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A Note on Certainty Effect and Self-Control*

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Abstract

Gul and Pesendorfer (2001) provide a model of temptation and self-control. In a theory of choice under risk, experimental evidence suggests a "certainty effect", that is, a decision maker tends to put more weight on a certain object in comparison with a lottery that is very likely but not completely certain. Thus what is certain may be more tempting, in other words, randomization may make objects less tempting. However, one of the axioms of Gul and Pesendorfer, called Independence, is not consistent with such a hypothesis. We provide an axiomatic model that can account for intuitive choice behavior under risk and temptation.

Keywords: temptation, self-control, independence axiom, certainty effect.
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1 Introduction

1.1 Objective

In a two-period setting, Gul and Pesendorfer [5] (Hereafter GP) model a decision maker who may be tempted by ex ante inferior items. That is, in the second period, the decision maker faces a conflict between two different perspectives - one is an ex ante ranking of items (what she should choose), and the other is a temptation ranking (what she would like to choose now). She may be able to resist temptation and choose an item which is

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desirable in terms of the ex ante ranking. Alternatively, she may not be able to exercise self-control and end up with choosing ex ante inferior items.

GP consider preference over menus of lotteries to rationalize the above story. Let C be a compact metric space of outcomes and $\Delta(C)$ be the set of lotteries over outcomes. Preference $\succsim$ is defined over non-empty compact subsets (called menus) of $\Delta(C)$, denoted by $\mathcal{K}(\Delta(C))$. GP provide an axiomatic foundation for the following representation: There exist mixture linear utility functions $u, v : \Delta(C) \to \mathbb{R}$ such that $\succsim$ over $\mathcal{D} \equiv \mathcal{K}(\Delta(C))$ is represented by

$$U(x) = \max_{l \in x} \left\{ u(l) - \left(\max_{l' \in x} v(l') - v(l) \right) \right\}. \tag{1}$$

The function u represents the DM's commitment preference and hence reflects her normative ranking of alternatives, while v is interpreted as the temptation ranking. The DM behaves as if she makes a choice so as to reconcile those two factors.

Suppose that a DM finds certain prospects more tempting than uncertain prospects – this is in the spirit of the 'certainty effect' (Allais [1] and Kahneman and Tversky [7]). Let s denote 'salad' and b denote 'burger'. Suppose that b is so tempting that the DM cannot exercise self-control, that is,

$$\{s\} \succ \{s, b\} \sim \{b\}. \tag{2}$$

Now suppose that each item is offered with probability $\lambda = .5$, with 0 (nothing) with remaining probability. Under the hypothesis that what is certain is more tempting, the lottery $b\lambda 0$ is not so tempting relative to $s\lambda 0$. The DM may resist the temptation in this case, and hence she exhibits:

$$\{s\lambda 0\} \succ \{s\lambda 0, b\lambda 0\} \succ \{b\lambda 0\}. \tag{3}$$

One can see that this is inconsistent with GP's Independence axiom: for all $x, y, z \in \mathcal{D}$ and $\lambda \in (0, 1)$,

$$x \succ y \;\Rightarrow\; \lambda x + (1-\lambda)z \succ \lambda y + (1-\lambda)z. \tag{4}$$

In particular, Independence precludes the possibility that randomization may make objects less tempting. Our objective is to weaken Independence so as to explain intuitive choice behavior under risk and temptation.

When C is finite, we axiomatize preference having the following representation: There exist mixture linear utility functions $u, v : \Delta(C) \to \mathbb{R}_+$ and a continuous, strictly increasing and (weakly) convex function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(0) = 0$ such that $\succsim$ over $\mathcal{D}$ is represented by

$$U(x) = \max_{l \in x} \left\{ u(l) - \varphi \left(\max_{l' \in x} v(l') - v(l) \right) \right\}. \tag{5}$$

As in GP, u and v are interpreted as commitment ranking and as temptation ranking, respectively. If $\varphi(w) = w$, this functional form is reduced to GP's model (1). Convexity of φ means that the marginal cost of self-control is increasing, whereas it is constant in GP's model. We show also uniqueness of the representation.

1.2 Related Literature

Several authors provide models of temptation without Independence. Gul and Pesendorfer [6] consider preference over finite menus of deterministic items, derive a temptation ranking from the preference, and characterize a representation. Epstein and Kopylov [2] adapt GP's model to explain cognitive dissonance. By extending choice objects to menus of acts, they characterize a functional form with a convex temptation utility.

Noor [9], Nehring [8], and Olszewski [10] provide foundations for functional forms closely related to ours. Noor [9] points out that the weak axiom of revealed preference (WARP) is not compelling under temptation. To accommodate choice behavior without WARP, he considers preference over menus of lotteries, as in GP, and provides a menu-dependent self-control model as follows:

$$U(x) = \lim_{k \to \kappa(x)} \left[\max_{l \in x} \left\{ u(l) + k \left(v(l) - \max_{l' \in x} v(l') \right) \right\} \right], \tag{6}$$

where u and v are expected utility functions on $\Delta(C)$ and $\kappa : \mathcal{K}(\Delta(C)) \to \mathbb{R}_+ \cup \{\infty\}$ is a lower semicontinuous function. Since our set of axioms is stronger than his, the preference we consider should also admit a representation of the form (6). Notice that our model also admits menu-dependent self-control even under the stronger axioms.

Nehring [8] considers "second-order preference" over extended outcomes (c, P), where c is an element of a finite outcome set C and P is a linear order on C. For a menu $x \subset C$, let $P(x)$ denote a P-maximal element in x. His model can induce preference over menus of outcomes as

$$U(x) = \max_{P \in \mathcal{L}(C)} V(P(x), P),$$

where $\mathcal{L}(C)$ is the set of all linear orders on C and V is a representation of the second-order preference. His model can accommodate a nonlinear generalization of GP's model. He mentions the same functional form as (5) as a special case of his model.

By considering preference over pairs of a finite menu x and a deterministic item l included in x, Olszewski [10] characterizes a functional form over the pairs (x, l), which induces the representation over menus as follows:

$$U(x) = \max_{l \in x} \left\{ u(l) - c \left(l, \max_{l' \in x} r(l, l') \right) \right\}, \tag{7}$$

where c satisfies $c(l, r(l, l)) = 0$. Our functional form is a special case of (7).

Fudenberg and Levine [4] take a different approach for modeling a self-control problem. They directly assume a long-run patient self and a sequence of short-run impulsive selves as primitives, and show that the equilibria of the game played by those selves can be regarded as the solution to a maximization problem. Though they have more general models, we focus on the case that the cost of self-control is convex. In our terminology, it can be expressed as the functional form

$$U(x) = \max_{l \in x} \left\{ u(l) - \gamma \left(\max_{l' \in x} v(l') - v(l) \right)^{\theta} \right\}, \tag{8}$$

where $\gamma > 0$ and $\theta > 1$. Some experimental findings suggest that self-control is a limited resource, that is, a DM tends not to resist temptation as cognitive load increases. Their model (8) can explain those evidence. Fudenberg and Levine also point out that Independence in GP, and hence the linear cost of self-control, is not compelling when lotteries are taken as choice objects.

2 Model

2.1 Domain

Let C be a compact metric space and $\Delta(C)$ be the set of all probability measures over C. Under the weak convergence topology, $\Delta(C)$ is a compact and metric space. Denote the metric by $d(l, l')$ for $l, l' \in \Delta(C)$.

Let $\mathcal{K}(\Delta(C))$ be the set of all non-empty compact subsets of $\Delta(C)$. Endow $\mathcal{K}(\Delta(C))$ with the Hausdorff metric. That is, for all $x, x' \in \mathcal{K}(\Delta(C))$, let

$$d_h(x, x') \equiv \max \left[\max_{l \in x} \min_{l' \in x'} d(l, l'), \max_{l' \in x'} \min_{l \in x} d(l, l') \right].$$

Preference $\succsim$ is defined on $\mathcal{D} \equiv \mathcal{K}(\Delta(C))$.

2.2 Convex Self-Control Representation

Take continuous mixture linear utility functions $u, v : \Delta(C) \rightarrow \mathbb{R}_+$ and a continuous, strictly increasing and (weakly) convex function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\varphi(0) = 0$. We consider the functional form $U : \mathcal{D} \rightarrow \mathbb{R}$ defined by

$$U(x) = \max_{l \in x} \left\{ u(l) - \varphi\left(\max_{l' \in x} v(l') - v(l) \right) \right\}. \tag{9}$$

Components u and v are interpreted as in GP. When committed to a lottery $l \in \Delta(C)$, the DM evaluates $\{l\}$ by $U(\{l\}) = u(l)$. Thus u is commitment ranking of lotteries. On the other hand, the function v is supposed to represent the DM's temptation ranking of lotteries. Taking into account the most tempting lottery within a menu at hand, the DM evaluates the menu by maximizing commitment utility $u(l)$ minus self-control cost $\varphi(\max_{l' \in x} v(l') - v(l))$. Since φ is convex, a self-control cost increases drastically as the difference between $\max_x v$ and $v(l)$ increases. In case of GP, $\varphi(w) = w$, and hence the difference, $\max_{l' \in x} v(l') - v(l)$, is directly interpreted as the self-control cost from choosing l when confronted with the most tempting lottery.

Definition 2.1. Preference $\succsim$ on $\mathcal{D}$ admits a convex self-control representation if there exists a functional form (9) with components (u, v, φ) that represents $\succsim$. Furthermore, $\succsim$ is said to admit a partial convex self-control representation if a functional form (9) represents $\succsim$ but φ is convex only on a non-degenerate interval $[0, \overline{w}]$.

Ex post choice suggested by a convex self-control model is

$$\mathcal{C}(x) = \arg\max_{l \in x} \left\{ u(l) - \varphi\left(\max_{l' \in x} v(l') - v(l) \right) \right\}. \tag{10}$$

Unlike GP, the ex post preference depends on the most tempting item in x, and hence exhibits menu-dependent self-control. Furthermore, since φ is convex, the menu-dependent utility function is concave in l. This has an interesting implication: choice between risky prospects may not be explicable by expected utility theory. In fact, the model may accommodate Allais-type behavior (Allais [1]) or the certainty effect (Kahneman and Tversky [7]).

Consider for instance the example surrounding the rankings (2) and (3). From ranking (2), the DM does not exhibit self-control at $\{s, b\}$, which implies

$$u(b) \geq u(s) - \varphi(v(b) - v(s)). \tag{11}$$

From ranking (3), she exercises self-control at $\{s\lambda 0, b\lambda 0\}$, and hence

$$u(s\lambda 0) - \varphi(v(b\lambda 0) - v(s\lambda 0)) > u(b\lambda 0). \tag{12}$$

Taking (11) and (12) together,

$$\varphi(v(b) - v(s)) > u(s) - u(b) \geq \frac{1}{\lambda}\varphi(\lambda(v(b) - v(s))). \tag{13}$$

If φ is strictly convex, we have $\lambda\varphi(v(b) - v(s)) > \varphi(\lambda(v(b) - v(s)))$ for all $\lambda \in (0, 1)$. Thus we can choose parameters so as to satisfy inequalities (13). For instance, it is easy to verify that $u(s) = 2$, $u(b) = 1$, $v(b) = 2$, $v(s) = 1$, $\varphi(w) = 2w^2$, and $\lambda = 1/2$ satisfy (13).

3 Foundations

3.1 Axioms

From now on, we use the following notation: For all $\lambda \in (0, 1)$ and $l, l' \in \Delta(C)$, let $l\lambda l' \equiv \lambda l + (1 - \lambda)l'$.

Say that $\succsim$ on $\mathcal{D}$ is a *self-control preference* if there exist $l, l' \in \Delta(C)$ with $\{l\} \succ \{l, l'\} \succ \{l'\}$.

The axioms which we consider on $\succsim$ are the following. The first two axioms are standard and need no explanation.

Axiom 1 (Order). $\succsim$ is complete and transitive.

Axiom 2 (Continuity). For all $x \in \mathcal{Z}$, $\{z \in \mathcal{Z} | x \succsim z\}$ and $\{z \in \mathcal{Z} | z \succsim x\}$ are closed.

The next is a key axiom of GP. It captures behavior of the DM who cares about the most tempting item within the menu at hand.

Axiom 3 **(Set Betweenness).** For all x, y, if $x \succsim y$, then $x \succsim x \cup y \succsim y$.

GP show that $\succsim$ satisfies Order, Continuity, Set Betweenness and Independence mentioned as (4) if and only if it admits a representation of the form (1).

Let $\mathcal{M}(C)$ be the set of all signed measures on $\Delta(C)$ and

$$\Theta \equiv \{\theta \in \mathcal{M}(C) | \theta(\Delta(C)) = 0\}$$

be the set of all translations. For all $l \in \Delta(C)$ and $\theta \in \Theta$, if $l + \theta \in \Delta(C)$, we can view $l + \theta$ as the lottery obtained by shifting l toward θ. For all $l \in \Delta(C)$, say that $\theta \in \Theta$ is admissible for l if $l + \theta \in \Delta(C)$.

The next axiom is a variant of Strong Continuity of Ergin and Sarver [3].

Axiom 4 **(Properness).** There exists $\theta \in \Theta$ such that for all $l, l' \in \Delta(C)$ with $\{l\} \succ \{l, l'\}$, $\varepsilon > 0$ and $\lambda \in (0,1)$, if $d(l\lambda l', l') < \varepsilon$ and $l + \varepsilon\theta, l' + \varepsilon\theta \in \Delta(C)$, then

$$\{l + \varepsilon\theta, l' + \varepsilon\theta\} \succ \{l, l\lambda l'\}.$$

The ranking $\{l\} \succ \{l, l'\}$ reveals that the DM is tempted by l'. Since $l\lambda l'$ would be preferred to l' and less tempting than l', the DM should rank $\{l, l\lambda l'\}$ over $\{l, l'\}$ for all $\lambda \in (0,1)$. Properness says that, for a sufficiently small λ, the difference between $\{l, l\lambda l'\}$ and $\{l, l'\}$ would be negligible, and hence the ranking can be reversed when $\{l, l'\}$ is slightly translated toward the normatively preferable direction θ.

Suppose that the DM has two rankings of lotteries in mind; one is the commitment ranking that reflects her normative view, and the other is the temptation ranking expressing her craving. If the DM knows that both rankings conform with the expected utility axioms, she should not care about shifting a menu x toward θ because such a translation does not alter the difference between temptation utility of the most tempting lottery and that of a lottery she will choose. Thus we introduce the following axiom:

Axiom 5 **(Translation Invariance).** For all $l, l', k, k' \in \Delta(C)$ and $\theta \in \Theta$, if θ is admissible for these lotteries, then

$$\{l, l'\} \succsim \{k, k'\} \;\Rightarrow\; \{l + \theta, l' + \theta\} \succsim \{k + \theta, k' + \theta\}.$$

Following Noor [9], for all $l \in \Delta(C)$, let

$$\begin{aligned} L_+(l) &\equiv \{l' \in \Delta(C) \,|\, \{l\} \succ \{l, l'\}\}, \\ L_-(l) &\equiv \{l' \in \Delta(C) \,|\, \{l\} \sim \{l, l'\} \succ \{l'\}\}. \end{aligned}$$

Axiom 6 **(Temptation Convexity).** For all $l \in \Delta(C)$, (i) $L_+(l)$ and $L_-(l)$ are convex, and, (ii) for all $\lambda \in (0,1)$,

$$\begin{aligned} &\{l\} \succ \{l, l'\} \;\Rightarrow\; \{l\} \succ \{l, l\lambda l'\}, \\ &\{l\} \sim \{l, l'\} \succ \{l'\} \;\Rightarrow\; \{l\} \sim \{l, l\lambda l'\} \succ \{l\lambda l'\}. \end{aligned}$$

Part (i) says that, if both l' and l'' tempt l, then $l'\lambda l''$ also tempts l. Moreover, if neither l' nor l'' tempts l, then $l'\lambda l''$ does not tempt l either. Part (ii) says that, if l' tempts l, then $l\lambda l'$ also tempts l, and if l' does not tempt l, then neither does $l\lambda l'$.

Axiom 7 (Temptation Consistency). For all $l, l', l'' \in \Delta(C)$ such that $\{l\} \succ \{l, l'\} \succ \{l'\}$ and $\{l\} \succ \{l, l''\}$, if either $\{l''\} \succ \{l', l''\}$ or $\{l'\} \sim \{l', l''\} \succ \{l''\}$, then $\{l, l''\} \succsim \{l, l'\}$.

To interpret the axiom, suppose that the DM exhibits self-control at $\{l, l'\}$. Either $\{l''\} \succ \{l', l''\}$ or $\{l'\} \sim \{l', l''\} \succ \{l''\}$ reveals that l' is at least as tempting as l''. If the DM chooses l over l'' at $\{l, l''\}$, she would incur a smaller self-control cost than she chooses l out of $\{l, l'\}$ because l'' is less tempting than l'. Thus the ex post choice from $\{l, l''\}$ should make the DM better off than that from $\{l, l'\}$, and hence $\{l, l''\}$ should be at least as good as $\{l, l'\}$.

For any $x \in \mathcal{D}$, if there exists a lottery $e(x) \in \Delta(C)$ satisfying $\{e(x)\} \sim x$, $e(x)$ is called a singleton equivalent of x.

Axiom 8 (Mixing Preserves Self-Control). For all $l, l', l'' \in \Delta(C)$ and $\lambda \in (0, 1)$,

(i) if $\{l\} \succ \{l, l'\} \succ \{l'\}$, then $\{l\lambda l''\} \succ \{l\lambda l'', l'\lambda l''\} \succ \{l'\lambda l''\}$, and

(ii) if $\{l, l'\}$ and $\{l, l''\}$ admit singleton equivalents, and if $\{l\} \succ \{l, l'\} \succ \{l'\}$ and $\{l\} \succ \{l, l''\} \succ \{l''\}$, then $\{l, l'\lambda l''\} \succsim \{e(\{l, l'\})\lambda e(\{l, l''\})\}$.

Part (i) says that, if the DM exhibits self-control at $\{l, l'\}$, then she does so at $\{l\lambda l'', l'\lambda l''\}$ because mixing of items may mitigate a self-control cost. As in part (ii), suppose that the DM exhibits self-control at both $\{l, l'\}$ and $\{l, l''\}$. Since a self-control cost at $\{l, l'\lambda l''\}$ will be less than the highest self-control cost between at $\{l, l'\}$ and $\{l, l''\}$, she should exhibit self-control at $\{l, l'\lambda l''\}$. Part (ii) says that a self-control cost from choosing l out of $\{l, l'\lambda l''\}$ is smaller than the average self-control cost between from choosing l out of $\{l, l'\}$ and that of $\{l, l''\}$. The role of part (ii) is to establish convexity of φ.

Suppose that l' and k' tempt l and k, respectively, and that the DM exhibits self-control at $\{l, l'\}$. Suppose that lotteries l, l', k, k' and another lottery h are mixed with the same weight λ, and translated so as to satisfy $l\lambda h + \theta = k\lambda h + \theta'$. If $\succsim$ satisfies MPSC and Translation Invariance, she exhibits self-control also at $\{l\lambda h + \theta, l'\lambda h + \theta\}$. Since $l\lambda h + \theta = k\lambda h + \theta'$, the ranking, $\{k\lambda h + \theta', k'\lambda h + \theta'\} \succsim \{l\lambda h + \theta, l'\lambda h + \theta\}$, implies that the self-control cost from choosing $l\lambda h + \theta$ out of $\{l\lambda h + \theta, l'\lambda h + \theta\}$ exceeds the cost of choice at $\{k\lambda h + \theta', k'\lambda h + \theta'\}$. This ranking should be preserved when $\{l, l'\}$ and $\{k, k'\}$ are translated so as to satisfy $l + \tilde{\theta} = k + \tilde{\theta}'$. That is, the ranking $\{k + \tilde{\theta}', k' + \tilde{\theta}'\} \succsim \{l + \tilde{\theta}, l' + \tilde{\theta}\}$ would still hold. This axiom has an appeal when the DM expects her temptation ranking to conform with the expected utility axioms.

Prior to formulating the above idea, notice that, for some $\{l, l'\}$ and $\{k, k'\}$, there exist no admissible translations at all satisfying $l + \tilde{\theta} = k + \tilde{\theta}'$. Thus, alternatively, consider singleton equivalents of $\{l, l'\}$ and of $\{k, k'\}$ as follows:

Axiom 9 (**Monotone Self-Control**). For all $l, l', k, k', h \in \Delta(C)$ such that $\{l\} \succ \{l, l'\} \succ \{l'\}$ and $\{k\} \succ \{k, k'\}$, if there exist $\lambda \in (0, 1)$ and admissible translations $\theta, \theta' \in \Theta$ with $l\lambda h + \theta = k\lambda h + \theta'$ such that

$$\{k\lambda h + \theta', k'\lambda h + \theta'\} \succsim \{l\lambda h + \theta, l'\lambda h + \theta\},$$

then $\{e(\{k, k'\}) + \tilde{\theta}'\} \succsim \{e(\{l, l'\}) + \tilde{\theta}\}$ for all admissible translations $\tilde{\theta}, \tilde{\theta}'$ satisfying $e(\{l\}) + \tilde{\theta} = e(\{k\}) + \tilde{\theta}'$.

3.2 Results

We are ready to state the main theorem.

Theorem 3.1.

(i) If a self-control preference $\succsim$ satisfies Order, Continuity, Set Betweenness, Translation Invariance, Temptation Convexity, Temptation Consistency, MPSC and Monotone Self-Control, then $\succsim$ admits a partial convex self-control representation.

(ii) Suppose that C is finite. A self-control preference $\succsim$ satisfies Order, Continuity, Set Betweenness, Translation Invariance, Temptation Convexity, Temptation Consistency, MPSC, Monotone Self-Control and Properness if and only if $\succsim$ admits a convex self-control representation.

Theorem 3.1 (i) says that there exists a partial convex self-control representation if $\succsim$ satisfies all the axioms except for Properness. The necessary part does not hold in general because φ may not be convex on the whole domain, which may violate MPSC and Monotone Self-Control. The reason why we cannot ensure convexity of φ is that the axioms are not enough to establish Lipschitz continuity of φ. As stated in Theorem 3.1 (ii), if we impose Properness in addition and assume that C is finite, then the set of axioms characterizes the convex self-control model.

We turn to uniqueness of convex self-control representations. For a convex self-control representation (u, v, φ), define

$$W(u, v, \varphi) = \{w \in \mathbb{R}_+ | w = v(l') - v(l) \text{ for some } l, l' \text{ with } \{l\} \succ \{l, l'\} \succ \{l'\}\}.$$

We call $W(u, v, \varphi)$ *the self-control region for* (u, v, φ).

Theorem 3.2. *Suppose that a self-control preference $\succsim$ admits a convex self-control representation U with components (u, v, φ). If $u' = \alpha_u u + \beta_u$ and $v' = \alpha_v v + \beta_v$ for some $\alpha_u, \alpha_v > 0$, $\beta_u, \beta_v \in \mathbb{R}$ and $\varphi'(\alpha_v w) = \alpha_u \varphi(w)$ for $w \in \mathbb{R}_+$, then the functional form with components (u', v', φ') also represent $\succsim$. Conversely, if another representation U' with components (u', v', φ') also represents $\succsim$, then there exist $\alpha_u, \alpha_v > 0$ and $\beta_u, \beta_v \in \mathbb{R}$ such that $u' = \alpha_u u + \beta_u$, $v' = \alpha_v v + \beta_v$, $W(u', v', \varphi') = \alpha_v W(u, v, \varphi)$, and $\varphi'(\alpha_v w) = \alpha_u \varphi(w)$ for all $w \in W(u, v, \varphi)$.*

From this theorem, we know that u and v are unique up to positive affine transformation. Moreover, if (u, v, φ) and $(\tilde{u}, \tilde{v}, \tilde{\varphi})$ represent the same preference and $\tilde{v} = \alpha_v v + \beta_v$, then for $\tilde{w} = \alpha_v w$ and $w \in W(u, v, \varphi)$,

$$\frac{\tilde{w}\tilde{\varphi}''(\tilde{w})}{\tilde{\varphi}'(\tilde{w})} = \frac{w\varphi''(w)}{\varphi'(w)}$$

where f' and f'' denote the first and the second derivatives of f, respectively. Thus the curvature of φ is uniquely determined within the self-control region.

4 Proofs

4.1 Sufficiency Part of Theorem 3.1

Throughout the proof, we assume that $\succsim$ satisfies all the axioms except for Properness and C is a compact metric space unless otherwise stated.

Lemma 6 of Ergin and Sarver [3] show that Order, Continuity, and Translation Invariance imply Commitment Independence: For all $l, l', l'' \in \Delta(C)$,

$$\{l\} \succsim \{l'\} \Leftrightarrow \lambda\{l\} + (1-\lambda)\{l''\} \succsim \lambda\{l'\} + (1-\lambda)\{l''\}.$$

Since $\succsim$ satisfies all the vNM axioms on $\Delta(C)$, there exists a continuous mixture linear function $u : \Delta(C) \to \mathbb{R}$ which represents $\succsim$ on $\Delta(C)$.

Since u is continuous on $\Delta(C)$, there exist a maximal and a minimal lotteries $c^+, c^- \in \Delta(C)$. Without loss of generality, we can assume $u(c^+) = 1$ and $u(c^-) = 0$. For any finite menu x, Set Betweenness implies that $\{c^+\} \succsim x \succsim \{c^-\}$. Continuity ensures that $\{c^+\} \succsim x \succsim \{c^-\}$ for all $x \in \mathcal{D}$. By Continuity, there exists a unique number $\alpha(x) \in [0,1]$ such that $x \sim \{\alpha(x)c^+ + (1-\alpha(x))c^-\}$. That is, all menus admit singleton equivalents. Define

$$U(x) = u(\alpha(x)c^+ + (1-\alpha(x))c^-).$$

Then U represents $\succsim$. Moreover, $U(\{l\}) = u(l)$ for all $l \in \Delta(C)$. This representation has the following properties:

Lemma 4.1.

(i) For all $\lambda \in (0,1)$ and all $l, l', l'' \in \Delta(C)$ such that $\{l\} \succ \{l, l'\} \succ \{l'\}$ and $\{l\} \succ \{l, l''\} \succ \{l''\}$, $U(\{l, l'\lambda l''\}) \geq \lambda U(\{l, l'\}) + (1-\lambda)U(\{l, l''\})$.

(ii) For all $l, l' \in \Delta(C)$ and $\theta \in \Theta$, if $l + \theta, l' + \theta \in \Delta(C)$, then $U(\{l + \theta, l' + \theta\}) = U(\{l, l'\}) + u(\theta)$.

Proof. (i) By definition of singleton equivalent, $U(\{l, l'\}) = u(e(\{l, l'\}))$ and $U(\{l, l''\}) = u(e(\{l, l''\}))$. By MPSC (ii) and mixture linearity of u,

$$\begin{aligned} U(\{l, l'\lambda l''\}) &\geq u(\lambda e(\{l, l'\}) + (1-\lambda)e(\{l, l''\})) = \lambda u(e(\{l, l'\})) + (1-\lambda)u(e(\{l, l''\})) \\ &= \lambda U(\{l, l'\}) + (1-\lambda)U(\{l, l''\}). \end{aligned}$$

(ii) By Set Betweenness, $\{l\} \succsim \{l, l'\} \succsim \{l'\}$. Since u is continuous, there exists $\lambda \in [0,1]$ such that $U(\{l, l'\}) = u(l\lambda l')$. If $l+\theta, l'+\theta \in \Delta(C)$, $l\lambda l' + \theta = (l+\theta)\lambda(l'+\theta) \in \Delta(C)$. Hence Translation Invariance implies that

$$U(\{l+\theta, l'+\theta)\}) = u(l\lambda l' + \theta) = u(l\lambda l') + u(\theta) = U(\{l, l'\}) + u(\theta).$$

□

Lemma 7 of Noor [9] shows that, under Translation Invariance and Temptation Convexity, there exists a continuous mixture linear function $v : \Delta(C) \to \mathbb{R}_+$ such that, if $\{l\} \succ \{l'\}$, then

$$\{l\} \succ \{l, l'\} \Leftrightarrow v(l') > v(l).$$

Without loss of generality, assume that $v(\Delta(C)) = [0,1]$. By his construction, we know that $L_+(l) \subset \{l' \in \Delta(C) | v(l') > v(l)\}$ and $L_-(l) \subset \{l' \in \Delta(C) | v(l') \le v(l)\}$.

Lemma 4.2. *For all $l, l', l'' \in \Delta(C)$, if $\{l\} \succ \{l, l'\} \succ \{l'\}$, $\{l\} \succ \{l, l''\}$ and $v(l') = v(l'')$, then $\{l, l''\} \succsim \{l, l'\}$.*

Proof. Case 1: $\{l''\} \succ \{l'\}$. Let l^M and l^m be a maximal and a minimal lottery with respect to v. Since v is not constant, $v(l^M) > v(l^m)$. By Continuity, for all $\lambda \in (0,1)$ close enough to one, $v(l'\lambda l^M) > v(l''\lambda l^m)$ and $u(l''\lambda l^m) > u(l'\lambda l^M)$. Thus $\{l''\lambda l^m\} \succ \{l''\lambda l^m, l'\lambda l^M\}$. Moreover, Continuity implies $\{l\} \succ \{l, l'\lambda l^M\} \succ \{l'\lambda l^M\}$ and $\{l\} \succ \{l, l''\lambda l^m\}$. Thus, by Temptation Consistency, $\{l, l''\lambda l^m\} \succsim \{l, l'\lambda l^M\}$. We have $\{l, l''\} \succsim \{l, l'\}$ as $\lambda \to 1$.

Case 2: $\{l'\} \succ \{l''\}$. If $\{l'\} \succ \{l', l''\}$, we have $v(l'') > v(l')$, which contradicts the assumption. Hence Set Betweenness implies $\{l'\} \sim \{l', l''\} \succ \{l''\}$. By Temptation Consistency, $\{l, l''\} \succsim \{l, l'\}$.

Let l^+ and l^- be a maximal and a minimal lottery with respect to u. Since u is not constant, $u(l^+) > u(l^-)$.

Case 3: $\{l''\} \sim \{l'\}$ and $v(l^+) = v(l^-)$. For all $\lambda \in (0,1)$ close enough to one, Continuity implies $\{l''\lambda l^+\} \succ \{l'\lambda l^-\}$, $\{l\} \succ \{l, l'\lambda l^-\} \succ \{l'\lambda l^-\}$ and $\{l\} \succ \{l, l''\lambda l^+\}$. Moreover, $v(l'\lambda l^+) = v(l''\lambda l^-)$. Thus, by Case 1, we have $\{l, l''\lambda l^+\} \succsim \{l, l'\lambda l^-\}$, and hence $\{l, l''\} \succsim \{l, l'\}$ as $\lambda \to 1$.

Case 4: $\{l''\} \sim \{l'\}$ and $v(l^+) > v(l^-)$. For all $\lambda \in (0,1)$, $v(l'\lambda l^+) > v(l''\lambda l^-)$ and $u(l'\lambda l^+) > u(l''\lambda l^-)$. Thus we must have $\{l'\lambda l^+\} \sim \{l'\lambda l^+, l''\lambda l^-\} \succ \{l''\lambda l^-\}$. Moreover, for all $\lambda \in (0,1)$ close enough to one, Continuity implies $\{l\} \succ \{l, l'\lambda l^+\} \succ \{l'\lambda l^+\}$ and $\{l\} \succ \{l, l''\lambda l^-\}$. By Temptation Consistency, $\{l, l''\lambda l^-\} \succsim \{l, l'\lambda l^+\}$, and hence $\{l, l''\} \succsim \{l, l'\}$ as $\lambda \to 1$.

Case 5: $\{l''\} \sim \{l'\}$ and $v(l^-) > v(l^+)$. For all $\lambda \in (0,1)$, $v(l'\lambda l^-) > v(l''\lambda l^+)$ and $u(l''\lambda l^+) > u(l'\lambda l^-)$. Thus we have $\{l''\lambda l^+\} \succ \{l''\lambda l^+, l'\lambda l^-\}$. Moreover, for all $\lambda \in (0,1)$ close enough to one, Continuity implies $\{l\} \succ \{l, l'\lambda l^-\} \succ \{l'\lambda l^-\}$ and $\{l\} \succ \{l, l''\lambda l^+\}$. By Temptation Consistency, $\{l, l''\lambda l^+\} \succsim \{l, l'\lambda l^-\}$, and hence $\{l, l''\} \succsim \{l, l'\}$ as $\lambda \to 1$. This completes the proof. □

Let

$$A = \{w \in [0,1] \mid w = v(l') - v(l), \text{ for some } l, l' \text{ such that } \{l\} \succ \{l,l'\} \succ \{l'\}\}.$$

Since $\succsim$ is a self-control preference, A is non-empty.

Lemma 4.3. *(i) A is an interval with* $\inf A = 0$. *(ii) A is open in* $[0,1]$.

Proof. (i) It suffices to show that, for all $w \in A$, $\lambda w \in A$ for all $\lambda \in (0,1)$. Let $w \in A$. There exist l, l' such that $w = v(l') - v(l)$ and $\{l\} \succ \{l,l'\} \succ \{l'\}$. By MPSC (i), $\{l\} \succ \{l, (1-\lambda)l + \lambda l'\} \succ \{(1-\lambda)l + \lambda l'\}$. Thus $\lambda w = \lambda(v(l') - v(l)) = v((1-\lambda)l + \lambda l') - v(l) \in A$.

(ii) Notice first that $0 \notin A$ by definition of v. If there exist l, l' such that $\{l\} \succ \{l,l'\} \succ \{l'\}$ and $v(l') - v(l) = 1$, we can conclude from (i) that $A = (0,1]$, and hence A is relatively open in $[0,1]$. Now suppose that $A = (0, \overline{w}]$ for some $\overline{w} < 1$. Then there exist l, l' such that $\{l\} \succ \{l,l'\} \succ \{l'\}$ and $v(l') - v(l) = \overline{w}$. We have either $\max_{\Delta(C)} v > v(l')$ or $\min_{\Delta(C)} v < v(l)$. In case of the former, Continuity implies that there exists l'' sufficiently close to l' such that $\{l\} \succ \{l,l''\} \succ \{l''\}$ and $v(l'') > v(l')$. Thus $\overline{w} < v(l'') - v(l) \in A$, which is a contradiction. In case of the latter, there exists l'' sufficiently close to l such that $\{l''\} \succ \{l'',l'\} \succ \{l'\}$ and $v(l) > v(l'')$. Thus $\overline{w} < v(l') - v(l'') \in A$, which is a contradiction either. Therefore A must be an open interval in $[0,1]$. □

Define $\varphi : A \to \mathbb{R}_{++}$ by

$$\varphi(w) \equiv u(l) - U(\{l,l'\}),$$

where l, l' satisfy $\{l\} \succ \{l,l'\} \succ \{l'\}$ and $w = v(l') - v(l)$.

Lemma 4.4. *φ is well-defined, that is, for any $l, l', k, k' \in \Delta(C)$ such that $\{l\} \succ \{l,l'\} \succ \{l'\}$ and $\{k\} \succ \{k,k'\} \succ \{k'\}$,*

$$v(k') - v(k) = v(l') - v(l) \quad \Rightarrow \quad u(k) - U(\{k,k'\}) = u(l) - U(\{l,l'\}).$$

Proof. Assume that $v(k') - v(k) = v(l') - v(l)$. Take a full support lottery $h \in \Delta(C)$. By MPSC (i), for all $\lambda \in (0,1)$,

$$\{l\lambda h\} \succ \{l\lambda h, l'\lambda h\} \succ \{l'\lambda h\}, \ \{k\lambda h\} \succ \{k\lambda h, k'\lambda h\} \succ \{k'\lambda h\}.$$

Let $\theta \equiv l\lambda h - k\lambda h \in \Theta$. For small λ, $k'\lambda h + \theta \in \Delta(C)$. Since v is mixture linear,

$$\begin{aligned} v(k'\lambda h + \theta) - v(l\lambda h) &= v(k'\lambda h + \theta) - v(k\lambda h + \theta) = \lambda(v(k') - v(k)) \\ &= \lambda(v(l') - v(l)) = v(l'\lambda h) - v(l\lambda h). \end{aligned}$$

Thus $v(k'\lambda h + \theta) = v(l'\lambda h)$. Furthermore, by Translation Invariance, $\{k\lambda h + \theta\} \succ \{k\lambda h + \theta, k'\lambda h + \theta\} \succ \{k'\lambda h + \theta\}$, that is, $\{l\lambda h\} \succ \{l\lambda h, k'\lambda h + \theta\} \succ \{k'\lambda h + \theta\}$. By Lemma 4.2, $\{l\lambda h, l'\lambda h\} \sim \{l\lambda h, k'\lambda h + \theta\}$. Let $e(\{l\}) = \alpha(\{l\})c^+ + (1-\alpha(\{l\}))c^-$, $e(\{k\}) = \alpha(\{k\})c^+ + (1-\alpha(\{k\}))c^-$, $e(\{l,l'\}) = \alpha(\{l,l'\})c^+ + (1-\alpha(\{l,l'\}))c^-$ and $e(\{k,k'\}) = \alpha(\{k,k'\})c^+ + (1-$

$\alpha(\{k,k'\}))c^-$. By definition, $u(l) = u(e(\{l\}))$, $u(k) = u(e(\{k\}))$, $U(\{l,l'\}) = u(e(\{l,l'\}))$, and $U(\{k,k'\}) = u(e(\{k,k'\}))$. Since these lotteries belong to

$$\{\alpha c^+ + (1-\alpha)c^- \in \Delta(C) \mid \alpha \in [0,1]\},$$

we can find $\theta_l, \theta_k \in \Theta$ such that $e(\{l\})+\theta_l = e(\{k\})+\theta_k$ and $e(\{l\})+\theta_l, e(\{k\})+\theta_k, e(\{l,l'\})+\theta_l, e(\{k,k'\}) + \theta_k \in \Delta(C)$. By Monotone Self-Control, $\{e(\{l,l'\}) + \theta_l\} \succsim \{e(\{k,k'\}) + \theta_k\}$, and hence

$$\begin{aligned} & u(e(\{l,l'\}) + \theta_l) \geq u(e(\{k,k'\}) + \theta_k) \\ \Leftrightarrow\ & u(k+\theta_k) - u(e(\{k,k'\}) + \theta_k) \geq u(l+\theta_l) - u(e(\{l,l'\}) + \theta_l) \\ \Leftrightarrow\ & u(k) - u(e(\{k,k'\})) \geq u(l) - u(e(\{l,l'\})) \\ \Leftrightarrow\ & u(k) - U(\{k,k'\}) \geq u(l) - U(\{l,l'\}). \end{aligned}$$

By the symmetric argument, we can show $u(l) - U(\{l,l'\}) \geq u(k) - U(\{k,k'\})$ as well. □

Lemma 4.5. *(i) φ is weakly convex, and (ii)* $\liminf_{w\to 0} \varphi(w) = 0$.

Proof. (i) Take any $w, w' \in A$ with $w > w'$ and $\lambda \in (0,1)$. There exist $l, l' \in \Delta(C)$ such that $w = v(l') - v(l)$ and $\{l\} \succ \{l,l'\} \succ \{l'\}$. Since $w > w' > 0$, there exists $\mu \in (0,1)$ satisfying $v(l\mu l') - v(l) = w'$. Define $l'' \equiv l\mu l'$. By MPSC (i), $\{l\} \succ \{l,l''\} \succ \{l''\}$. By definition,

$$\varphi(w) = u(l) - U(\{l,l'\}),\ \varphi(w') = u(l) - U(\{l,l''\}). \tag{14}$$

By mixture linearity of v,

$$\lambda w + (1-\lambda)w' = \lambda(v(l') - v(l)) + (1-\lambda)(v(l'') - v(l)) = v(l'\lambda l'') - v(l).$$

Since $v(l'\lambda l'') > v(l)$ and $u(l) > u(l'\lambda l'')$, $\{l\} \succ \{l, l'\lambda l''\}$. Furthermore, by MPSC (ii),

$$\begin{aligned} U(\{l,l'\lambda l''\}) &\geq u(\lambda e(\{l,l'\}) + (1-\lambda)e(\{l,l''\})) = \lambda u(e(\{l,l'\})) + (1-\lambda)u(e\{l,l''\}) \\ &> \lambda u(l') + (1-\lambda)u(l'') = u(l'\lambda l''). \end{aligned}$$

Thus by definition of φ,

$$\varphi(\lambda w + (1-\lambda)w') = \varphi(v(l'\lambda l'') - v(l)) = u(l) - U(\{l, l'\lambda l''\}). \tag{15}$$

Taking (14), (15) and Lemma 4.1 together,

$$\begin{aligned} \lambda\varphi(w) + (1-\lambda)\varphi(w') &= \lambda u(l) + (1-\lambda)u(l) - \lambda U(\{l,l'\}) - (1-\lambda)U(\{l,l''\}) \\ &\geq u(l) - U(\{l,l'\lambda l''\}) = \varphi(\lambda w + (1-\lambda)w'). \end{aligned}$$

(ii) Take any $w \in A$. There exist l, l' such that $w = v(l') - v(l)$ and $\{l\} \succ \{l,l'\} \succ \{l'\}$. By MPSC (i), $\{l\} \succ \{l, (1-\lambda)l + \lambda l'\} \succ \{(1-\lambda)l + \lambda l'\}$ for all $\lambda \in (0,1)$. Since

$$\varphi(\lambda w) = \varphi(v((1-\lambda)l + \lambda l') - v(l)) = u(l) - U(\{l, (1-\lambda)l + \lambda l'\}),$$

$0 = \liminf_{\lambda\to 0} \varphi(\lambda w) \geq \liminf_{w\to 0} \varphi(w) \geq 0$, and hence $\liminf_{w\to 0} \varphi(w) = 0$. □

Denote the closure of A by $\overline{A}$. By Lemma 4.3 (i), $\overline{A}$ is a closed interval including 0.

Lemma 4.6. *There exists a unique continuous convex extension of φ to $\overline{A}$. Moreover, the extension $\overline{\varphi}$ is strictly increasing with $\overline{\varphi}(0) = 0$.*

Proof. Theorem 10.3 [11, p.85] ensures the existence of a unique continuous convex extension $\overline{\varphi}$. By Lemma 4.5 (ii),

$$0 = \liminf_{w \to 0} \varphi(w) = \liminf_{w \to 0} \overline{\varphi}(w) = \lim_{w \to 0} \overline{\varphi}(w) = \overline{\varphi}(0).$$

Finally, we claim that $\overline{\varphi}$ is strictly increasing, that is, $w' > w$ implies $\overline{\varphi}(w') > \overline{\varphi}(w)$. Since $\overline{\varphi}(w') > 0 = \overline{\varphi}(0)$ for all $w' \in A$, the claim holds when $w = 0$ and $w' \in A$. Suppose that there exist $w, w' \in A$ such that $w' > w$ and $\overline{\varphi}(w') \leq \overline{\varphi}(w)$. There exists $\lambda \in (0,1)$ with $w = \lambda w'$. Since $\overline{\varphi}$ is weakly convex,

$$\begin{aligned} \overline{\varphi}(w) &= \overline{\varphi}(\lambda w') = \overline{\varphi}(\lambda w' + (1-\lambda)0) \leq \lambda \overline{\varphi}(w') + (1-\lambda)\overline{\varphi}(0) \\ &= \lambda \overline{\varphi}(w') \leq \lambda \overline{\varphi}(w) < \overline{\varphi}(w), \end{aligned}$$

which is a contradiction. Thus $\overline{\varphi}$ is strictly increasing on $A \cup \{0\}$. Let $\overline{w} \equiv \sup A$. Since $\varphi(w) < \sup\{\varphi(w) | w \in A\} = \overline{\varphi}(\overline{w})$ for all $w \in A \cup \{0\}$, $\overline{\varphi}$ is strictly increasing on $\overline{A}$. □

Lemma 4.7. *Assume in addition that C is finite and $\succsim$ satisfies Properness.*

(i) There exists $K > 0$ such that, for all $l, l' \in \Delta(C)$ with $\{l\} \succ \{l, l'\}$ and $\lambda \in (0,1)$, $U(\{l, l\lambda l'\}) - U(\{l, l'\}) \leq K d(l\lambda l', l')$.

(ii) There exists $\overline{K} > 0$ such that, for all $w, w' \in \overline{A}$ with $w' > w$,

$$\frac{\varphi(w') - \varphi(w)}{w' - w} \leq \overline{K}.$$

Proof. (i) Since $\{l\} \succ \{l, l'\}$, we have $v(l') > v(l)$ and $u(l) > u(l')$, and hence $\{l\} \succ \{l, l\lambda l'\}$ for all $\lambda \in (0,1)$. If $\{l\} \succ \{l, l'\} \succ \{l'\}$, MPSC (i) implies $\{l\} \succ \{l, l\lambda l'\} \succ \{l\lambda l'\}$. Since $v(l') - v(l) > v(l\lambda l') - v(l)$ and $\overline{\varphi}$ is strictly increasing,

$$u(l) - U(\{l, l\lambda l'\}) = \overline{\varphi}(v(l\lambda l') - v(l)) < \overline{\varphi}(v(l') - v(l)) = u(l) - U(\{l, l'\}),$$

that is, $U(\{l, l\lambda l'\}) > U(\{l, l'\})$. If $\{l\} \succ \{l, l'\} \sim \{l'\}$, by Set Betweenness,

$$\{l, l\lambda l'\} \succsim \{l\lambda l'\} \succ \{l'\} \sim \{l, l'\}.$$

In either case, we have $U(\{l, l\lambda l'\}) > U(\{l, l'\})$ for all $\lambda \in (0,1)$.

We adapt the argument in Ergin and Sarver [3, Lemma 16]. Since C is finite, the metric $d(a,b)$ on $\Delta(C)$ is understood to be the metric induced by a norm in $\mathbb{R}^{\#C}$. That is, $d(a,b) = \|a - b\|$. Let $\theta \in \Theta$ be the object guaranteed by Properness. Let $K \equiv u(\theta)$.

Step 1: For all $\varepsilon > 0$ and $\lambda \in (0,1)$, if $l + \varepsilon\theta, l' + \varepsilon\theta \in \Delta(C)$ and $d(l\lambda l', l') < \varepsilon$, then $U(\{l, l\lambda l'\}) - U(\{l, l'\}) \leq Kd(l\lambda l', l')$.

Take any $\varepsilon' > 0$ with $d(l\lambda l', l') < \varepsilon' < \varepsilon$. Then $l + \varepsilon'\theta, l' + \varepsilon'\theta \in \Delta(C)$. By Properness and Lemma 4.1 (ii),

$$U(\{l, l\lambda l\}) < U(\{l + \varepsilon'\theta, l' + \varepsilon'\theta\}) = U(\{l, l'\}) + \varepsilon' u(\theta) = U(\{l, l'\}) + K\varepsilon'.$$

Hence $U(\{l, l\lambda l'\}) - U(\{l, l'\}) < K\varepsilon'$. Since this inequality holds for all ε' with $d(l\lambda l', l') < \varepsilon' < \varepsilon$, $U(\{l, l\lambda l'\}) - U(\{l, l'\}) \leq Kd(l\lambda l', l')$.

Since $U(\{l, l\lambda l'\}) > U(\{l, l'\})$, K is positive.

Step 2: For all $\varepsilon > 0$ and $\lambda \in (0,1)$, if $l + \varepsilon\theta, l' + \varepsilon\theta \in \Delta(C)$, then $U(\{l, l\lambda l'\}) - U(\{l, l'\}) \leq Kd(l\lambda l', l')$.

For all $\alpha \in [0,1]$, let $l(\alpha) = \alpha(l\lambda l') + (1-\alpha)l'$. Notice that $\{l\} \succ \{l, l(\alpha)\}$. Since $l + \varepsilon\theta, l' + \varepsilon\theta \in \Delta(C)$,

$$l(\alpha) + \varepsilon\theta = \alpha(l\lambda l') + (1-\alpha)l' + \theta = \alpha\lambda(l + \varepsilon\theta) + (1 - \alpha\lambda)(l' + \varepsilon\theta) \in \Delta(C).$$

By continuity of $l(\alpha)$, for all α, we can find an open interval $O(\alpha) \subset [0,1]$ such that $\alpha \in O(\alpha)$ and $d(l(\alpha), l(\alpha')) < \frac{\varepsilon}{2}$ for all $\alpha' \in O(\alpha)$. Then $\{O(\alpha) | \alpha \in [0,1]\}$ is an open cover of $[0,1]$. Since $[0,1]$ is compact, there exists a finite open cover $\{O(\alpha_1), \cdots, O(\alpha_k)\}$. Without loss of generality, assume $\alpha_1 \leq \cdots \leq \alpha_k$. Let $\alpha_0 \equiv 0$ and $\alpha_{k+1} \equiv 1$. Since $\{l\} \succ \{l, l(\alpha_i)\}$, $l(\alpha_i) + \varepsilon\theta \in \Delta(C)$ and $d(l(\alpha_{i+1}), l(\alpha_i)) < \varepsilon$, Step 1 implies

$$U(\{l, l(\alpha_{i+1})\}) - U(\{l, l(\alpha_i)\}) \leq Kd(l(\alpha_{i+1}), l(\alpha_i)).$$

Since $d(l(\alpha_{i+1}), l(\alpha_i)) = \|l(\alpha_{i+1}) - l(\alpha_i)\| = (\alpha_{i+1} - \alpha_i)\|l\lambda l' - l'\|$ and $l(\alpha_0) = l'$ and $l(\alpha_{k+1}) = l\lambda l'$,

$$\begin{aligned} U(\{l, l\lambda l'\}) - U(\{l, l'\}) &= \sum_{i=0}^{k} (U(\{l, l(\alpha_{i+1})\}) - U(\{l, l(\alpha_i)\}) \leq K \sum_{i=0}^{k} d(l(\alpha_{i+1}), l(\alpha_i)) \\ &= Kd(l\lambda l', l') \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) = Kd(l\lambda l', l'). \end{aligned}$$

Step 3: For all l, l' with $\{l\} \succ \{l, l'\}$ and $\lambda \in (0,1)$, $U(\{l, l\lambda l'\}) - U(\{l, l'\}) \leq Kd(l\lambda l', l')$.

Take a lottery l^* in the relative interior of $\Delta(C)$. There exists $\kappa > 0$ such that $l^* + \kappa\theta \in \Delta(C)$. Let $l^n = (1 - \frac{1}{n})l + \frac{1}{n}l^*$, $l'^n = (1 - \frac{1}{n})l' + \frac{1}{n}l^*$ and $\varepsilon^n = \frac{\kappa}{n}$. Since $v(l') > v(l)$ and $u(l) > u(l')$, we have $v(l'^n) > v(l^n)$ and $u(l^n) > u(l'^n)$, and hence $\{l^n\} \succ \{l^n, l'^n\}$. Furthermore, $l^n + \varepsilon^n\theta, l'^n + \varepsilon^n\theta \in \Delta(C)$. Thus Step 2 implies, for all $\lambda \in (0,1)$,

$$U(\{l^n, l^n\lambda l'^n\}) - U(\{l^n, l'^n\}) \leq Kd(l^n\lambda l'^n, l'^n).$$

Since U is continuous, we have $U(\{l, l\lambda l'\}) - U(\{l, l'\}) \leq Kd(l\lambda l', l')$.

(ii) Take a monotone sequence $w^n \to \overline{w}$ with $w^n \in A$. There exist a^n, b^n such that $w^n = v(b^n) - v(a^n)$ and $\{a^n\} \succ \{a^n, b^n\} \succ \{b^n\}$. Since $\{a^n\}_{n=0}^{\infty}$ and $\{b^n\}_{n=0}^{\infty}$ are sequences in $\Delta(C)$, assume $a^n \to a^0$ and $b^n \to b^0$ without loss of generality. For all $\lambda \in (0,1)$, $\{a^n\} \succ \{a^n, a^n\lambda b^n\} \succ \{a^n\lambda b^n\}$ and $v(a^n\lambda b^n) - v(a^n) = (1-\lambda)(v(b^n) - v(a^n)) = (1-\lambda)w^n$. We have

$$\begin{aligned}\frac{\varphi(\overline{w}) - \varphi((1-\lambda)\overline{w})}{\lambda\overline{w}} &= \lim_{n\to\infty} \frac{\varphi(w^n) - \varphi((1-\lambda)w^n)}{\lambda w^n} \\ &= \lim_{n\to\infty} \frac{U(\{a^n, a^n\lambda b^n\}) - U(\{a^n, b^n\})}{\lambda w^n} \\ &= \frac{U(\{a^0, a^0\lambda b^0\}) - U(\{a^0, b^0\})}{\lambda\overline{w}}. \end{aligned} \tag{16}$$

Since the LHS of (16) is positive, $U(\{a^0, a^0\lambda b^0\}) > U(\{a^0, b^0\})$. Since $v(b^0) - v(a^0) = \overline{w} > 0$, $v(b^0) > v(a^0)$. By continuity of u, $u(a^0) \geq u(b^0)$. If $u(a^0) = u(b^0)$, Set Betweenness implies $U(\{a^0, a^0\lambda b^0\}) = U(\{a^0, b^0\})$. Thus we must have $u(a^0) > u(b^0)$. By definition of v, $\{a^0\} \succ \{a^0, b^0\}$. Let $K > 0$ be the number guaranteed by (i). For all $\lambda \in (0,1)$,

$$U(\{a^0, a^0\lambda b^0\}) - U(\{a^0, b^0\}) \leq Kd(a^0\lambda b^0, b^0) = K\|b^0 - a^0\lambda b^0\| = \lambda K\|b^0 - a^0\|. \tag{17}$$

Taking (16) and (17) together, $\overline{K} \equiv K\|b^0 - a^0\|/\overline{w}$ satisfies

$$\sup_{\lambda\to 0} \frac{\varphi(\overline{w}) - \varphi((1-\lambda)\overline{w})}{\lambda\overline{w}} \leq \overline{K}. \tag{18}$$

Since φ is strictly increasing and convex, (18) implies the desired property. □

Lemma 4.8. *There exists a strictly increasing continuous extension of $\overline{\varphi}$ to $\mathbb{R}_+$ such that, for all $a, b \in \Delta(C)$,*

$$U(\{a,b\}) = \max_{l\in\{a,b\}} \left(u(l) - \overline{\overline{\varphi}}\left(\max_{\{a,b\}} v - v(l)\right)\right).$$

Furthermore, if C is finite and $\succsim$ satisfies Properness, then the extension $\overline{\overline{\varphi}}$ can be taken to be convex.

Proof. By Lemma 4.6, there exists a strictly increasing continuous extension of $\overline{\varphi}$ to $\mathbb{R}_+$. Without loss of generality, assume $\{a\} \succsim \{b\}$. By Set Betweenness, $\{a\} \succsim \{a,b\} \succsim \{b\}$. There are four cases: (i) $\{a\} \succ \{a,b\} \succ \{b\}$; (ii) $\{a\} \succ \{a,b\} \sim \{b\}$; (iii) $\{a\} \sim \{a,b\} \succ \{b\}$; and (iv) $\{a\} \sim \{a,b\} \sim \{b\}$. We first show that, in cases of (i), (iii) and (iv), the desired result holds for any increasing extension $\overline{\overline{\varphi}}$.

(i) $\{a\} \succ \{a,b\} \succ \{b\}$. In this case, $v(b) > v(a)$. By definition of φ, $U(\{a,b\}) = u(a) - \varphi(v(b) - v(a)) > u(b)$. Thus $U(\{a,b\})$ can be expressed as the desired form.

(iii) $\{a\} \sim \{a,b\} \succ \{b\}$. By definition, $b \in L_-(a)$. By construction of v, $v(a) \geq v(b)$. Since $U(\{a,b\}) = u(a) > u(b) - \overline{\overline{\varphi}}(v(a) - v(b))$, $U(\{a,b\})$ is represented by the desired form.

(iv) $\{a\} \sim \{a,b\} \sim \{b\}$. If $v(b) \geq v(a)$, $U(\{a,b\}) = u(b) \geq u(a) - \overline{\overline{\varphi}}(v(b) - v(a))$. If $v(a) \geq v(b)$, we have $U(\{a,b\}) = u(a) \geq u(b) - \overline{\overline{\varphi}}(v(a) - v(b))$. In either case, $U(\{a,b\})$ is represented by the desired form.

Now turn to case (ii) $\{a\} \succ \{a,b\} \sim \{b\}$. This ranking implies $v(b) > v(a)$.

Step 1: If $v(b) - v(a) \in A$, then $u(b) \geq u(a) - \varphi(v(b) - v(a))$.

There exists a', b' such that $\{a'\} \succ \{a',b'\} \succ \{b'\}$ and $v(b') - v(a') = v(b) - v(a)$. Since $\varphi(v(b) - v(a)) = \varphi(v(b') - v(a')) = u(a') - U(\{a',b'\})$, it suffices to show that $u(a') - U(\{a',b'\}) \geq u(a) - U(\{a,b\})$. Take a full support lottery $h \in \Delta(C)$. For all $\lambda \in (0,1)$, since $v(b\lambda h) > v(a\lambda h)$ and $u(a\lambda h) > u(b\lambda h)$, $\{a\lambda h\} \succ \{a\lambda h, b\lambda h\}$. By MPSC (i), $\{a'\lambda h\} \succ \{a'\lambda h, b'\lambda h\} \succ \{b'\lambda h\}$. Let $\overline{\theta} \equiv a'\lambda h - a\lambda h \in \Theta$. For sufficiently small λ, $b\lambda h + \overline{\theta} \in \Delta(C)$. By Translation Invariance,

$$\{a'\lambda h\} = \{a\lambda h + \overline{\theta}\} \succ \{a\lambda h + \overline{\theta}, b\lambda h + \overline{\theta}\} = \{a'\lambda h, b\lambda h + \overline{\theta}\}.$$

Moreover, since

$$v(b\lambda h + \overline{\theta}) - v(a'\lambda h) = \lambda(v(b) - v(a)) = \lambda(v(b') - v(a')) = v(b'\lambda h) - v(a'\lambda h),$$

$v(b\lambda h + \overline{\theta}) = v(b'\lambda h)$. By Lemma 4.2, $\{a'\lambda h, b\lambda h + \overline{\theta}\} \succsim \{a'\lambda h, b'\lambda h\}$. Let $e(\{a\}) = \alpha(\{a\})c^+ + (1-\alpha(\{a\}))c^-$, $e(\{a'\}) = \alpha(\{a'\})c^+ + (1-\alpha(\{a'\}))c^-$, $e(\{a,b\}) = \alpha(\{a,b\})c^+ + (1-\alpha(\{a,b\}))c^-$ and $e(\{a',b'\}) = \alpha(\{a',b'\})c^+ + (1-\alpha(\{a',b'\}))c^-$. By definition, $u(a) = u(e(\{a\}))$, $u(a') = u(e(\{a'\}))$, $U(\{a,b\}) = u(e(\{a,b\}))$, and $U(\{a',b'\}) = u(e(\{a',b'\}))$. There exist feasible $\theta, \theta' \in \Theta$ such that $e(\{a\}) + \theta = e(\{a'\}) + \theta'$. By Monotone Self-Control, $\{e(\{a,b\}) + \theta\} \succsim \{e(\{a',b'\}) + \theta'\}$. Thus

$$\begin{aligned}
& u(a' + \theta') - u(e(\{a',b'\}) + \theta') \geq u(a + \theta) - u(e(\{a,b\}) + \theta) \\
\Leftrightarrow \;& u(a') - u(e(\{a',b'\})) \geq u(a) - u(e(\{a,b\})) \\
\Leftrightarrow \;& u(a') - U(\{a',b'\}) \geq u(a) - U(\{a,b\}).
\end{aligned}$$

By Step 1, when $A = (0,1]$, we complete the proof. Now consider the case that $A = (0, \overline{w})$ for some $\overline{w} \leq 1$. Let

$$B = \{w \in [\overline{w}, 1] \,|\, w = v(l') - v(l), \text{ for some } l, l' \text{ such that } \{l\} \succ \{l,l'\} \sim \{l'\}\}.$$

Step 2: *(i)* $\overline{w} \in B$, *(ii)* B *is convex, and (iii) if* $\overline{w} < 1$, *there exists* $w' \in B$ *with* $w' > \overline{w}$.

(i) Since $A = (0, \overline{w})$, there exists a monotone sequence $w^n \to \overline{w}$ such that $w^n = v(b^n) - v(a^n) < \overline{w}$ and $\{a^n\} \succ \{a^n, b^n\} \succ \{b^n\}$. Since $\{a^n\}_{n=1}^{\infty}$ and $\{b^n\}_{n=1}^{\infty}$ are sequences in $\Delta(C)$, we can assume $a^n \to a^0$ and $b^n \to b^0$ without loss of generality. Since v is continuous, $\overline{w} = v(b^0) - v(a^0)$. Moreover, since φ is increasing, $u(a^n) - u(b^n) > \varphi(v(b^n) - v(a^n)) = u(a^n) - U(\{a^n, b^n\}) \geq u(a^1) - U(\{a^1, b^1\}) > 0$. By continuity, $u(a^0) - u(b^0) > 0$. Rankings $\{a^0\} \succ \{b^0\}$ and $v(b^0) > v(a^0)$ imply $\{a^0\} \succ \{a^0, b^0\}$. Set Betweenness and the definition of A imply $\{a^0\} \succ \{a^0, b^0\} \sim \{b^0\}$. Thus $\overline{w} \in B$.

(ii) Take $w, w' \in B$. There exist l, l', k, k' such that $w = v(l') - v(l)$, $w' = v(k') - v(k)$ and $\{l\} \succ \{l, l'\} \sim \{l'\}$ and $\{k\} \succ \{k, k'\} \sim \{k'\}$. Then $v(l'\lambda k') - v(l\lambda k) = \lambda w + (1-\lambda)w'$. Since $v(l'\lambda k') > v(l\lambda k)$ and $u(l\lambda k) > u(l'\lambda k')$, we have $\{l\lambda k\} \succ \{l\lambda k, l'\lambda k'\}$. Set Betweenness implies $\{l\lambda k\} \succ \{l\lambda k, l'\lambda k'\} \succsim \{l'\lambda k'\}$. Moreover, since $\lambda w + (1-\lambda)w' \geq \overline{w}$, $\{l\lambda k\} \succ \{l\lambda k, l'\lambda k'\} \sim \{l'\lambda k'\}$. Thus $\lambda w + (1-\lambda)w' \in B$.

(iii) Let l, l' satisfy $\overline{w} = v(l') - v(l)$ and $\{l\} \succ \{l, l'\} \sim \{l'\}$. Since $\overline{w} < 1$, we have either $1 > v(l')$ or $v(l) > 0$. In case of the former, let $l^M \in \Delta(C)$ be a maximal element of v. By Continuity, for all λ sufficiently close to 1, $\{l\} \succ \{l, l'\lambda l^M\}$. By Set Betweenness, $\{l\} \succ \{l, l'\lambda l^M\} \succsim \{l'\lambda l^M\}$. Moreover, $v(l'\lambda l^M) - v(l) > v(l') - v(l) = \overline{w}$. By definition of A, we must have $\{l\} \succ \{l, l'\lambda l^M\} \sim \{l'\lambda l^M\}$. Hence $\overline{w} < w' \equiv v(l'\lambda l^M) - v(l) \in B$. The symmetric argument works for the latter case.

By Lemma 4.3 (i) and Step 2, $A \cup B$ is an interval in $[0,1]$ with $\overline{w} \in A \cup B$. For all $w \in A \cup B$, define

$$F(w) \equiv \sup \{u(l) - u(l') \mid w = v(l') - v(l) \text{ for some } \{l\} \succ \{l, l'\}\}.$$

Step 3: *(i) F is weakly concave on $A \cup B$, (ii) $F(\overline{w}) \leq \overline{\varphi}(\overline{w})$, and (iii) if $\overline{w} < 1$, $F(\overline{w}) = \overline{\varphi}(\overline{w})$.*

(i) Take $w, w' \in A \cup B$ and $\lambda \in (0,1)$. There exist $a^n, b^n, l^n, k^n \in \Delta(C)$ such that $\{a^n\} \succ \{a^n, b^n\}$, $\{l^n\} \succ \{l^n, k^n\}$, $v(b^n) - v(a^n) = w$, $u(k^n) - u(l^n) = w'$, $u(a^n) - u(b^n) \to F(w)$, and $u(l^n) - u(k^n) \to F(w')$. Since $v(b^n) > v(a^n)$, $v(k^n) > v(l^n)$, $u(a^n) > u(b^n)$ and $u(l^n) > u(k^n)$, we have $v(b^n\lambda k^n) > v(a^n\lambda l^n)$ and $u(a^n\lambda l^n) > u(b^n\lambda k^n)$. Thus $\{a^n\lambda l^n\} \succ \{a^n\lambda l^n, b^n\lambda k^n\}$. Furthermore,

$$\lambda w + (1-\lambda)w' = \lambda(v(b^n) - v(a^n)) + (1-\lambda)(v(k^n) - v(l^n)) = v(b^n\lambda k^n) - v(a^n\lambda l^n).$$

Thus

$$\begin{aligned} F(\lambda w + (1-\lambda)w') &\geq \limsup u(a^n\lambda l^n) - u(b^n\lambda k^n) \\ &= \limsup \lambda(u(a^n) - u(b^n)) + (1-\lambda)(u(l^n) - u(k^n)) \\ &= \lambda F(w) + (1-\lambda)F(w'). \end{aligned}$$

(ii) Suppose $F(\overline{w}) > \overline{\varphi}(\overline{w}) = \sup\{\varphi(w) | w \in A\}$. For all $w \in A$ and $l, l' \in \Delta(C)$ such that $w = v(l') - v(l)$ and $\{l\} \succ \{l, l'\} \sim \{l'\}$, we have shown in Step 1 that $\varphi(w) \geq u(l) - u(l')$. Thus there exist sequences $w^n \to \overline{w}$, $\{a^n\}_{n=1}^{\infty}$ and $\{b^n\}_{n=1}^{\infty}$ such that $w^n = v(b^n) - v(a^n) \in A$, $\{a^n\} \succ \{a^n, b^n\} \succ \{b^n\}$, and $u(a^n) - u(b^n) > c > \sup\{\varphi(w) | w \in A\}$, where $c > 0$ is a constant number. Since $\{a^n\}_{n=1}^{\infty}$ and $\{b^n\}_{n=1}^{\infty}$ are sequences in $\Delta(C)$, we can assume $a^n \to a^0$ and $b^n \to b^0$ without loss of generality. Since

$$u(a^n) - u(b^n) > c > \varphi(v(b^n) - v(a^n)) = u(a^n) - U(\{a^n, b^n\}),$$

continuity implies $u(a^0) - u(b^0) > u(a^0) - U(\{a^0, b^0\})$, that is, $U(\{a^0, b^0\}) > u(b^0)$. On the other hand, since $\overline{w} = v(b^0) - v(a^0) > 0$ and $u(a^0) > u(b^0)$, we have $\{a^0\} \succ \{a^0, b^0\}$. Hence $\{a^0\} \succ \{a^0, b^0\} \succ \{b^0\}$, which contradicts $\overline{w} \notin A$.

(iii) By Step 2 (ii) and (iii), $[\overline{w}, w'] \subset B$ for some $w' > \overline{w}$. Since a convex function is continuous on the relative interior of the domain, F is continuous around $\overline{w}$.

Suppose $F(\overline{w}) < \sup\{\varphi(w) | w \in A\}$. By continuity of F and φ, there exists $w < \overline{w}$ such that $F(w) < \varphi(w)$. There exist a, b such that $\{a\} \succ \{a, b\} \succ \{b\}$ and $w = v(b) - v(a)$. Since

$$u(a) - U(\{a, b\}) = \varphi(v(b) - v(a)) > F(w) \geq u(a) - u(b),$$

we have $u(b) > U(\{a, b\})$, which is a contradiction. Hence $F(\overline{w}) = \overline{\varphi}(\overline{w})$.

By Step 3 (ii), when $\overline{w} = 1$, for all a, b with $\{a\} \succ \{a, b\} \sim \{b\}$ with $v(b) - v(a) = 1$, we have $\overline{\varphi}(v(b) - v(a)) \geq F(v(b) - v(a)) \geq u(a) - u(b)$. Thus, taking Step 1 together, any strictly increasing continuous extension $\overline{\overline{\varphi}} : \mathbb{R}_+ \to \mathbb{R}_+$ is the desired object.

Now assume $\overline{w} < 1$. By Step 3 (i) and (iii), there exists an affine function $f_1 : \mathbb{R} \to \mathbb{R}$ supporting F at $\overline{w}$. If its slope is negative, then take any positive affine function f_2 with $f_2(\overline{w}) = \overline{\varphi}(\overline{w})$. Now define

$$\overline{\overline{\varphi}}(w) \equiv \begin{cases} \overline{\varphi}(w) & \text{if } w \in \overline{A}, \\ \max[f_1, f_2] & \text{elsewhere.} \end{cases}$$

Since $\overline{\varphi}(\overline{w}) = f_1(\overline{w}) = f_2(\overline{w})$, $\overline{\overline{\varphi}}$ is continuous and strictly increasing.

We can verify that $\overline{\overline{\varphi}}(v(b) - v(a)) \geq u(a) - u(b)$ for all a, b such that $\{a\} \succ \{a, b\} \sim \{b\}$ and $v(b) - v(a) \geq \overline{w}$. Indeed, by definition of f_1,

$$\overline{\overline{\varphi}}(v(b) - v(a)) \geq f_1(v(b) - v(a)) \geq F(v(b) - v(a)) \geq u(a) - u(b).$$

Therefore $U(\{a, b\})$ can be written as the desired form in all the cases.

Finally, suppose in addition that C is finite and $\succsim$ satisfies Properness. Then by Lemma 4.7 (ii), there exists $\overline{K} > 0$ satisfying the statement of the lemma. Let f_3 be the affine function which has slope $\overline{K}$ and $f_3(\overline{w}) = \overline{\varphi}(\overline{w})$. Define

$$\overline{\overline{\varphi}}(w) \equiv \begin{cases} \overline{\varphi}(w) & \text{if } w \in \overline{A}, \\ \max[f_1, f_2, f_3] & \text{elsewhere.} \end{cases}$$

Then $\overline{\overline{\varphi}}$ is convex. By the same argument above, $\overline{\overline{\varphi}}(v(b) - v(a)) \geq u(a) - u(b)$ for all a, b such that $\{a\} \succ \{a, b\} \sim \{b\}$ and $v(b) - v(a) \geq \overline{w}$. This completes the proof. □

Lemma 4.9. *For all finite menus $x \in \mathcal{D}$,*

$$U(x) = \max_{l \in x} \left(u(l) - \overline{\overline{\varphi}} \left(\max_x v - v(l) \right) \right).$$

Proof. First of all, Gul and Pesendorfer [5] show that, if $\succsim$ satisfies Set Betweenness, for all finite menus $x \in \mathcal{D}$,

$$U(x) = \max_{a \in x} \min_{b \in x} U(\{a, b\}) = \min_{b \in x} \max_{a \in x} U(\{a, b\}). \tag{19}$$

Define $\psi : \mathbb{R}_- \to \mathbb{R}_-$ by $\psi(w) = -\overline{\overline{\varphi}}(-w)$. Since $\overline{\overline{\varphi}}$ is increasing, so is ψ. Fix $a \in x$ arbitrarily. Since ψ is increasing,

$$\begin{aligned}
\min_{b\in x} U(\{a,b\}) &= \min_{b\in x}\max_{l\in\{a,b\}}\left(u(l) - \overline{\overline{\varphi}}\left(\max_{\{a,b\}} v - v(l)\right)\right)\\
&= \min_{b\in x}\max_{l\in\{a,b\}}\left(u(l) + \psi\left(v(l) + \min_{\{a,b\}}(-v)\right)\right)\\
&\geq \min_{b\in x}\max_{l\in\{a,b\}}\left(u(l) + \psi\left(v(l) + \min_{b\in x}\min_{\{a,b\}}(-v)\right)\right)\\
&= \max_{l\in\{a,b^a\}}\left(u(l) + \psi\left(v(l) + \min_{l'\in x}(-v)\right)\right),
\end{aligned}$$

where b^a is a minimizer of the associated minimization problem. Since the above inequality holds for all $a \in x$, if follows from (19) that

$$\begin{aligned}
U(x) &\geq \max_{a\in x}\max_{l\in\{a,b^a\}}\left(u(l) + \psi\left(v(l) + \min_{l'\in x}(-v)\right)\right)\\
&= \max_{l\in x}\left(u(l) + \psi\left(v(l) + \min_{l'\in x}(-v)\right)\right)\\
&= \max_{l\in x}\left(u(l) - \overline{\overline{\varphi}}\left(\max_{l'\in x} v - v(l)\right)\right). \qquad (20)
\end{aligned}$$

Similarly, for all fixed $b \in x$,

$$\begin{aligned}
\max_{a\in x} U(\{a,b\}) &= \max_{a\in x}\max_{l\in\{a,b\}}\left(u(l) - \overline{\overline{\varphi}}\left(\max_{\{a,b\}} v - v(l)\right)\right)\\
&= \max_{a\in x}\max_{l\in\{a,b\}}\left(u(l) + \psi\left(v(l) + \min_{\{a,b\}}(-v)\right)\right)\\
&\leq \max_{a\in x}\max_{l\in\{a,b\}}\left(u(l) + \psi\left(v(l) + \max_{a\in x}\min_{\{a,b\}}(-v)\right)\right)\\
&= \max_{l\in x}\left(u(l) + \psi\left(v(l) + \min_{\{a^b,b\}}(-v)\right)\right),
\end{aligned}$$

where a^b is a maximizer of the associated maximization problem. Since the above inequality holds for all $b \in x$, if follows from (19) that

$$\begin{aligned}
U(x) &\leq \min_{b\in x}\max_{l\in x}\left(u(l) + \psi\left(v(l) + \min_{\{a^b,b\}}(-v)\right)\right)\\
&= \max_{l\in x}\left(u(l) + \psi\left(v(l) + \min_{b\in x}\min_{\{a^b,b\}}(-v)\right)\right)\\
&= \max_{l\in x}\left(u(l) + \psi\left(v(l) + \min_{l'\in x}(-v)(l')\right)\right)\\
&= \max_{l\in x}\left(u(l) - \overline{\overline{\varphi}}\left(\max_{l'\in x} v - v(l)\right)\right). \qquad (21)
\end{aligned}$$

Taking (20) and (21) together, the desired result holds. □

Lemma 4.10. *For all $x \in \mathcal{D}$, U can be written as the desired form.*

Proof. By Lemma 0 of Gul and Pesendorfer [5, p.1421], there exists a sequence of subsets x^n of x such that each x^n is finite and $x^n \to x$ in the Hausdorff metric. By Lemma 4.9,

$$U(x^n) = \max_{l \in x^n} \left(u(l) - \overline{\overline{\varphi}} \left(\max_{x^n} v - v(l) \right) \right). \tag{22}$$

Since $\overline{\overline{\varphi}}$ is continuous, the maximum theorem implies that the RHS of (22) converges to

$$\max_{l \in x} \left(u(l) - \overline{\overline{\varphi}} \left(\max_{x} v - v(l) \right) \right).$$

On the other hand, by Continuity, $U(x^n) \to U(x)$. This completes the proof. □

4.2 Proof of Theorem 3.2

By assumption, for all $w' \in \mathbb{R}_+$, $\varphi'(w') = \alpha_u \varphi(\frac{w'}{\alpha_v})$. Since

$$\begin{aligned} U'(x) &= \max_{l \in x} \left(\alpha_u u(l) + \beta_u - \alpha_u \varphi \left(\frac{\alpha_v (\max_x v - v(l))}{\alpha_v} \right) \right) \\ &= \alpha_u \left\{ \max_{l \in x} \left(u(l) - \varphi \left(\max_x v - v(l) \right) \right) \right\} + \beta_u \\ &= \alpha_u U(x) + \beta_u, \end{aligned}$$

U' and U represent the same preference.

We turn to the converse. Since mixture linear functions $U(\{l\}) = u(l)$ and $U'(\{l\}) = u'(\{l\})$ represent the same preference over $\Delta(C)$, they are cardinally equivalent by the standard argument. That is, there exist $\alpha_u > 0$ and $\beta_u \in \mathbb{R}$ such that $u' = \alpha_u u + \beta_u$.

Next we claim that mixture linear functions v and v' represent the identical preference over $\Delta(C)$.

Lemma 4.11. *For all $l, l' \in \Delta(C)$, $\{l\} \succ \{l, l'\}$ if and only if $u(l) > u(l')$ and $v(l') > v(l)$. Similarly, $\{l\} \succ \{l, l'\}$ if and only if $u'(l) > u'(l')$ and $v'(l') > v'(l)$.*

Proof. If part: By assumption, $u(l) > u(l) - \varphi(v(l') - v(l))$ and $u(l) > u(l')$. Thus the associated representation implies $u(l) > U(\{l, l'\})$. Only-if part: Since the representation satisfies Set Betweenness, $u(l) > U(\{l, l'\}) \geq u(l')$, that is, $u(l) > u(l')$. Suppose $v(l) \geq v(l')$. Then the representation implies $u(l) = u(l) - \varphi(v(l) - v(l)) > u(l') - \varphi(v(l) - v(l'))$, that is, $U(\{l, l'\}) = u(l)$. This is a contradiction. The exactly same argument works for (u', v', φ'). □

Suppose by contradiction that v and v' are not identical. Then there exist l, l' satisfying $v(l') > v(l)$ and $v'(l') \leq v'(l)$. Let l^M and l^m be a maximal lottery and a minimal lottery with respect to v'. Since $\succsim$ is a self-control preference, there exist $k, k' \in \Delta(C)$ such that $\{k\} \succ \{k, k'\} \succ \{k'\}$. Thus by Lemma 4.11, $v'(l^M) \geq v'(k') > v'(k) \geq v'(l^m)$. Since v is continuous and v' is mixture linear, we can find small $\lambda > 0$ such that $v'(l^m \lambda l') < v'(l^M \lambda l)$ and $v(l^m \lambda l') > v(l^M \lambda l)$. Redefine l and l' as $l^M \lambda l$ and $l^m \lambda l'$, respectively. If either $u(l) > u(l')$ or $u(l') > u(l)$, this contradicts Lemma 4.11 because $v(l') > v(l)$ and $v'(l) > v'(l')$. Suppose $u(l) = u(l')$. Since $\{k\} \succ \{k'\}$, u is not degenerate. Let l^+ and l^- be a maximal lottery and a minimal lottery with respect to u. Since v, v' are continuous and u is mixture linear, there exists a small $\mu > 0$ such that $u(l^+ \mu l) > u(l^- \mu l')$, $v(l^- \mu l') > v(l^+ \mu l)$ and $v'(l^+ \mu l) > v'(l^- \mu l')$. Again, this contradicts Lemma 4.11. Thus v and v' induce the identical preference over $\Delta(C)$. Since v and v' are mixture linear, there exist $\alpha_v > 0$ and $\beta_v \in \mathbb{R}$ such that $v' = \alpha_v v + \beta_v$.

Since $v' = \alpha_v v + \beta_v$,

$$\begin{aligned} W(u', v', \varphi') &= \{v'(l') - v'(l) \in \mathbb{R}_+ \mid l, l' \in \Delta(C) \text{ with } \{l\} \succ \{l, l'\} \succ \{l'\}\} \\ &= \{\alpha_v(v(l') - v(l)) \in \mathbb{R}_+ \mid l, l' \in \Delta(C) \text{ with } \{l\} \succ \{l, l'\} \succ \{l'\}\} \\ &= \alpha_v W(u, v, \varphi). \end{aligned}$$

Lemma 4.12. *If U and U' represent the same preference and $u' = \alpha_u u + \beta_u$, then $U' = \alpha_u U + \beta_u$.*

Proof. Let l^+ and l^- be a maximal lottery and a minimal lottery with respect to u. By Set Betweenness and Continuity, $\{l^+\} \succsim x \succsim \{l^-\}$ for all $x \in \mathcal{D}$. Thus there exists a unique $\lambda(x) \in [0, 1]$ such that $x \sim \{\lambda(x) l^+ + (1 - \lambda(x)) l^-\}$. Since u and u' are mixture linear,

$$\begin{aligned} U'(x) &= u'(\lambda(x) l^+ + (1 - \lambda(x)) l^-) = \lambda(x) u'(l^+) + (1 - \lambda(x)) u'(l^-) \\ &= \lambda(x)(\alpha_u u(l^+) + \beta_u) + (1 - \lambda(x))(\alpha_u u(l^-) + \beta_u) \\ &= \alpha_u(\lambda(x) u(l^+) + (1 - \lambda(x)) u(l^-)) + \beta_u = \alpha_u U(x) + \beta_u. \end{aligned}$$

□

Since $u' = \alpha_u u + \beta_u$ and $v' = \alpha_v v + \beta_v$, for all $x \in \mathcal{D}$,

$$\begin{aligned} U'(x) &= \max_{l \in x} \left(u'(l) - \varphi' \left(\max_{l' \in x} v'(l') - v'(l) \right) \right) \\ &= \max_{l \in x} \left(\alpha_u u(l) + \beta_u - \varphi' \left(\alpha_v \left(\max_{l' \in x} v(l') - v(l) \right) \right) \right) \\ &= \alpha_u \left\{ \max_{l \in x} \left(u(l) - \frac{1}{\alpha_u} \varphi' \left(\alpha_v \left(\max_{l' \in x} v(l') - v(l) \right) \right) \right) \right\} + \beta_u. \end{aligned}$$

By Lemma 4.12,

$$\begin{aligned} U(x) &= \max_{l \in x} \left(u(l) - \varphi \left(\max_{l' \in x} v(l') - v(l) \right) \right) \\ &= \max_{l \in x} \left(u(l) - \frac{1}{\alpha_u} \varphi' \left(\alpha_v \left(\max_{l' \in x} v(l') - v(l) \right) \right) \right). \end{aligned} \tag{23}$$

Take any $w \in W(u,v,\varphi)$. There exist l, l' such that $\{l\} \succ \{l,l'\} \succ \{l'\}$ and $w = v(l') - v(l)$. Since $v(l') > v(l)$ and $u(l) > u(l')$, it follows from (23) that

$$u(l) - \varphi(v(l') - v(l)) = U(\{l,l'\}) = u(l) - \frac{1}{\alpha_u}\varphi'(\alpha_v(v(l') - v(l))),$$

and hence $\varphi'(\alpha_v w) = \alpha_u \varphi(w)$.

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