ-field of a set Ω, and μ a σ-finite, complete and nonatomic measure of $\mathscr{F}$. For each $1 \le p < \infty$, let $L^p(\Omega, \mathscr{F}, \mu; \mathbb{R}^n)$ be the set of measurable function f from Ω to $\mathbb{R}^n$ with $\int_\Omega |f|^p d\mu < \infty$ endowed with the L^p-norm $\|f\|_p = (\int_\Omega |f|^p d\mu)^{1/p}$, where $|\cdot|$ is the Euclidean norm of $\mathbb{R}^n$. Since $\mathscr{F}$ is countably generated, $L^p(\Omega, \mathscr{F}, \mu; \mathbb{R}^n)$ is a separable Banach space (see Billingsley 1995, Theorem 19.2).

An element in $L^p(\Omega, \mathscr{F}, \mu; \mathbb{R}^n)$ is called a *trajectory*. Let χ_A be a characteristic function of $A \in \mathscr{F}$, that is, $\chi_A(t) = 1$ if $t \in A$ and $\chi_A(t) = 0$ otherwise. If x is a trajectory, then $x\chi_A$ denotes a trajectory taking its values $x(t)$ on A and zero on $\Omega \setminus A$. Thus, if x and y are trajectories and $A \cap B = \emptyset$, then $x\chi_A + y\chi_B$ is a "patched" trajectory taking its values $x(t)$ a.e. $t \in A$ and $y(t)$ a.e. $t \in B$, and vanishing on $\Omega \setminus (A \cup B)$.

Definition 2.1. A subset $\mathscr{X}$ of $L^p(\Omega, \mathscr{F}, \mu; \mathbb{R}^n)$ is *admissible* if the following conditions are satisfied: (i) $0 \in \mathscr{X}$; (ii) $x, y \in \mathscr{X}$ and $A \cap B = \emptyset$ imply $x\chi_A + y\chi_B \in \mathscr{X}$.

Let $\mathscr{X}$ be an admissible set of trajectories. Then $x\chi_A \in \mathscr{X}$ for every $x \in \mathscr{X}$ and $A \in \mathscr{F}$, and hence $\mathscr{X}_A := \{x\chi_A \mid x \in \mathscr{X}\}$ is contained in $\mathscr{X}$. A *preference relation* $\succsim$ on $\mathscr{X}$ is a complete transitive binary relation on $\mathscr{X}$.

We introduce the following axioms on the preference relation.

- *Continuity:* For every $x \in \mathscr{X}$, the upper contour set $\{y \in \mathscr{X} \mid y \succsim x\}$ and the lower contour set $\{y \in \mathscr{X} \mid x \succsim y\}$ are closed in $L^p(\Omega, \mathscr{F}, \mu; \mathbb{R}^n)$.
- *Sensitivity:* For every $A \in \mathscr{F}$ with $\mu(A) > 0$, there exist $x, y \in \mathscr{X}$ such that $x\chi_A \succ y\chi_A$.
- *Separability:* For every $A, B \in \mathscr{F}$ with $A \cap B = \emptyset$, $x\chi_A \succsim y\chi_A$ implies $x\chi_A + z\chi_B \succsim y\chi_A + z\chi_B$ for every $z \in \mathscr{X}$.

The continuity axiom is a standard condition of the continuity of preference relations on topological spaces. The sensitivity of $\succsim$ rules out the situation in which the induced preference relation on $\mathscr{X}_A$ with $\mu(A) > 0$ is "degenerate" in that every element in $\mathscr{X}_A$ is indifferent. The separability of $\succsim$ implies that $x\chi_A \succsim y\chi_A$ if and only if $x\chi_A + z\chi_B \succsim y\chi_A + z\chi_B$ for every $z \in \mathscr{X}$ with $A \cap B = \emptyset$. Thus, $\succsim$ induces a preference relation on $\mathscr{X}_A$ by restricting $\succsim$ to $\mathscr{X}_A$.

Let $I = \{1, \dots, m\}$ be a finite set of natural numbers and $\{\Omega_1, \dots, \Omega_m\}$ be a partition of Ω such that each Ω_i has a positive measure. Define $\mathscr{X}_i = \mathscr{X}_{\Omega_i}$ for each $i \in I$. Since every trajectory $x \in \mathscr{X}$ is identified with the element $(x\chi_{\Omega_1}, \dots, x\chi_{\Omega_m})$ in the product space $\prod_{i\in I} \mathscr{X}_i$ and every element $(x_1, \dots, x_m) \in \prod_{i\in I} \mathscr{X}_i$ is identified with its algebraic sum $\sum_{i\in I} x_i \in \mathscr{X}$, it follows that $\mathscr{X} = \prod_{i\in I} \mathscr{X}_i = \sum_{i\in I} \mathscr{X}_i$, where $\sum_{i\in I} \mathscr{X}_i$ is the algebraic sum of $\mathscr{X}_1, \dots, \mathscr{X}_m$.

Lemma 2.1. Let $\mathscr{X}$ be a admissible set of trajectories. Then $\mathscr{X}$ is a separable metric space. If $\mathscr{X}$ is connected, then $\mathscr{X}_i$ is connected and separable for each $i \in I$.

Proof. Since $\mathscr{X}$ is a subset of the separable Banach space $L^p(\Omega, \mathscr{F}, \mu; \mathbb{R}^n)$, it is also separable. Suppose that $\mathscr{X}$ is connected. Let pr_i be the projection from $\mathscr{X}$ into $\mathscr{X}_i$. Since pr_i is continuous and $\mathscr{X}_i = \mathrm{pr}_i(\mathscr{X})$, it follows that $\mathscr{X}_i$ is a connected set as the image of the connected set by the continuous mapping. To show the separability of $\mathscr{X}_i$, choose $x \in \mathscr{X}_i$ arbitrarily. Note that $\mathscr{X}_i$ is a subset of $\mathscr{X}$ since $\mathscr{X}$ is admissible. Then there exists a sequence $\{x^\nu\}$ in $\mathscr{X}$ such that $x^\nu \to x$ by the separability of $\mathscr{X}$. Therefore, x is the cluster point of the sequence $\{\mathrm{pr}_i(x^\nu)\}$ in $\mathscr{X}_i$ in view of $\mathrm{pr}_i(x^\nu) \to \mathrm{pr}_i(x) = x$. □

Suppose that $m \geq 3$. Let N be an arbitrary subset of I. Since $\succsim$ satisfies separability, $\succsim$ induces on the product space $\prod_{i\in N} \mathscr{X}_i$ a preference relation

$\succsim_N$ by

$$(x_i)_{i\in N} \succsim_N (y_i)_{i\in N} \overset{\text{def}}{\Longleftrightarrow} [(x_i)_{i\in N}, (z_i)_{i\in I\setminus N}] \succsim [(x_i)_{i\in I}, (z_i)_{i\in I\setminus N}] \ \forall z \in \mathscr{X}. \tag{2.1}$$

We denote $\succsim_{\{i\}}$ by $\succsim_i$. Thus for every subset N of I, the preference relation $\succsim_N$ on $\prod_{i\in N} \mathscr{X}_i$ is independent of any $(z_i)_{i\in I\setminus N} \in \prod_{i\in I\setminus N} \mathscr{X}_i$. By the sensitivity of $\succsim$, there exist $x_i, y_i \in \mathscr{X}_i$ such that $x_i \succ_i y_i$ for each $i \in I$. By Lemma 2.1, we can apply the theorem of Debreu–Gorman (Debreu 1960; Gorman 1968) to obtain an additive separable utility function representing $\succsim$.

Theorem 2.1. *Let $\mathscr{X}$ be a connected admissible set of trajectories. If $\succsim$ satisfies continuity, separability and sensitivity, then for each $i \in I$, there exists a continuous function U_i on $\mathscr{X}_i$ such that*

$$x \succsim y \Longleftrightarrow \sum_{i\in I} U_i(x_i) \geq \sum_{i\in I} U_i(y_i).$$

This representation of $\succsim$ is unique up to increasing linear transformation of $\sum_{i\in I} U_i$.

Remark 2.1. The general result of Debreu (1960) on the additive separable representation of preference relations on product topological spaces were extended by Gorman (1968), who demonstrated that the separability axiom (2.1) can be replaced with the weaker condition. The terminologies for the above axioms are different from those of Debreu (1960) and Gorman (1968). We follow the usage of the expositive article by Koopmans (1972a). Note that the requirement $m \geq 3$ is crucial for the additive separable representation. Koopmans (1972a) gave a counter example such that for $m = 2$, every preference relation on a connected separable topological space $\mathscr{X}_1 \times \mathscr{X}_2$ that satisfies continuity, separability and sensitivity cannot be represented by an additive separable utility function!

3 Integral Representation

We introduce the following axioms on the preference relation.

- *Substitutability:* For every $x \in \mathscr{X}$ and $A \in \mathscr{F}$ with $\mu(A) > 0$, there exists some $y \in \mathscr{X}$ such that $x \sim y\chi_A$.
- *Additivity:* For every $x, y \in \mathscr{X}$ and $A, B, E, F \in \mathscr{F}$ satisfying $A \cap B = E \cap F = \emptyset$, $x\chi_A \sim y\chi_E$ and $x\chi_B \sim y\chi_F$ imply $x\chi_A + x\chi_B \sim y\chi_E + y\chi_F$.

- *Lower boundedness:* There exists some $x_0 \in \mathscr{X}$ such that $x \succsim x_0$ for every $x \in \mathscr{X}$.

When maximal elements with respect to $\succsim$ exist, substitutability becomes a somewhat strong requirement because it necessarily implies the existence of multiple maximal elements. In particular, if $\mathscr{X}$ is convex, then substitutability excludes the strict convexity of $\succsim$, which guarantees a unique maximal element. However, we do not assume the compactness of $\mathscr{X}$, and hence substitutability is not a strong restriction when maximal elements are nonexistent. The lower boundedness of $\succsim$ excludes that a utility function representing $\succsim$ is identically equal to $-\infty$, which is an innocuous requirement.

In essence, additivity implies separability; More precisely, additivity implies the following weaker form of the separability:

- *Indifferent separability:* For every $A, B \in \mathscr{F}$ with $A \cap B = \emptyset$, $x\chi_A \sim y\chi_A$ implies $x\chi_A + z\chi_B \sim y\chi_A + z\chi_B$ for every $z \in \mathscr{X}$.

To demonstrate this claim, let $x, y, z \in \mathscr{X}$ and $A \cap B = \emptyset$. Suppose that $x\chi_A \sim y\chi_A$. Define $v = x\chi_A + z\chi_B$ and $w = y\chi_A + z\chi_B$. Since $v\chi_A = x\chi_A$, $y\chi_A = w\chi_A$ and $v\chi_B = w\chi_B$ by construction, we have $v\chi_A \sim w\chi_A$ and $v\chi_B \sim w\chi_B$. The additivity of $\succsim$ implies $v\chi_A + v\chi_B \sim w\chi_A + w\chi_B$, which is equivalent to $x\chi_A + z\chi_B \sim y\chi_A + z\chi_B$, from which indifferent separability follows.

Theorem 3.1. *Let $\mathscr{X}$ be an admissible set of trajectories that is connected and closed in $L^p(\Omega, \mathscr{F}, \mu; \mathbb{R}^n)$. If $\succsim$ satisfies continuity, separability, sensitivity, substitutability, additivity and lower boundedness, then there exists a unique extended real-valued function $f : \Omega \times \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ with the following properties:*

(i) *$f(t, \cdot)$ is upper semicontinuous on $\mathbb{R}^n$ a.e. $t \in \Omega$ and $f(\cdot, v)$ is measurable on Ω for every $v \in \mathbb{R}^n$.*

(ii) *There exist some $\alpha \in L^1(\Omega, \mathscr{F}, \mu)$ and $\beta \geq 0$ such that $f(t, v) \leq \alpha(t) + \beta|v|^p$ a.e. $t \in \Omega$ for every $v \in \mathbb{R}^n$.*

(iii) *For every $A \in \mathscr{F}$, $x\chi_A \succsim y\chi_A$ if and only if $\int_A f(t, x(t))d\mu(t) \geq \int_A f(t, y(t))d\mu(t)$.*

A function $g : \Omega \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a *normal integrand* if $-g$ satisfies condition (i) of Theorem 3.1. Thus condition (i) states that $-f$ is a normal integrand, which we say that f is *upper semicontinuous integrand* in the sequel. Condition (ii) is called *growth condition* in optimal control theory. The meaning of the uniqueness of f is as follows: If g is another upper semicontinuous integrand satisfying the conditions of Theorem 3.1, then $g(t,v) = f(t,v)$ a.e. $t \in \Omega$ for every $v \in \mathbb{R}^n$.

Proof of Theorem 3.1. By virtue of Theorem 2.1, there exists a continuous utility function U on $\mathscr{X}$ which represents $\succsim$ with the form $U(x) = \sum_{i\in I} U_i(x_i)$. Without loss of generality one may assume that $U_i(0) = 0$ for each $i \in I$. We shall show that U is *disjointly additive* on $\mathscr{X}$, that is, $A \cap B = \emptyset$ and $x, y \in \mathscr{X}$ imply $U(x\chi_A + y\chi_B) = U(x\chi_A) + U(y\chi_B)$.

To this end, take any $x \in \mathscr{X}$ and $A, B \in \mathscr{F}$ with $A \cap B = \emptyset$. Let $E, F \in \mathscr{F}$ be such that $E \subset \bigcup_{j\in J} \Omega_j$ and $F \subset \bigcup_{k\in K} \Omega_k$ for some partition $\{J, K\}$ of N, and let E and F have positive measure. Then E and F are disjoint. By the substitutability of $\succsim$, there exist $u, v \in \mathscr{X}$ such that $x\chi_A \sim u\chi_E$ and $x\chi_B \sim v\chi_F$. Define $y = u\chi_E + v\chi_F$. Since $\mathscr{X}$ is admissible, we have $y \in \mathscr{X}$. Note that $y\chi_E = u\chi_E$ and $y\chi_F = v\chi_F$. We thus have $x\chi_A \sim y\chi_E$ and $x\chi_B \sim y\chi_F$. By the additivity of $\succsim$, we have $x\chi_A + x\chi_B \sim y\chi_E + y\chi_F$. Define $E_i = E \cap \Omega_i$ and $F_i = F \cap \Omega_i$ for each $i \in N$. Then $E \cup F$ is decomposed into an n-tuple of pairwise disjoints sets $\{(E_j)_{i\in J}, (F_k)_{k\in K}\}$ with $E_k = \emptyset$ for $k \in K$ and $F_j = \emptyset$ for $j \in J$. Since $y\chi_E \in \mathscr{X}$ and $y\chi_{E_i} = (x\chi_E)\chi_{\Omega_i}$, we have $y\chi_{E_i} \in \mathscr{X}_i$, and similarly $y\chi_{F_i} \in \mathscr{X}_i$. Thus, we have $y\chi_E = (y\chi_{E_1}, \dots, y\chi_{E_n}) \in \prod_{i\in N} \mathscr{X}_i$ with $y\chi_{E_k} = 0$ for $k \in K$ and $y\chi_F = (y\chi_{F_1}, \dots, y\chi_{F_n}) \in \prod_{i\in N} \mathscr{X}_i$ and $y\chi_{F_j} = 0$ for $j \in J$. Therefore, $U(x\chi_A) = U(y\chi_E) = \sum_{j\in J} U_j(y\chi_{E_j})$, $U(x\chi_B) = U(y\chi_F) = \sum_{k\in K} U_k(y\chi_{F_k})$ and $U(x\chi_A + y\chi_B) = U(y\chi_E + y\chi_F) = \sum_{j\in J} U_j(y\chi_{E_j}) + \sum_{k\in K} U_k(y\chi_{F_k})$, and hence $U(x\chi_A + x\chi_B) = U(x\chi_A) + U(x\chi_B)$. From this condition, we can derive the disjoint additivity of U. To demonstrate this, let $x, y \in \mathscr{X}$ and $A \cap B = \emptyset$. Define $z = x\chi_A + y\chi_B$. We then have $z \in \mathscr{X}$ since $\mathscr{X}$ is admissible, and $z\chi_A + z\chi_B = x\chi_A + y\chi_B$ by construction. Thus, $U(x\chi_A + y\chi_B) = U(z\chi_A + z\chi_B) = U(z\chi_A) + U(z\chi_B) = U(x\chi_A) + U(y\chi_B)$.

Define the functional $\Phi : L^p(\Omega, \mathscr{F}, \mu; \mathbb{R}^n) \times \mathscr{F} \to \mathbb{R} \cup \{-\infty\}$ by

$$\Phi(x, A) = \begin{cases} U(x\chi_A) & \text{if } x \in \mathscr{X}, \\ -\infty & \text{otherwise.} \end{cases}$$

By construction, Φ satisfies the following properties:

- $\Phi(\,\cdot\,, \Omega)$ is upper semicontinuous on $L^p(\Omega, \mathscr{F}, \mu; \mathbb{R}^n)$.
- Φ is *finitely additive* on $\mathscr{F}$, that is, $A, B \in \mathscr{F}$ and $A \cap B = \emptyset$ imply $\Phi(x, A \cup B) = \Phi(x, A) + \Phi(x, B)$ for every $x \in L^p(\Omega, \mathscr{F}, \mu; \mathbb{R}^n)$.
- Φ is *local* on $\mathscr{F}$, that is, $x, y \in L^p(\Omega, \mathscr{F}, \mu; \mathbb{R}^n)$ and $x\chi_A = y\chi_A$ imply $\Phi(x, A) = \Phi(y, A)$.

- $-\infty < \Phi(x_0, A)$ for every $A \in \mathscr{F}$.

Then by the representation theorem of Buttazzo and Dal Maso (1983), there exists a unique upper semicontinuous integrand $f : \Omega \times \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ with the following properties:

(a) There exist some $\alpha \in L^1(\Omega, \mathscr{F}, \mu)$ and $\beta \geq 0$ such that $f(t, v) \leq \alpha(t) + \beta|v|^p$ a.e. $t \in \Omega$ for every $v \in \mathbb{R}^n$.

(b) $\Phi(x, A) = \int_A f(t, x(t))d\mu(t) + \Phi(x_0, A)$ for every $x \in L^p(\Omega, \mathscr{F}, \mu; \mathbb{R}^n)$ and $A \in \mathscr{F}$.

Conditions (i) and (ii) of the theorem follows from this result. Since an additive constant does not affect the representation of $\succsim$, it follows from condition (b) that $x\chi_A \succsim y\chi_A$ if and only if $\int_A f(t, x(t))d\mu(t) \geq \int_A f(t, y(t))d\mu(t)$, which shows condition (iii) in the above theorem. □

Example 3.1. Suppose that the admissible set $\mathscr{X}$ is a positive cone of $L^p(\Omega, \mathscr{F}, \mu; \mathbb{R}^n)$ given by

$$\mathscr{X} = \{x \in L^p(\Omega, \mathscr{F}, \mu; \mathbb{R}^n) \mid x(t) \geq 0 \text{ a.e. } t \in \Omega\}.$$

Let x^* be a continuous linear functional on the Banach space $L^p(\Omega, \mathscr{F}, \mu; \mathbb{R}^n)$ such that $\langle x, x^* \rangle \geq 0$ for each $x \in \mathscr{X}$ and $\ker x^* = \{0\}$, where the duality relation is denoted by $x^*(x) = \langle x, x^* \rangle$ for each $x \in L^p(\Omega, \mathscr{F}, \mu; \mathbb{R}^n)$. Suppose that $\succsim$ is represented by the restriction of x^* to $\mathscr{X}$, that is, $x \succsim y$ if and only if $\langle x, x^* \rangle \geq \langle y, x^* \rangle$. It is evident that $\succsim$ satisfies continuity, separability and additivity. The lower bound of $\succsim$ is the origin of $\mathscr{X}$. Since $x \neq 0$ implies $\langle x, x^* \rangle > 0$, for every $A \in \mathscr{F}$ with positive measure, it follows that $\langle x\chi_A, x^* \rangle > 0$ by choosing $x \in \mathscr{X}$ with $x(t) > 0$ on A. Thus, $\succsim$ satisfies sensitivity. To show the substitutability of $\succsim$, take any $x \in \mathscr{X}$ and A with positive measure. Let $y \in \mathscr{X}$ be such that $y(t) > 0$ on A. We then have $\langle y\chi_A, x^* \rangle > 0$. Consider the continuous increasing function on $[0, \infty)$ defined by $\lambda \mapsto \langle \lambda y\chi_A, x^* \rangle$. Then there exists some $\lambda \geq 0$ such that $\langle \lambda y\chi_A, x^* \rangle = \langle x, x^* \rangle$. Since $\mathscr{X}$ is a positive cone and $y\chi_A \in \mathscr{X}$, we have $\lambda y\chi_A \in \mathscr{X}$. This demonstrates the substitutability of $\succsim$.

Therefore, by Theorem 3.1, there exists a unique upper semicontinuous function $f(t, \cdot)$ on $\mathbb{R}^n$ such that $\langle x, x^* \rangle = \int_\Omega f(t, x(t))d\mu(t)$ for every $x \in \mathscr{X}$. On the other hand, the Riesz representation theorem implies that there exists a unique $\varphi \in L^q(\Omega, \mathscr{F}, \mu; \mathbb{R}^n)$ with $\frac{1}{p} + \frac{1}{q} = 1$ such that $\langle x, x^* \rangle = \int_\Omega \langle x(t), \varphi(t) \rangle d\mu(t)$ for every $x \in \mathscr{X}$, where $\langle x(t), \varphi(t) \rangle$ is the inner product of $\mathbb{R}^n$. By the uniqueness of f, we obtain $f(t, v) = \langle v, \varphi(t) \rangle$ for every $v \in \mathbb{R}^n$ a.e. $t \in \Omega$.

Convexity of Preferences

We introduce the convexity axiom of the preferences.

- *Convexity:* Let $\mathscr{X}$ be a convex admissible set. For every $x \in \mathscr{X}$, the upper contour set $\{y \in \mathscr{X} \mid y \succsim x\}$ is convex.

Theorem 3.2. *Suppose that $\succsim$ satisfies the axioms in Theorem 3.1 replacing the strong continuity with the weak continuity of the weak topology of $L^p(\Omega, \mathscr{F}, \mu; \mathbb{R}^n)$. Then the integrand in Theorem 3.1 is a concave integrand, and hence $\succsim$ is convex.*

Proof. The weak continuity of $\succsim$ implies that the preference relation is represented by a weakly continuous utility function. Thus, the functional Φ defined in the proof of Theorem 3.1 is weakly upper semicontinuous on $L^p(\Omega, \mathscr{F}, \mu; \mathbb{R}^n)$. The representation theorem of Buttazzo and Dal Maso (1983) guarantees the concavity of the integrand $f(t, \cdot)$. □

Even if the convexity of $\succsim$ is not assumed explicitly, the weak continuity of $\succsim$ necessarily implies the convexity of $\succsim$!

Stationarity of Preferences

Let X be a subset of $\mathbb{R}^n$ such that $x(t) \in X$ for every $x \in \mathscr{X}$ a.e. $t \in \Omega$. For each $v \in X$ and $A \in \mathscr{F}$ with $\mu(A) > 0$, we say that $v\chi_A$ is a *locally constant trajectory* in X.

- *Stationarity:* Let $\mathscr{X}$ be an admissible set that contains every locally constant trajectory in X. For every $A, B \in \mathscr{F}$ with $A \cap B = \emptyset$, $\mu(A) = \mu(B)$ implies $v\chi_A \sim v\chi_B$ for every $v \in X$.

Theorem 3.3. *Let $\mathscr{F}$ be the Borel σ-field of $\Omega = [0, \infty)$ and μ be a regular Borel measure. Suppose that $\succsim$ satisfies the axioms in Theorem 3.1. Furthermore, if $\succsim$ satisfies stationarity, then the integrand f is independent of $t \in \Omega$ on X, that is, there exists a unique upper semicontinuous function $g : X \to \mathbb{R} \cup \{-\infty\}$ such that $f(t, v) = g(v)$ a.e. $t \in \Omega$ for every $v \in X$.*

Proof. Let $s, t \in \Omega$ with $s < t$ be arbitrary, and let $I_\varepsilon(s) = (s-\varepsilon, s+\varepsilon) \cap (0, \infty)$ and $I_{\varepsilon'}(t) = (t - \varepsilon', t + \varepsilon')$ be disjoint open intervals with $\varepsilon, \varepsilon' > 0$ and $\mu(I_\varepsilon(s)) = \mu(I_{\varepsilon'}(t))$. By the stationarity of $\succsim$, we have $v\chi_{I_\varepsilon(s)} \sim v\chi_{I_{\varepsilon'}(t)}$ for every $v \in X$ and hence $\int_{I_\varepsilon(s)} f(\tau, v) d\mu(\tau) = \int_{I_{\varepsilon'}(s)} f(\tau, v) d\mu(\tau)$ for every

$v \in X$. Thus, by the Lebesgue-Besicovitch differentiation theorem (Evans and Gariepy, 1992, Theorem 1.7.1), we have

$$\begin{aligned} f(t,v) &= \lim_{\varepsilon \to 0} \frac{1}{\mu(I_\varepsilon(s))} \int_{I_\varepsilon(s)} f(\tau, v) d\mu(\tau) \\ &= \lim_{\varepsilon' \to 0} \frac{1}{\mu(I_{\varepsilon'(t)})} \int_{I_{\varepsilon'}(t)} f(\tau, v) d\mu(\tau) = f(s,v). \end{aligned}$$

Therefore, $f(t,v)$ is constant a.e. $t \in \Omega$ for arbitrarily fixed $v \in X$. □

4 TAS Representation with Myopia

Let $\Omega = [0, \infty)$ and $\mathscr{F}$ be the Borel σ-field of Ω. Let ρ be a Lebesgue integrable continuous function on Ω with positive values and let μ_ρ be a nonatomic finite measure of a measurable space $(\Omega, \mathscr{F})$ given by $\mu_\rho(A) = \int_A \rho(t) dt$ for $A \in \mathscr{F}$.

Recursive Utility

Suppose that the admissible set of trajectories is $\mathscr{X} = L^p(\Omega, \mathscr{F}, \mu_\rho; \mathbb{R}^n_+)$ with $1 \leq p < \infty$. A preference relation $\succsim$ on $\mathscr{X}$ is given by the following recursive integral functional

$$\begin{aligned} \forall x, y \in \mathscr{X} : x \succsim y \Longleftrightarrow & \int_\Omega f(t, x(t)) F\left(t, \int_0^s r(s, x(s)) ds\right) dt \\ & \geq \int_\Omega f(t, y(t)) F\left(t, \int_0^s r(s, y(s)) ds\right) dt, \end{aligned} \tag{4.1}$$

where f and r are measurable functions on $\Omega \times \mathbb{R}^n_+$ and F is a measurable function on $\Omega \times \mathbb{R}$.

Assumption 4.1. (i) $f(t, \cdot)$ is continuous on $\mathbb{R}^n_+$ a.e. $t \in \Omega$ and $f(\cdot, v)$ is measurable on Ω for every $v \in \mathbb{R}^n_+$.

(ii) There exist some $\alpha \in L^1(\Omega, \mathscr{F}, \mu_\rho)$ and $a > 0$ such that

$$|f(t,v)| \leq \alpha(t) + a|v|^p \quad \text{for every } (t,v) \in \Omega \times \mathbb{R}^n_+.$$

(iii) $F(t, \cdot)$ is continuous on $\mathbb{R}$ a.e. $t \in \Omega$ and $F(\cdot, z)$ is measurable on Ω for every $z \in \mathbb{R}$.

(iv) $r(t, \cdot)$ is continuous on $\mathbb{R}^n_+$ a.e. $t \in \Omega$ and $r(\cdot, v)$ is measurable on Ω for every $v \in \mathbb{R}^n_+$.

(v) There exists some $\beta \in L^1_{\text{loc}}(\Omega, \mathscr{F}, \mu_\rho)$ such that

$$|r(t,v)| \leq \beta(t) \quad \text{a.e. } t \in \Omega \text{ for every } v \in \mathbb{R}^n_+$$

and

$$\left| F\left(t, \int_0^t \beta(s)ds\right) \right| \leq \rho(t) \quad \text{a.e. } t \in \Omega.$$

(vi) $f(t,0)F(t, \int_0^t r(s,0)ds) = 0$ a.e. $t \in \Omega$.

Assumption 4.2. (i) $f(t,x) \geq 0$ a.e. $t \in \Omega$ for every $x \in \mathbb{R}^n_+$.

(ii) $F(t,z) \geq 0$ a.e. $t \in \Omega$ for every $z \in \mathbb{R}$ and $F(t,\cdot)$ is decreasing on $\mathbb{R}$ a.e. $t \in \Omega$.

(iii) $f(t,\cdot)F(t,\cdot)$ is concave on $\mathbb{R}^n_+ \times \mathbb{R}$ a.e. $t \in \Omega$.

(iv) $r(t,\cdot)$ is concave on $\mathbb{R}^n_+$ a.e. $t \in \Omega$.

It is easy to verify that by growth conditions (ii) and (v) of Assumption 4.1, we have

$$\left| f(t,x(t))F\left(t, \int_0^t r(s,x(s))ds\right) \right| \leq (\alpha(t) + a|x(t)|^p)\rho(t)$$

for every $x \in \mathscr{X}$ a.e. $t \in \Omega$ and the right-hand side of the above inequality is Lebesgue integrable over Ω for every $x \in \mathscr{X}$. Thus, the preference relation given above is well defined.

By the similar argument developed by Sagara (2007), under Assumption 4.1, one can show the continuity of the recursive integral functional

$$x \mapsto \int_\Omega f(t,x(t))F\left(t, \int_0^t r(s,x(s))ds\right) dt$$

on $\mathscr{X}$, and hence the continuity axiom of $\succsim$ is satisfied. It is easy to verify that separability, additivity, indifferent separability are satisfied. If, in addition, Assumption 4.2 is satisfied, then the recursive integral functional is concave on $\mathscr{X}$.

Theorem 4.1 (Sagara). *Let $\succsim$ be a preference relation on $\mathscr{X}$ defined by* (4.1). *Suppose that Assumption 4.1 is satisfied. Then there exists a unique upper semicontinuous integrand g on $\Omega \times \mathbb{R}^n$ such that*

$$\forall x, y \in \mathscr{X} : x \succsim y \Longleftrightarrow \int_\Omega g(t,x(t))\rho(t)dt \geq \int_\Omega g(t,y(t))\rho(t)dt.$$

If, moreover, Assumption 4.2 is satisfied, then g is a concave integrand.

There is a degree of freedom for the choice of ρ. By choosing $\rho(t) = \exp(-\rho t)$, one obtains a TAS utility function with exponential discounting.

TAS Utility

We denote by $L^\infty(\Omega, \mathscr{F}; \mathbb{R}^n)$ the set of essentially bounded functions on Ω to $\mathbb{R}^n$ with respect to the Lebesgue measure. In view of the inclusion

$$L^\infty(\Omega, \mathscr{F}; \mathbb{R}^n) \subset L^\infty(\Omega, \mathscr{F}, \mu_\rho; \mathbb{R}^n) \subset L^p(\Omega, \mathscr{F}, \mu_\rho; \mathbb{R}^n) \quad \text{for } p \geq 1,$$

it is legitimate to endow $L^\infty(\Omega, \mathscr{F}; \mathbb{R}^n)$ with the relative L^p-norm topology from $L^p(\Omega, \mathscr{F}, \mu_\rho; \mathbb{R}^n)$, instead of the essential sup (ess. sup) norm topology of L^∞. By changing the ess. sup norm of $L^\infty(\Omega, \mathscr{F}; \mathbb{R}^n)$ to the L^p-norm, we can deal with $L^\infty(\Omega, \mathscr{F}; \mathbb{R}^n)$ as an admissible set of trajectories in $L^p(\Omega, \mathscr{F}, \mu_\rho; \mathbb{R}^n)$.

The following main result of this paper strengthens Theorem 4.1 under the alternative hypotheses on the preference relation.

Theorem 4.2. *Let $\mathscr{X}$ be an admissible set of trajectories closed and convex in $L^p(\Omega, \mathscr{F}, \mu_\rho; \mathbb{R}^n)$. If $\succsim$ satisfies continuity, sensitivity, separability, substitutability, additivity, lower boundedness, stationarity, then there exists a unique upper semicontinuous integrand g on $\mathbb{R}^n$ such that*

$$\forall x, y \in \mathscr{X} : x \succsim y \Longleftrightarrow \int_\Omega g(x(t))\rho(t)dt \geq \int_\Omega g(y(t))\rho(t)dt.$$

If, moreover, $\succsim$ satisfies convexity, then g is a concave integrand.

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