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Super-replication cost for a single large investor

Takashi Adachi

Fukushima Medical University,
Fukushima City, Fukushima 960-1295, JAPAN
(*E-mail*: ada@fmu.ac.jp)

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Abstract

We consider the cost of hedging contingent claims in a financial market where the trades of a large investor can move market prices. We provide a characterization of a super-replication cost in terms of an associated stochastic control problem. We also prove that the super-replication cost is a viscosity solution of a corresponding dynamic programming equation in the case of a Markov market model.

Key words: large investor, super-replication cost, dynamic programming equation.
JEL Classification: G12, G13.

1 Introduction

Our concern is to examine the cost of hedging contingent claims in a financial market where the trades of a large investor can move market prices, and the purpose of this paper is to provide a characterization of a super-replication cost in terms of an associated stochastic control problem.

1.1 General large investor problem

Let $T > 0$ be a finite time horizon and $\{W(t),\ 0 \leq t \leq T\}$ a standard d-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with a filtration $\mathbb{F} = \{\mathcal{F}_t,\ 0 \leq t \leq T\}$ which is the $\mathbb{P}$-augmentation of the filtration generated by the Brownian motion W. Let $\mathcal{P}$ denote the set of all $\mathbf{R}^n$-valued, $\mathbb{F}$-progressively measurable processes $p(\cdot)$ such that $\int_0^T |p(t)|^2\, dt < \infty$ a.s. Here $n \leq d$.

1.1.a *Price dynamics in presence of large investor*

We assume that there is a single large investor I in a financial market where one bank account and n stocks are traded continuously up to the time T. Let $\Pi \subset \mathcal{P}$ be a set of admissible trading strategies $\pi(\cdot)$ of the investor I. We assume $0 \in \Pi$.

Consider a model for price fluctuations as follows: If the investor I starts at time 0 with an initial capital $x \in \mathbf{R}$ and holds $\pi_j(t)$ shares of the j-th stock at time $t \in [0,T]$, $j = 1, \dots, n$, then the price processes $B^\pi(\cdot)$ of the bank account and $\widehat{S}^\pi(\cdot)$ of the stocks evolve according to the stochastic differential equation (SDE, for short)

$$dB(t) = B(t)r^\pi(t)dt, \qquad B(0) = 1,$$
$$d\widehat{S}(t) = diag[\widehat{S}(t)] \left\{ b^\pi(t)dt + \sigma^\pi(t)^\top dW(t) \right\}, \qquad \widehat{S}(0) = s \in (0,\infty)^n,$$

and the discounted wealth process $X^{x,\pi}(\cdot)$ of the investor I is given as

$$X(t) = x + \int_0^t \pi(u)^\top dS^\pi(u), \qquad t \in [0,T], \tag{1.1}$$

where $\top$ denotes the transpose operation; $diag[s]$ is the $n \times n$-diagonal matrix with diagonal elements $s_1, \dots, s_n$; $S^\pi(\cdot) := B^\pi(\cdot)^{-1}\widehat{S}^\pi(\cdot)$ is the discounted price process of stocks; $\{r^\pi(t),\ 0 \le t \le T\}$, $\{b^\pi(t) = (b_1^\pi(t), \dots, b_n^\pi(t))^\top,\ 0 \le t \le T\}$ and $\{\sigma^\pi(t) = (\sigma_1^\pi(t) \cdots \sigma_n^\pi(t)),\ 0 \le t \le T\}$ are bounded $\mathbb{F}$-progressively measurable processes taking values in $\mathbf{R}_+$, $\mathbf{R}^n$ and $\mathbf{R}^d \otimes \mathbf{R}^n$, respectively. Here the superscript π means that the process $h^\pi(t,\omega)$ ($h = r, b, \sigma$, for instance) with the superscript π depends on the path $\{\pi(u,\omega),\ 0 \le u \le t\}$ for a.e. $(t,\omega) \in [0,T] \times \Omega$ and $\pi \in \Pi$. Therefore the price dynamics are influenced by the actions of the investor I. It is for this reason that I is called the *large investor*. We also remark that the integral in (1.1) is well-defined by means of $\int_0^T |\pi(t)|^2\, dt < \infty$ a.s. and of the boundedness of the coefficients of market.

1.1.b *Contingent claim and super-replication cost*

A contingent claim $\{B^\pi C^\pi, \mathcal{T}^\pi\}$ consists of an $\mathbb{F}$-adapted, non-negative process $\{C^\pi(t),\ 0 \le t \le T\}$ and some class $\mathcal{T}^\pi$ of $\mathbb{F}$-stopping times. We assume $T \in \mathcal{T}^\pi$. Let us consider now the following situation: At time $t = 0$, two agents (the "buyer" and "seller") enter into an agreement. The seller I agrees to provide the buyer with the random payment $B^\pi(\tau(\omega),\omega)C^\pi(\tau(\omega),\omega)$ at time $t = \tau(\omega)$, where τ is an element of $\mathcal{T}^\pi$ and at the disposal of the buyer.

The objective of the seller I is to find a portfolio strategy $\pi \in \Pi$ which enables him to fulfill his obligation whenever the buyer decides to ask for the payment. Hence the super-replication cost h_{up} is defined as

$$h_{up} := \inf \left\{ x \ge 0 \;\middle|\; \exists \pi \in \Pi \quad \text{s.t.} \quad X^{x,\pi}(\tau) \ge C^\pi(\tau) \quad \text{a.s.}, \quad \forall \tau \in \mathcal{T}^\pi. \right\}.$$

1.2 Existent studies dealing with analogous models

In a standard market of a small investor model (the coefficients r, b and σ do not depend on π), there exists an $\mathbb{F}$-progressively measurable process $\theta : [0,T] \times \Omega \to \mathbf{R}^d$ such that

$$b(t,\omega) - r(t,\omega)\mathbf{1}_n = -\sigma(t,\omega)^\top \theta(t,\omega) \qquad \text{a.e. } (t,\omega) \in [0,T] \times \Omega \tag{1.2}$$

and the stochastic exponential process

$$Z_\theta(t) := \exp\left\{\int_0^t \theta(u)^\top dW(u) - \frac{1}{2}\int_0^t |\theta(u)|^2 du\right\}, \qquad 0 \le t \le T \tag{1.3}$$

is a martingale, where $\mathbf{1}_n = (1,\ldots,1)^\top \in \mathbf{R}^n$. Further, if the standard market is complete, by the martingale representation theorem and the Bayes' rule, we then have

$$\mathbb{E}\left[\frac{Z_\theta(T)}{Z_\theta(t)}C(T)\middle|\mathcal{F}_t\right] = \mathbb{E}\left[Z_\theta(T)C(T)\right] + \int_0^t \pi(u)^\top dS(u), \qquad t \in [0,T] \tag{1.4}$$

for some hedging portfolio π. Therefore the super-replication cost[1] of European contingent claim $\{B(T)C(T), \{T\}\}$ is given by $h_{up} = \mathbb{E}[Z_\theta(T)C(T)]$.

In some special cases, we can also use the martingale duality approach to study the replication of European contingent claims by the large investor. Cuoco & Liu[6] has provided the dual formulation for the case that r^π, σ^π and C^π are independent of the trading strategy π, and b^π satisfies

$$q(t,\omega)^\top b^\pi(t,\omega) = q(t,\omega)^\top \mu(t,\omega) + h(t, q(t,\omega), \omega) \qquad \text{a.e. } (t,\omega) \in [0,T] \times \Omega,$$

where $q(t) = X(t)^{-1} diag[S(t)]\pi(t)$, μ is a bounded $\mathbb{F}$-progressively measurable process taking values in $\mathbf{R}^n$, and a function $h(t,q,\omega)$ on $[0,T] \times \mathbf{R}^n \times \Omega$ satisfies: $h(\cdot, q, \cdot)$ is an optional process for each $q \in \mathbf{R}^n$; $h(t,\cdot,\omega)$ is Lipschitz uniformly in $(t,\omega) \in [0,T] \times \Omega$; $h(t,\cdot,\omega)$ is concave and upper semicontinuous for all $(t,\omega) \in [0,T] \times \Omega$; $h(t,0,\omega) = 0$ for all $(t,\omega) \in [0,T] \times \Omega$. Bank & Baum[3] dealt with a general semimartingale market model with the large investor. They presented a characterization of the super-replication cost for European contingent claims in terms of an associated stochastic control problem under the condition that (1.2) was satisfied for some process θ which did not depend on the large investor's position π despite of the dependence of r, b and σ upon π.

In the case of the general large investor model, however, it is difficult to use the martingale duality approach in order to show the existence of a portfolio π satisfying (1.4) because of the dependence of θ, C and S upon π. Hence the martingale duality approach has not been successful to solve the general large investor problem. Therefore the previous studies have provided several treatments of this problem which avoid the passage from the dual formulation. These studies dealt with Markov market models with the large investor as follows:

[1] For a standard asset pricing theory, see the usual textbooks, e.g. Duffie[9], Karatzas[12] and Karatzas & Shreve[13].

(i) Cvitanić & Ma[8]: For $h = b, \sigma$,

$$h^{\pi}(t) = h(t, S(t), \pi(t), X(t)), \qquad r^{\pi}(t) = r(t, diag[S(t)]\pi(t), X(t)).$$

(ii) Soner & Touzi[18]: For $h = b, \sigma$ and $q(t) = X(t)^{-1} diag[S(t)]\pi(t)$,

$$h^{\pi}(t) = h(t, S(t), q(t)), \qquad r^{\pi}(t) \equiv 1.$$

(iii) Frey[10]: In one-dimensional case ($n = d = 1$),

$$S^{\pi}(t) = \psi(t, Z_{\eta}(t), \pi(t)), \qquad r^{\pi}(t) \equiv 1, \qquad \pi(t) = \phi(t, Z_{\eta}(t)),$$

where ψ is a smooth reaction function, the stochastic exponential Z_{η} defined as (1.3) with a constant η is a fundamental state variable process, and the trading strategy ϕ is selected from among smooth functions. In Platen & Schweizer[14] and Frey & Stremme[11] the state variable Z_{η} and the reaction function ψ have been obtained from equilibrium considerations.

Cvitanić & Ma [8] characterized the cost and portfolio of hedging European option $B(T)C(T) = g(S(T))$ as a solution of a forward-backward SDE corresponding to their Markov model, and proved the existence and uniqueness of the solution of this equation under regularity conditions on r, b, σ and g. Frey[10] characterized the hedging portfolio ϕ of European option $C(T) = g(S(T))$ as a solution of an associated quasi-linear partial differential equation and provided results on existence and uniqueness of the solution to this equation under regularity conditions on ψ and g. Soner & Touzi[18] used a new dynamic programming principle established in Soner & Touzi[16] to characterize the super-replication cost for European option $C(T) = g(S(T))$ as a viscosity solution of a corresponding dynamic programming equation under suitable conditions on b, σ and g.

Since r, b and σ in our model do not depend on the value of the large investor's wealth X, our model does not include those of Cvitanić & Ma[8] and Soner & Touzi[18]. However we can apply our approach to the study of the replication in the model of Soner & Touzi[18], and we can treat Example 5.1 of Cvitanić & Ma[8] in our framework (see Appendix B in the author[1]). Extending the set of admissible portfolios in the model of Frey[10] to the set of controlled semimartingales

$$d\pi(t) = \alpha(t)dt + \beta(t)dW(t) \qquad \text{(where } \alpha \text{ and } \beta \text{ are controls)},$$

we also have the application of our approach to the study of the replication in the model of Frey[10].

The remainder of this paper is organized in the following way: In the next section we characterize the super-replication cost in terms of associated stochastic control problem. In §3, we derive a corresponding dynamic programming equation from the representation obtained in §2, and characterize the super-replication cost as a viscosity solution of this equation in the case of a Markov market model. For the proofs of assertions stated in §2 and §3, see the author's paper[1].

2 Main result

In order to characterize the super-replication cost in terms of the stochastic control problem, we need the notion of the change of measure. Let $\mathcal{D}_m$ be the class of all $\mathbf{R}^d$-valued, $\mathbb{F}$-progressively measurable processes $\nu(\cdot)$ such that $|\nu(t,\omega)| \le m$ a.e., and $\mathcal{D} := \bigcup_{m=1}^{\infty} \mathcal{D}_m$. Then the stochastic exponential process

$$Z_\nu(t) := \exp\left\{\int_0^t \nu(u)^\top dW(u) - \frac{1}{2}\int_0^t |\nu(u)|^2 du\right\}, \qquad 0 \le t \le T$$

is a martingale for each $\nu \in \mathcal{D}$, and

$$\mathbb{P}_\nu(\Lambda) := \mathbb{E}\left[Z_\nu(T)\mathbf{1}_\Lambda\right], \qquad \Lambda \in \mathcal{F}_T$$

is a probability measure, where $\mathbf{1}$ is the indicator function.

When the seller I receives the amount $x > h_{up}$ from the buyer, he can cover his obligation at any time $\tau \in \mathcal{T}^\pi$ *without risk*, i.e.

$$\inf_{\pi\in\Pi}\sup_{\tau\in\mathcal{T}^\pi}\sup_{\nu\in\mathcal{D}}\mathbb{E}_\nu\left[(C^\pi(\tau) - X^{x,\pi}(\tau))^+\right] = 0,$$

where $\mathbb{E}_\nu$ denotes the expectation operator under $\mathbb{P}_\nu$ and $a^+ := \max\{a, 0\}$. Formally, we calculate

$$\begin{aligned}
0 &= \inf_{\pi\in\Pi}\sup_{\tau\in\mathcal{T}^\pi}\sup_{\nu\in\mathcal{D}}\mathbb{E}_\nu\left[(C^\pi(\tau) - X^{x,\pi}(\tau))^+\right] \\
&\text{“}\overset{?}{=}\text{”} \inf_{\pi\in\Pi}\sup_{\tau\in\mathcal{T}^\pi}\sup_{\nu\in\mathcal{D}}\mathbb{E}_\nu\left[C^\pi(\tau) - X^{x,\pi}(\tau)\right] \\
&= \inf_{\pi\in\Pi}\sup_{\tau\in\mathcal{T}^\pi}\sup_{\nu\in\mathcal{D}}\mathbb{E}_\nu\left[C^\pi(\tau) - X^{0,\pi}(\tau)\right] - x \\
&\text{“}\overset{?}{=}\text{”} \inf_{\pi\in\Pi}\sup_{\tau\in\mathcal{T}^\pi}\sup_{\nu\in\mathcal{D}}\mathbb{E}_\nu\left[\left(C^\pi(\tau) - X^{0,\pi}(\tau)\right)^+\right] - x.
\end{aligned}$$

Letting $x \downarrow h_{up}$, we conjecture

$$h_{up} = \inf_{\pi\in\Pi}\sup_{\tau\in\mathcal{T}^\pi}\sup_{\nu\in\mathcal{D}}\mathbb{E}_\nu\left[\left(C^\pi(\tau) - X^{0,\pi}(\tau)\right)^+\right]. \tag{2.1}$$

Indeed, we have

Theorem 2.1 *The super-replication cost h_{up} is expressed as* (2.1). *Moreover, if $\mathbb{E}[|X^{0,\pi}(\tau)|^p] < \infty$ for any $\pi \in \Pi$, $\tau \in \mathcal{T}^\pi$ and some constant $p = p(\pi,\tau) > 1$, then*

$$h_{up} = \left(\inf_{\pi\in\Pi}\sup_{\tau\in\mathcal{T}^\pi}\sup_{\nu\in\mathcal{D}}\mathbb{E}_\nu\left[C^\pi(\tau) - X^{0,\pi}(\tau)\right]\right)^+. \tag{2.2}$$

Remark 2.2 When we defer to the suggestion of Bank & Baum[3], the discounted wealth process X^π should be replaced by

$$\widetilde{X}^\pi(t) = X^\pi(t) - \frac{L^\pi(t)}{B^\pi(t)}, \qquad\qquad 0 \le t \le T,$$

where $\{L^\pi(t),\ 0 \le t \le T\}$ is a right-continuous, $\mathbb{F}$-adapted increasing process with $L^\pi(0) = 0$. Here $L^\pi(t)$ has the interpretation of the cumulative cost of the liquidity risk up to time $t \in [0, T]$. As seen in the proof stage, however, it is clear that if we replace X^π with $\widetilde{X}^\pi$ in the equations (2.1)-(2.2), the assertions in the previous theorem remain to be true without additional assumptions on L^π.

In order to obtain further sharp results, we are now in a position to make some assumptions:

Assumption 2.3

(i) For all $\pi \in \Pi$ there exists $\theta^\pi \in \mathcal{D}$ such that

$$b^\pi(t,\omega) - r^\pi(t,\omega)\mathbf{1}_n = -\sigma^\pi(t,\omega)^\top \theta^\pi(t,\omega) \qquad \text{a.e. } (t,\omega) \in [0,T] \times \Omega, \tag{2.3}$$

where $\mathbf{1}_n = (1, \dots, 1)^\top \in \mathbf{R}^n$.

(ii) For all $\pi \in \Pi$ there exists a constant $p > 1$ such that

$$\mathbb{E}\left[\int_0^T |\pi(t)|^{2p}\, dt\right] < \infty. \tag{2.4}$$

In the case of a small investor model, the process $-\theta$ of (2.3) is called the market price of risk process and the risk-neutral equivalent martingale measure $\mathbb{P}_\theta$ plays an important role for the pricing theory, as stated in §1.2. Moreover the conditions (2.3)-(2.4) guarantees that there is no arbitrage opportunity in a standard market of the small investor model. Therefore it seems natural to assume (2.3)-(2.4).

Corollary 2.4 *Under Assumption 2.3, we have*

$$h_{up} = \inf_{\pi\in\Pi} \sup_{\tau\in\mathcal{T}^\pi} \sup_{\nu\in\mathcal{D}} \mathbb{E}_\nu \left[C^\pi(\tau) - X^{0,\pi}(\tau)\right]. \tag{2.5}$$

3 Dynamic programming equation

3.1 Markov market model

In order to adapt the arguments developed by Soner & Touzi[16]-[18] and Bensoussan *et al.*[4] to our large investor model, we now focus on the Markov case:

$$h^\pi(t) = h(t, B(t), S(t), \pi(t)), \qquad \text{for } h = r, b, \sigma,$$

where r, b and σ are $\mathbf{R}_+, \mathbf{R}^n$ and $\mathbf{R}^d \otimes \mathbf{R}^n$-valued, bounded functions defined on $[0,T] \times \mathbf{R}_+ \times \mathbf{R}_+^n \times \mathbf{R}^n$. We further assume that r, b and σ are Lipschitz functions in the (β, s, π) variable, uniformly in t. We consider the special case of European contingent claim:

$$C^\pi(T) = g(B(T), S(T)) \qquad \text{and} \qquad \mathcal{T}^\pi = \{T\},$$

where a non-negative function g on $(0,\infty) \times \mathbf{R}_+^n$ satisfies the polynomial growth condition:

$$g(\beta, s) \leq c_0(\beta^{-l} + \beta^l + |s|^l), \qquad (\beta, s) \in (0,\infty) \times \mathbf{R}_+^n$$

for certain constants $c_0, l > 0$.

Let $K \subset \mathbf{R}^n$ be a compact convex subset which contain the origin. We assume that Π is the set of all processes $\pi \in \mathcal{P}$ such that $\pi(t,\omega) \in K$ a.e. Let δ denote the support function $\delta(q) := \sup_{p \in K}(p^\top q)$, $q \in \mathbf{R}^n$. Define

$$\begin{aligned} \mathcal{H}(p) &:= \inf\left\{\delta(q) - q^\top p \ : \ |q| = 1\right\}, && p \in \mathbf{R}^n, \\ \widehat{h}(\beta, s) &:= \sup_{q \in \mathbf{R}_+^n}\left\{h(\beta, q) - \delta(q - s)\right\}, && (\beta, s) \in (0,\infty) \times \mathbf{R}_+^n, \end{aligned}$$

for each function $h : (0,\infty) \times \mathbf{R}_+^n \to \mathbf{R}$. It is well known that the support function δ is non-negative, convex and positively homogeneous, and

$$\text{“} p \in K \iff \mathcal{H}(p) \geq 0 \text{”} \qquad \text{and} \qquad \text{“} p \in \operatorname{int} K \iff \mathcal{H}(p) > 0 \text{”}. \tag{3.1}$$

Furthermore we assume that σ satisfies the uniform ellipticity condition:

$$|\sigma(y,\pi)\xi| \geq c|\xi|, \qquad y \in [0,T] \times \mathbf{R}_+^{n+1}, \ \pi \in K, \ \xi \in \mathbf{R}^n, \tag{3.2}$$

for some constant $c > 0$. This condition guarantees that there exists a bounded function $\theta : [0,T] \times \mathbf{R}_+^{n+1} \times K \to \mathbf{R}^d$ such that

$$-\sigma(y,\pi)^\top \theta(y,\pi) = b(y,\pi) - r(y,\pi)\mathbf{1}_n,$$

and hence the condition (2.3) holds.

3.2 Stochastic control problem and dynamic programming equation

Thanks to Girsanov's theorem, we have

$$\mathbb{E}_{\nu+\theta}\left[X^{0,\pi}(T)\right] = \mathbb{E}_{\nu+\theta}\left[\int_0^T \pi(u)^\top diag[S(u)]\sigma(Y(u),\pi(u))^\top \nu(u)du\right], \qquad \pi \in \Pi, \nu \in \mathcal{D},$$

where $Y(u) := (u, B(u), S(u))^\top$ and $\theta(u) = \theta(Y(u), \pi(u))$. From (2.5), therefore, we can derive the stochastic control problem:

$$U(y) := \inf_{\pi\in\Pi}\sup_{\nu\in\mathcal{D}} \mathbb{E}^y\left[g(B^\pi(T), S^{\pi,\nu}(T)) - \int_t^T \pi(u)^\top diag[S^{\pi,\nu}(u)]\sigma(a(u))^\top \nu(u)du\right] \tag{3.3}$$

for $y = (t, \beta, s) \in [0, T] \times (0, \infty) \times (0, \infty)^n$, where $a(u) = (Y^{\pi,\nu}(u), \pi(u))^\top$, $S^{\pi,\nu}$ is a unique solution of the equation

$$dS(u) = diag[S(u)]\sigma(a(u))^\top\{\nu(u)du + dW(u)\}, \qquad t \le u \le T,$$

and the suffix $y = (t, \beta, s)$ of $\mathbb{E}$ means that we have specified the data $(B^\pi(t), S^{\pi,\nu}(t)) = (\beta, s)$.

Then, since $\{\sigma_i(y, \pi)\}_i$ is linearly independent by means of (3.2), the dynamic programming equation (DPE, for short) for (3.3) is given as follows:

$$\begin{aligned} 0 = U_t(y) + \inf_{\pi\in K}\sup_{\nu\in\mathbf{R}^d}\Big\{ & r(y,\pi)\beta U_\beta(y) + (DU(y) - \pi)^\top diag[s]\sigma(y,\pi)^\top \nu \\ & + \frac{1}{2}\mathbf{Tr}\left[\left\{diag[s]\sigma^\top\sigma(y,\pi)diag[s]\right\}D^2U(y)\right]\Big\} \\ = \begin{cases} \mathcal{G}^{DU(y)}U(y), & \text{if } DU(y) \in K, \\ +\infty \quad, & \text{if } DU(y) \notin K, \end{cases} \end{aligned} \tag{3.4}$$

for $y = (t, \beta, s) \in [0, T) \times (0, \infty)^{n+1}$, where $D\varphi$ and $D^2\varphi$ are the first and second order differentials of φ with respect to the variable s,

$$\mathcal{G}^\pi\varphi(y) = \varphi_t(y) + r(y,\pi)\beta\varphi_\beta(y) + \frac{1}{2}\mathbf{Tr}\left[\left\{diag[s]\sigma^\top\sigma(y,\pi)diag[s]\right\}D^2\varphi(y)\right]$$

and $\mathbf{Tr}[\mathrm{MN}] = \sum_{i,j=1}^n m_{ij}n_{ij}$ for symmetric matrices $\mathrm{M} = [m_{ij}]$, $\mathrm{N} = [n_{ij}]$. Combining (3.4) with (3.1), we have the DPE

$$\min\left\{-\mathcal{G}^{DU(y)}U(y),\ \mathcal{H}(DU(y))\right\} = 0, \qquad y \in [0, T) \times (0, \infty)^{n+1}. \tag{3.5}$$

We are now in the position to provide some conditions on the payoff function g and convex set K.

Assumption 3.1

(i) *There are constants* $c_0, l > 0$ *and* $\gamma_0 \in K$ *such that*

$$g(\beta, s) \leq c_0(\beta^l + \beta^{-l}) + \gamma_0^\top s, \qquad (\beta, s) \in (0, \infty) \times \mathbf{R}^n_+. \quad (3.6)$$

(ii) *Either one of the following conditions holds:*

$$\bullet\ g \text{ is continuous}, \quad \text{or} \quad \bullet\ \widehat{g} \text{ is continuous and } \widehat{g} = \widehat{g_*}\ , \qquad (3.7)$$

where

$$g_*(z) := \liminf_{\varepsilon \downarrow 0} \{ g(z') \ : \ z' \in (0, \infty) \times \mathbf{R}^n_+ \ \text{and}\ |z - z'| \leq \varepsilon \}, \quad z \in \mathbf{R}^{n+1}_+.$$

(iii) *For any* $q, q' \in \mathbf{R}^n$ *satisfying* $q' - q \in \mathbf{R}^n_+$ *and* $|q_k| = |q'_k|$, $k = 1, \ldots, n$, *we have*

$$\delta(q) \geq \delta(q'). \qquad (3.8)$$

Example 3.2 Let us consider the following two examples.

(i) K is the closed ball $\mathrm{B}_\rho(0)$ centered at 0 with radius $\rho > 0$. Then $\delta(q) = \rho|q|$ satisfies (3.8).

(ii) (*Rectangular constraints*) $K = J_1 \times \cdots \times J_n$ with $J_k = [-\eta_k, \xi_k]$, $0 \leq \xi_k \leq \eta_k < \infty$. Then $\delta(q) = \sum_{k=1}^n (\xi_k q_k^+ + \eta_k q_k^-)$ satisfies (3.8).

The following theorem characterizes the value function U as a viscosity solution of the DPE (3.5). For the notion and general theory of viscosity solutions, we recommend readers to refer to the User's Guide by Crandall *et al.*[5].

Theorem 3.3 *Let* (3.2) *and* (3.6) *hold. Then* U *satisfies the following expressions:*

(i) (Growth condition) *For all* $(t, \beta, s) \in [0, T] \times (0, \infty) \times (0, \infty)^n$,

$$0 \leq U(t, \beta, s) \leq c_0(\beta^l + \beta^{-l}) e^{l\|r\|_\infty T} + \gamma_0^\top s.$$

(ii) (Supersolution) *For any smooth test function* φ *and local minimizer* $y = (t, \beta, s) \in [0, T) \times \mathbf{R}^{n+1}_+$ *of* $(U_* - \varphi)$ *on* $[0, T] \times \mathbf{R}^{n+1}_+$, *we have*

$$\min\Big\{ -\mathcal{G}^{D\varphi(y)}\varphi(y), \ \sup_{p \in \mathbf{R}^n} \mathcal{H}(D^{s,p}\varphi(y)) \Big\} \geq 0.$$

(iii) (Subsolution) *For any smooth test function* φ *and local maximizer* $y = (t, \beta, s) \in [0,T) \times \mathbf{R}_+^{n+1}$ *of* $(U^* - \varphi)$ *on* $[0,T] \times \mathbf{R}_+^{n+1}$, *we have*

$$\min\Big\{-\mathcal{G}^{D\varphi(y)}\varphi(y),\ \widetilde{\mathcal{H}}(D\varphi(y):s)\Big\} \le 0.$$

(iv) (Terminal condition) $\quad U_*(T,z) \ge \widehat{g_*}(z), \qquad z \in (0,\infty)^{n+1}.$
Moreover if γ_0 *in* (3.6) *is an element of* $int(K \cap \mathbf{R}_+^n)$ *and* (3.7)-(3.8) *are satisfied, then* $\quad U_*(T,z) = U^*(T,z) = \widehat{g}(z), \qquad z \in (0,\infty)^{n+1}.$

Here the upper (resp. lower) semicontinuous envelope U^* *(resp.* $U_* := -(-U)^*$*) of* U *is defined as*

$$U^*(y) := \lim_{\varepsilon \downarrow 0} \sup\{U(y') \ :\ |y - y'| \le \varepsilon,\ y' \in [0,T) \times (0,\infty)^{n+1}\}, \quad y \in [0,T] \times \mathbf{R}_+^{n+1},$$

$$D^{s,p}\varphi := (D_1^{s,p}\varphi, \dots, D_n^{s,p}\varphi)^\top \qquad \text{with} \qquad D_j^{s,p}\varphi := D_{s_j}\varphi \mathbf{1}_{\{s_j>0\}} + p_j \mathbf{1}_{\{s_j=0\}},$$

and $\widetilde{\mathcal{H}}(p:s) := \mathcal{H}(p)\mathbf{1}_{\{s\in(0,\infty)^n\}} + \infty \mathbf{1}_{\{s\in\partial\mathbf{R}_+^n\}}$.

We conclude the paper with mention of a verification theorem for the DPE (3.5).

Corollary 3.4 (Verification Theorem) *Let* (3.2) *and* (3.6) *be satisfied, and assume* $g \le \widehat{g_*}$. *Let* $u \in C^{1,1,2}([0,T) \times (0,\infty) \times (0,\infty)^n) \cap C([0,T] \times (0,\infty)^{n+1})$ *be solution of*

$$\begin{aligned}
\mathcal{G}^{Du(y)}u(y) &= 0, && y \in [0,T) \times (0,\infty)^{n+1},\\
Du(y) &\in K, && y \in [0,T) \times (0,\infty)^{n+1},\\
u(T,z) &= \widehat{g_*}(z), && z \in (0,\infty)^{n+1},\\
u(t,z) &\le c_0\Big(1 + |z|^l + \prod_{j=0}^{n} z_j^{-l}\Big), && z \in (0,\infty)^{n+1},
\end{aligned}$$

where c_0, $l > 0$ *are constants. Then* $u = U$ *on* $[0,T) \times (0,\infty)^{n+1}$.

References

[1] Adachi, T.: Hedging costs for two large investors. Preprint, (2007). (available from **http://www.fmu.ac.jp/home/mathema/mathema-e/index.html**)

[2] Adachi, T.: Option on a unit-type closed-end investment fund. Adv. Math. Econ. **9**, 1-23 (2006).

[3] Bank, P., Baum, D.: Hedging and portfolio optimization in financial markets with a large trader. Math. Finance **14**, 1-18 (2004).

[4] Bensoussan, A., Touzi, N., Menaldi, J.-L.: Penalty approximation and analytical characterization of the problem of super-replication under portfolio constraints. Asymptotic Analysis **41**, 311-330 (2005).

[5] Crandall, M.G., Ishii, H., Lions, P.-L.: User's guide to viscosity solutions of second order partial differential equations. Bull. Amer. Math. Soc. **27**, no.1, 1-67 (1992).

[6] Cuoco, D., Liu, H.: A martingale characterization of consumption choices and hedging costs with margin requirements. Math. Finance **10**, 355-385 (2000).

[7] Cvitanić, J., Karatzas, I., Soner, H.M.: Backward stochastic differential equations with constraints on the gains-process. Ann. Probab. **26**, 1522-1551 (1999).

[8] Cvitanić, J., Ma, J.: Hedging options for a large investor and forward-backward SDE's. Ann. Appl. Probab. **6**, no.2, 370-398 (1996).

[9] Duffie, D.: Dynamic asset pricing theory. 3rd ed., Princeton Univ. Press 2001.

[10] Frey, R.: Perfect option hedging for a large trader. Finance & Stochast. **2**, 115-141 (1998).

[11] Frey, R., Stremme, A.: Market volatility and feedback effects from dynamic hedging. Math. Finance **7**, 351-374 (1997).

[12] Karatzas, I.: Lectures on the mathematics of finance. CRM Monographs **8**, AMS 1997.

[13] Karatzas, I., Shreve, S.E.: Methods of mathematical finance. Springer-Verlag, NY 1998.

[14] Platen, E., Schweizer, M.: On feedback effects from hedging derivatives. Math. Finance **8**, 67-84 (1998).

[15] Sekine, J.: On superhedging under delta constraints. Appl. Math. Finance **9**, 103-121 (2002).

[16] Soner, H.M., Touzi, N.: Dynamic programming for stochastic target problems and geometric flows. J. European Math. Soc. **4**, 201-236 (2002).

[17] Soner, H.M., Touzi, N.: Stochastic target problems, dynamic programming and viscosity solutions. SIAM J. Control & Opt. **41**, 404-424 (2002).

[18] Soner, H.M., Touzi, N.: The problem of super-replication under constraints. Paris-Princeton Lectures on Mathematical Finance 2002, LNM **1814**, Springer-Verlag, 133-172 (2003).