A Note on the Stewart's Secretary Problem

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1　Introduction

Here we reconsider the continuous arrival time secretary problem with unknown number of rankable applicants firstly treated by Stewart(1981). Let $X_1, X_2, \ldots, X_N$ be continuous i.i.d. random variables with values in $[0, T]$ and common c.d.f. F, where N is an integer-valued random variable independent of $X_k's$. X_k is thought of as the arrival time of the kth best applicant and N represents the total number of applicants. It is easy to see that, given $N = n$, the arrival orders of n applicants are equally likely. The objective of the problem is to find a policy that maximizes the probability of selecting the best applicant among N all applicants. Stewart adopted a Bayesian approach to this problem by assuming a prior distribution $p_n = P\{N = n\}$ on N and examined the limiting case ($M \to \infty$) of the following uniform prior, i.e.,

$$p_n = \frac{1}{M}, \quad 1 \leq n \leq M \tag{1}$$

In this note, we examine the optimal policy in the case of finite M.

Bruss and Samuels(1987) also reconsidered the Stewart problem in greater generality, but the exact solution of our problem is not included. For related works, see Cowan and Zabczyk(1978), Bruss(1984, 1987, 1988), Bruss and Samuels(1990) and Bruss and Rogers(1991).

2　Optimal Policy

For the following, it is convenient to introduce a change of time $Z_k = F(X_k), 1 \leq k \leq N$ such that $Z_1, Z_2, \ldots, Z_N$ are i.i.d. uniform on $[0, 1]$. Let $N(1) = N$ and

$$N(t) = \#\{Z_k : Z_k \leq t\}, \quad 0 \leq t \leq 1$$

and focus our attention on the posterior distribution $P\{N = n \mid ß_t\}$ where $ß_t$ denotes the σ-algebra generated by $\{N(s) : s \leq t\}$. The posterior distribution depends on the prior p_n, parameter t and the observation $N(t)$ because of the i.i.d. assumption of the arrival times.

A straightforward application of Bayes formulae yields (see,e.g., Bruss and Rogers (1991))

$$P\{n \mid \beta_t\} = \frac{\binom{n}{N(t)}t^{N(t)}(1-t)^{n-N(t)}p_n}{\sum_{m=N(t)}^{\infty}\binom{m}{N(t)}t^{N(t)}(1-t)^{m-N(t)}p_m}$$

$$= \frac{\binom{n}{N(t)}(1-t)^n p_n}{\sum_{m=N(t)}^{\infty}\binom{m}{N(t)}(1-t)^m p_m} \tag{2}$$

Let (k,t) denote the state in which we are facing at time $1-t$ the kth applicant who is relatively best, referred to as a *candidate* hereafter (note that t is not an elapsed time but represents the remaining time). Since, in our case, a prior is given by (1), the posterior in state (k,t) is updated, from (2), to

$$p(n \mid k,t) = \frac{\binom{n}{k}t^n}{C(k,t)} \tag{3}$$

where $C(k,t) = \sum_{m=k}^{M}\binom{m}{k}t^m$.

Let $P(k,t)$ be the probability of choosing the best if we select a current candidate in state (k,t). Since, conditional on $N=n$, success probability is k/n, we have

$$P(k,t) = \sum_{n=k}^{M}\frac{k}{n}p(n \mid k,t)$$

$$= \frac{1}{C(k,t)}\sum_{n=k}^{M}\binom{n-1}{k-1}t^n \tag{4}$$

On the other hand, let $Q(k,t)$ be the corresponding probability if we reject the current applicant and then select the first candidate that appears. Conditional on $N=n$, the success probability is known to be given by $\frac{k}{n}\sum_{j=k+1}^{n}\frac{1}{j-1}$ for $n>k$. Hence, unconditioning with respect to the posterior distribution gives

$$Q(k,t) = \sum_{n=k+1}^{M}\left(\frac{k}{n}\sum_{j=k+1}^{n}\frac{1}{j-1}\right)p(n \mid k,t)$$

$$= \frac{1}{C(k,t)}\sum_{n=k+1}^{M}\left(\sum_{j=k+1}^{n}\frac{1}{j-1}\right)\binom{n-1}{k-1}t^n \tag{5}$$

Now let

$$G = \{(k,t) : P(k,t) \geq Q(k,t)\} \tag{6}$$

Hence, G represents the set of states for which stopping immediately is at least as good as continuing for exactly one more transition and then

stopping. The policy that stops the first time the process enters a state in G is called the OLA (one-stage look-ahead) policy. It is well known that, if G is *closed* in a sense that, once $(k, t) \in G$, then $(k + j, s) \in G$ for $j \geq 1, s < t$, then the OLA policy is optimal (see Ross(1983)). Chow, Robbins and Siegmund(1971) called this case *monotone case*. From (4) and (5), $P(k, t) \geq Q(k, t)$ is equivalent to

$$\sum_{n=k}^{M} \binom{n-1}{k-1} t^n \geq \sum_{n=k+1}^{M} \left(\sum_{j=k+1}^{n} \frac{1}{j-1} \right) \binom{n-1}{k-1} t^n \tag{7}$$

Remark: Let M tend to infinity in (7). If we use an identity (see Stewart(1981) or Bruss(1987)),

$$\sum_{n=k+1}^{\infty} \left(\sum_{j=k+1}^{n} \frac{1}{j-1} \right) \binom{n-1}{k-1} t^k (1-t)^{n-k} = -\log t,$$

for $k = 1, 2, \ldots$, and $t \in (0, 1)$. then (7) can be reduced to

$$\left(\frac{t}{1-t} \right)^k \geq - \left(\frac{t}{1-t} \right)^k \log(1-t)$$

or equivalently $t \leq 1 - e^{-1}$ and G is expressed as

$$G = \left\{ (k, t) : t \leq 1 - e^{-1}, \text{ irrespective of } k \right\}$$

Since G is closed, G gives an optimal stopping region. This is just the result Stewart obtained.

We now return to finite M and find the range of t which satisfies the inequality (7). Since both sides of (7) are continuous in t, the solution to this inequality is found by solving the equality. Before doing so, note that (7) can be written as

$$\sum_{n=k+1}^{M} f_{k,n}(t) \leq 1 \tag{8}$$

if we define

$$f_{k,n}(t) = (b_{k,n} - 1) \binom{n-1}{k-1} t^{n-k}, \quad k+1 \leq n \leq M$$

where $b_{k,n} = \sum_{j=k+1}^{n} \frac{1}{j-1}$. We also define, for a given M,

$$s^* = s^*(M) = \min \left\{ s \geq 1 : b_{s,M} \leq 1 \right\}$$

Lemma 1 Let $F_k(t) = \sum_{n=k+1}^{M} f_{k,n}(t)$. Then we have the following:

(i) $k \geq s^*$,
$$F_k(t) \leq 0, \quad \text{for} \quad 0 \leq t \leq 1$$

(ii) $k \leq s^* - 1$,

If $F_k(1) > 1$, then $F_k(t) = 1$ has a unique root $t_k^* \in (0,1)$ such that $F_k(t) \leq 1, 0 \leq t \leq t_k^*$ and $F_k(t) > 1, t_k^* < t \leq 1$. If $F_k(1) \leq 1$, then $F_k(t) \leq 1$ for $0 \leq t \leq 1$.

Proof. (i) Obvious from the definition of s^*. (ii) For given k, there exists an integer r such that $b_{k,n} > 1$ for $n > r$ and $b_{k,n} \leq 1$ for $n \leq r$. Thus

$$F_k(t) = \sum_{n=r+1}^{M} (b_{k,n} - 1)\binom{n-1}{k-1}t^{n-k} - \sum_{n=k+1}^{r} (b_{k,n} - 1)\binom{n-1}{k-1}t^{n-k}$$

with both summations containing only non-negative coefficients. The mth derivative of $F_k(t)$ is given by

$$F_k^{(m)}(t) = \sum_{n=k+m}^{M} (b_{k,n} - 1)\binom{n-1}{k-1}\frac{(n-k)!}{(n-k-m)!}t^{n-k-m}, \quad 1 \leq m \leq M - k$$

First note that $F_k^{(r+1-k)}(t) > 0$ for $0 \leq t \leq 1$. This implies that $F_k^{(r-k)}(t)$ is increasing and has the following property due to $F_k^{(r-k)}(0) < 0$:

(a) If $F_k^{(r-k)}(1) \geq 0$, then there exists a unique root $a \in (0,1)$ such that $F_k^{(r-k)}(a) = 0$ and $F_k^{(r-k)}(t) \leq 0$, for $t \leq a$ whereas $F_k^{(r-k)}(t) > 0$, for $a < t \leq 1$.

(b) If $F_k^{(r-k)}(1) < 0$, then $F_k^{(r-k)}(t) < 0$, for $0 \leq t \leq 1$.

In case (a), $F_k^{(r-k)}(a) = 0$ and $F_k^{(r+1-k)}(a) > 0$ implies that $F_k^{(r-1-k)}(t)$ achieves its minimum at a. Since $F_k^{(r-1-k)}(0) < 0$ and $F_k^{(r-1-k)}(t)$ is increasing for $t \geq a$, $F_k^{(r-1-k)}(t)$ also has the above property. In case (b), obviously $F_k^{(r-1-k)}(t) < 0$ for $0 \leq t \leq 1$, and has the above property. Thus we have shown that $F_k^{(r-1-k)}(t)$ has the same property as $F_k^{(r-k)}(t)$. This argument is repeated to show that $F_k^{(1)}(t)$ also has the same property. Since $F_k(0) = 0$, if $F_k^{(1)}(t)$ satisfies the case (b), $F_k(t)$ is decreasing and so $F_k(t) \leq 0$. If $F_k^{(1)}(t)$ satisfies the case (a), $F_k(t)$ attains its minimum at a and then increases. Hence, if $F_k(1) > 1$, then $F_k(t) = 1$ has a unique root. Thus the proof is complete.

For convenience, define $t_k^* = 1$ if $F_k(t) < 1$, for $0 \le t \le 1$. To show that the set G is closed, it suffices to show that the sequence t_k^*'s are non-decreasing, i.e., $t_1^* \le t_2^* \le \cdots \le t_M^*$. To show this, we need the following lemma.

Lemma 2 We have, for $2 \le k \le M - 1$,

$$F_{k-1}(t) - F_k(t) = t\{1 - F_k(t)\} + \{f_{k-1,M}(t) + tf_{k,M}(t)\}$$

Proof. It is easy to see

$$F_{k-1}(t) - F_k(t) = f_{k-1,M}(t) + \sum_{n=k+1}^{M} \{f_{k-1,n-1}(t) - f_{k,n}(t)\} \qquad (9)$$

From the definition, we have

$$f_{k-1,n-1}(t) - f_{k,n}(t)$$

$$= \left[\left\{\binom{n-1}{k-1} - \binom{n-2}{k-2}\right\} + \left\{\binom{n-2}{k-2}b_{k-1,n-1} - \binom{n-1}{k-1}b_{k,n}\right\}\right] t^{n-k} \quad (10)$$

Applying to (10) the following easily verifiable results

$$\binom{n-1}{k-1} - \binom{n-2}{k-2} = \binom{n-2}{k-1}$$

$$\binom{n-2}{k-2}b_{k-1,n-1} - \binom{n-1}{k-1}b_{k,n} = -\binom{n-2}{k-1}b_{k,n-1},$$

we have

$$
\begin{aligned}
f_{k-1,n-1}(t) - f_{k,n}(t) &= -\binom{n-2}{k-1}(b_{k,n-1} - 1)t^{n-k} \\
&= -tf_{k,n-1}(t) \qquad (11)
\end{aligned}
$$

Substituting (11) into (9) yields

$$
\begin{aligned}
F_{k-1}(t) - F_k(t) &= f_{k-1,M}(t) - t\sum_{n=k+1}^{M} f_{k,n-1}(t) \\
&= f_{k-1,M}(t) - t\{F_k(t) - 1 - f_{k,M}(t)\} \\
&= t\{1 - F_k(t)\} + f_{k-1,M}(t) + tf_{k,M}(t),
\end{aligned}
$$

which is the desired rsult.

Lemma 3 The sequence $\{t_k^*\}$ is non-decreasing in k. That is,

$$t_1^* \le t_2^* \le \cdots \le t_M^*$$

Proof. Since $t_k^* = 1$ for $k \geq s^*$, we only show $t_{k-1}^* \leq t_k^*$ for $k \leq s^* - 1$. from Lemma 2, to show this, it suffices to show that

$$F_{k-1}(t) \geq F_k(t), \quad 0 \leq t \leq t_k^* \qquad (12)$$

Since $1 \geq F_k(t)$ for $0 \leq t \leq t_k^*$, (12) is immediate from Lemma 2 because $f_{k,M}(t) \geq 0, f_{k-1,M}(t) \geq 0$ for $k \leq s^* - 1$.

We can now summarize our main result.

Theorem 4 The optimal policy of the problem is to choose a candidate in state (k, t) if $t \leq t_k^*$, where t_k^* is the unique root of the equation $F_k(t) = 1$ if $F_k(1) \geq 1$, otherwise t_k^* is defined as 1.

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