数理解析研究所講究録
第 1561 巻 2007 年 61-70

Note on Information Transmission in Quantum Systems

Noboru Watanabe

**Department of Information Sciences,
Tokyo University of Science
Noda City, Chiba, 278-8510, Japan
E-mail: watanabe@is.noda.tus.ac.jp**

December 31, 2006

Abstracts The mutual entropy (information) denotes an amount of information transmitted correctly from the input system to the output system through a channel. The (semi-classical) mutual entropies for classical input and quantum output were defined by several researchers. The fully quantum mutual entropy, which is called Ohya mutual entropy, for quantum input and output by using the relative entropy was defined by Ohya in 1983. In this paper, we compare with mutual entropy-type measures and show some resuls for quantum capacity.

1 Introduction

The development of communication theory is closely connected with study of entropy theory. The signal of the input system is carried through a physical device, which is called a channel. The mathematical representation of the channel is a mapping from the input state space to the output state space. In classical communication theory, the mutual entropy was formulated by using the joint probability distribution between the input system and the output system. The (semi-classical) mutual entropies for classical input and quantum output were defined by several researchers [5, 6]. In fully quantum system, there does not exist the joint probability distribution in general. Instead of the joint probability distribution, Ohya [8] invented the quantum (Ohya) compound state, and he introduced the fully quantum mutual entropy (information), which is called Ohya mutual entropy, for quantum input and output systems, describes the amount of information correctly sent from the quantum input system to the quantum output system through the quantum channel.

Recently Shor [21] and Bennet et al [2, 3, 19, 20] took the coherent entropy and defined the mutual type entropy to discuss a sort of coding theorem for communication processes.

In this paper, we compare with mutual entropy-type measures and show some resuls for quantum capacity for the attenuation channel. $\mathcal{H}$

2 Quantum Channels

The concept of channel has been carried out an important role in the progress of the quantum communication theory. In particular, an attenuation channel introduced in [8] is one of the most inportaint model for discussing the information transmission in quantum optical communication. Here we review the definition of the quantum channels.

Let $\mathcal{H}_1, \mathcal{H}_2$ be the complex separable Hilbert spaces of an input and an output systems, respectively, and let $\mathbf{B}(\mathcal{H}_k)$ be the set of all bounded linear operators on $\mathcal{H}_k$. We denote the set of all density operators on $\mathcal{H}_k \quad (k = 1, 2)$ by

$$\mathfrak{S}(\mathcal{H}_k) \equiv \{\rho \in \mathbf{B}(\mathcal{H}_k); \rho \geq 0, tr\rho = 1\}. \tag{1}$$

A map Λ^* from the quantum input system to the quantum output system is called a **(fully) quantum channel**.

1. Λ^* is called a **linear channel** if it satisfies the affine property, i.e.,

$$\sum_k \lambda_k = 1 \, (\forall \lambda_k \geq 0) \Rightarrow \Lambda^* \left(\sum_k \lambda_k \rho_k \right) = \sum_k \lambda_k \Lambda^* (\rho_k), \forall \rho_k \in \mathfrak{S}(\mathcal{H}_1). \tag{2}$$

2. $\Lambda^* : \mathfrak{S}(\mathcal{H}_1) \to \mathfrak{S}(\mathcal{H}_2)$ is called a **completely positive (CP) channel** if its dual map Λ satisfies

$$\sum_{j,k=1}^{n} B_j^* \Lambda (A_j^* A_k) B_k \geq 0 \tag{3}$$

for any $n \in \mathbf{N}$, any $B_j \in \mathbf{B}(\mathcal{H}_1)$ and any $A_k \in \mathbf{B}(\mathcal{H}_2)$, where the dual map $\Lambda : \mathbf{B}(\mathcal{H}_2) \to \mathbf{B}(\mathcal{H}_1)$ of $\Lambda^* : \mathfrak{S}(\mathcal{H}_1) \to \mathfrak{S}(\mathcal{H}_2)$ satisfies $tr\rho\Lambda(A) = tr\Lambda^*(\rho) A$ for any $\rho \in \mathfrak{S}(\mathcal{H}_1)$ and any $A \in \mathbf{B}(\mathcal{H}_2)$.

2.1 Attenuation channel

Let us consider the communication processes including noise and loss systems. Let $\mathcal{K}_1, \quad \mathcal{K}_2$ be the complex separable Hilbert spaces for the noise and the loss systems, respectively. The quantun communication channel

$$\Lambda_0^*(\rho) \equiv tr_{\mathcal{K}_2} \pi_0^* (\rho \otimes \xi_0), \quad \xi_0 \equiv |0\rangle \langle 0| \text{ and } \pi_0^* (\cdot) \equiv V_0 (\cdot) V_0^* \tag{4}$$

is called the attenuation channel, where $|0\rangle\langle 0|$ is vacuum state in $\mathcal{H}_1$ and V_0 is a linear mapping from $\mathcal{H}_1 \otimes \mathcal{K}_1$ to $\mathcal{H}_2 \otimes \mathcal{K}_2$ given by

$$V_0\left(|n\rangle \otimes |0\rangle\right) \equiv \sum_{j=0}^{n} C_j^n \, |j\rangle \otimes |n-j\rangle\,, \quad C_j^n = \sqrt{\frac{n!}{j!(n-j)!}} \alpha^j \bar{\beta}^{n-j} \tag{5}$$

for any $|n\rangle$ in $\mathcal{H}_1$ and α,β are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$. $\eta = |\alpha|^2$ is the transmission rate of the channel. $\pi_0^*$is called a beam splittings, which means that one beam comes and two beams appear after passing throughπ_0^*. This attenuation channel is generalized by Ohya and Watanabe such as noisy optical channel [14, 15]. After that, Accardi and Ohya [1] reformulated it by using liftings, which is the dual map of the transition expectation by mean of Accardi. It contains the concept of beam splittings, which is extended by Fichtner, Freudenberg and Libsher [4] concerning the mappings on generalized Fock spaces. For the attenuation channel Λ_0^*, one can obtain the following theorem:

Theorem 1 *The attenuation channel Λ_0^* is described by*

$$\Lambda_0^*\left(\rho\right) = \sum_{i=0}^{\infty} O_i V_0 Q \rho Q^* V_0^* O_i^*, \tag{6}$$

where $Q \equiv \sum_{l=0}^{\infty} \left(|y_l\rangle \otimes |0\rangle\right) \langle y_l|\,,\ O_i \equiv \sum_{k=0}^{\infty} |z_k\rangle \left(\langle z_k| \otimes \langle i|\right)$, $\{|y_l\rangle\}$ is a CONS in$\mathcal{H}_1$, $\{|z_k\rangle\}$ is a CONS in $\mathcal{H}_2$ and $\{|i\rangle\}$ is the set of number states in $\mathcal{K}_2$.

3 Ohya Mutual Entropy and Capacity

The quantum entropy was introduced by von Neumann around 1932 [7], which is defined by

$$S\left(\rho\right) \equiv -tr \rho \log \rho$$

for any density operators ρ in $S\left(\mathcal{H}_1\right)$. It denotes the amount of information of the quantum state ρ.

In order to define such a quantum mutual entropy, we need the quantum relative entropy and the joint state, which is called a compound state, describing the correlation between an input state ρ and the output state $\Lambda^*\rho$ through a channel Λ^*. For a state $\rho \in \mathfrak{S}(\mathcal{H}_1)$,

$$\rho = \Sigma_k \lambda_k E_k, \tag{7}$$

is called a Schatten decomposition [18] of ρ, where E_k is the one-dimensional projection associated with λ_k. This Schatten decomposition is not unique unless every eigenvalue is non-degenerated. For $\rho \in \mathfrak{S}(\mathcal{H}_1)$ and $\Lambda^* : \mathfrak{S}(\mathcal{H}_1) \to \mathfrak{S}(\mathcal{H}_2)$, the compound states are define by

$$\sigma_E = \sum_n \lambda_n E_n \otimes \Lambda^* E_n, \quad \sigma_0 = \varphi \otimes \Lambda^* \varphi. \tag{8}$$

The first compound state, which is called a Ohya compund state associating the Schatten decomposition $\rho = \Sigma_k \lambda_k E_k$, generalizes the joint probability in classical dynamical system and it exhibits the correlation between the initial state ρ and the final state $\Lambda^* \rho$.

Ohya mutual entropy with respect to ρ and Λ^* is defined by

$$I(\rho; \Lambda^*) \equiv \sup \{ S(\sigma_E, \sigma_0); E = \{E_n\} \}, \tag{9}$$

where $S(\sigma_E, \sigma_0)$ is Umegaki's relative entropy [22]. $I(\rho; \Lambda^*)$ satisfies the Shannon's type inequality : $0 \le I(\rho, \Lambda^*) \le \min \{ S(\rho), S(\Lambda^* \rho) \}$.

3.1 Quantum Capacity

The capacity means the ability of the information transmission of the channel, which is used as a measure for construction of channels. The fully quantum capacity is formulated by taking the supremum of the fully quantum mutual entropy with respect to a certain subset of the initial state space. The capacity of purely quantum channel was studied in [11, 12, 13, 14].

Let $\mathcal{S}$ be the set of all input states satisfying some physical conditions. Let us consider the ability of information transmission for the quantum channelΛ^*. The answer of this question is the capacity of quantum channel Λ^* for a certain set $\mathcal{S} \subset S(\mathcal{H}_1)$ defined by

$$C_q^{\mathcal{S}}(\Lambda^*) \equiv \sup \{ I(\rho; \Lambda^*); \rho \in \mathcal{S} \}. \tag{10}$$

When $\mathcal{S} = S(\mathcal{H}_1)$, the capacity of quantum channel Λ^* is denoted by $C_q(\Lambda^*)$. Then the following theorem for the attenuation channel was proved in [16].

Theorem 2 *For a subset $\mathcal{S}_n \equiv \{ \rho \in S(\mathcal{H}_1); \dim s(\rho) = n \}$, the capacity of the attenuation channel Λ_0^* satisfies*

$$C_q^{\mathcal{S}_n}(\Lambda_0^*) = \log n,$$

where $s(\rho)$ is the support projection of ρ.

When the mean energy of the input state vectors $\{|\tau\theta_k\rangle\}$ can be taken infinite, i.e., $\lim_{\tau\to\infty} |\tau\theta_k|^2 = \infty$, the above theorem tells that the quantum capacity for the attenuation channel Λ_0^* with respect to $\mathcal{S}_n$ becomes $\log n$. It is a natural result, however it is impossible to take the mean energy of input state vector infinite.

3.2 Semi-classical mutual entropy

When the input system is classical, the state φ is a probability distribution and the Schatten-von Neumann decomposition is unique with delta measures δ_n such that $\varphi = \sum_n \lambda_n \delta_n$. In this case we need to code the classical state φ by a quantum state ψ, whose process is a quantum coding described by a channel

Γ^* such that $\Gamma^*\delta_n = \psi_n$ (quantum state) and$\psi \equiv \Gamma^*\varphi = \sum_n \lambda_n \psi_n$. Then Ohya mutual entropy $I(\varphi; \Lambda^* \circ \Gamma^*)$ becomes Holevo's one, that is,

$$I(\varphi; \Lambda^* \circ \Gamma^*) = S(\Lambda^*\psi) - \sum_n \lambda_n S(\Lambda^*\psi_n) \tag{11}$$

when $\sum_n \lambda_n S(\Lambda^*\psi_n)$ is finite. These Ohya mutual entropy are completely quantum, namely, it represents the information transmission from a quantum input to a quantum output. The quantum system is described by a noncommutative structure. The classical system is expressed by a commutative construction. In the mathematical point of view, the commutative systems are contained in the noncommutative framework. One can obtain the diagram in Figure 1.

	↙	**Semi-Classical Mutual Entropy**	←	**Ohya Mutual Entropy**
Shannon's Mutual Entropy				↑
	↖	**Mutual Entropy (GKY)**	←	**Ohya Mutual Entropy for GQS**

4 Quantum Mutual Type Entropies

Recently Shor [21] and Bennet et al [2, 3] took the coherent entropy and defined the mutual type entropy to discuss a sort of coding theorem for quantum communication. In this section, we compare these mutual types entropy.

Let us discuss the entropy exchange [19]. For a stateρ, a channel Λ^* is denoted by using an operator valued measure $\{A_j\}$ such as

$$\Lambda^*(\cdot) \equiv \sum\nolimits_j A_j^* \cdot A_j, \tag{12}$$

which is called a Stinespring-Sudarshan-Kraus form. Then one can define a matrix $W = (W_{ij})_{i,j}$ with

$$W_{ij} \equiv trA_i^* \rho A_j, \tag{13}$$

by which the entropy exchange is defined by

$$S_e(\rho, \Lambda^*) = -trW \log W. \tag{14}$$

By using the entropy exchange, two mutual type entropies are defined as follows:

$$I_C(\rho; \Lambda^*) \equiv S(\Lambda^*\rho) - S_e(\rho, \Lambda^*), \tag{15}$$

$$I_L(\rho; \Lambda^*) \equiv S(\rho) + S(\Lambda^*\rho) - S_e(\rho, \Lambda^*). \tag{16}$$

The first one is called the coherent entropy $I_C(\rho; \Lambda^*)$ [20] and the second one is called the Lindblad entropy $I_L(\rho; \Lambda^*)$ [3]. By comparing these mutual entropies for quantum information communication processes, we have the following theorem [16, 17]:

Theorem 3 *Let $\{A_j\}$ be a projection valued measure with $dimA_j = 1$. For arbitrary state ρ and the quantum channel $\Lambda^*(\cdot) \equiv \sum_j A_j \cdot A_j^*$, one has*
(1) $0 \leq I(\rho; \Lambda^*) \leq \min\{S(\rho), S(\Lambda^*\rho)\}$ *(Ohya mutual entropy),*
(2) $I_C(\rho; \Lambda^*) = 0$ *(coherent entropy),*
(3) $I_L(\rho; \Lambda^*) = S(\rho)$ *(Lindblad entropy).*

For the attenuation channel Λ_0^*, one can obatain the following theorems [16]:

Theorem 4 *For any state $\rho = \sum_n \lambda_n |n\rangle\langle n|$ and the attenuation channel Λ_0^* with $|\alpha|^2 = |\beta|^2 = \frac{1}{2}$, one has*
(1) $0 \leq I(\rho; \Lambda_0^*) \leq \min\{S(\rho), S(\Lambda_0^*\rho)\}$ *(Ohya mutual entropy),*
(2) $I_C(\rho; \Lambda_0^*) = 0$ *(coherent entropy),*
(3) $I_L(\rho; \Lambda_0^*) = S(\rho)$ *(Lindblad entropy).*

5 Numerical Calculation of Quantum Mutal Type Measures

Based on the results [11, 11, 13], let us compute the quantum mutual type entropies with respect to the attenuation channel Λ_0^* and an input state

$$\rho = \lambda|0\rangle\langle 0| + (1-\lambda)|\theta\rangle\langle\theta|, \tag{17}$$

where $|0\rangle$ is a vacuum state vector in $\mathcal{H}$ and $|\theta\rangle$ is a coherent state vector in$\mathcal{H}$. Then the Schatten decomposion of ρ is uniquely determined by

$$\rho = \lambda_0 E_0^{0,\theta} + \lambda_1 E_1^{0,\theta}, \tag{18}$$

where the eigenvalues λ_0 , $\lambda_1 = 1 - \lambda_0$ of ρ are

$$\lambda_j = \frac{1}{2}\left\{1 + (-1)^j \sqrt{1 - 4\lambda(1-\lambda)\left(1 - \exp\left(-|\theta|^2\right)\right)}\right\} \quad (j = 0, 1). \tag{19}$$

And two projections $E_0^{0,\theta}, E_1^{0,\theta}$ and the eigenvectors $\left|e_j^{0,\theta}\right\rangle$ of λ_j $(j = 0, 1)$ are given by

$$E_j^{0,\theta} = \left|e_j^{0,\theta}\right\rangle\left\langle e_j^{0,\theta}\right|, \tag{20}$$

$$\left|e_j^{0,\theta}\right\rangle = a_j^{0,\theta}|0\rangle + b_j^{0,\theta}|\theta\rangle, \quad (j = 0, 1), \tag{21}$$

where

$$\left|b_j^{0,\theta}\right|^2 = \frac{1}{\tau_{j,0,\theta}^2 + 2\exp\left(-\frac{1}{2}|\theta|^2\right)\tau_{j,0,\theta} + 1}, \tag{22}$$

$$\left|a_j^{0,\theta}\right|^2 = \tau_{j,0,\theta}^2 \left|b_j^{0,\theta}\right|^2, \tag{23}$$

$$a_j^{0,\theta}\overline{b_j^{0,\theta}} = \overline{a_j^{0,\theta}}b_j^{0,\theta} = \tau \tag{24}$$

$$\begin{aligned}\tau_{j,0,\theta} &= \frac{-(1-2\lambda)}{2(1-\lambda)\exp\left(-\frac{1}{2}|\theta|^2\right)} \\ &\quad + (-1)^j \frac{\sqrt{1-4\lambda(1-\lambda)\left(1-\exp\left(-|\theta|^2\right)\right)}}{2(1-\lambda)\exp\left(-\frac{1}{2}|\theta|^2\right)} \qquad (j=0,1).\end{aligned} \tag{25}$$

For the above input state ρ, one can obtain the output state for the attenuation channel Λ_0^* as follows:

$$\Lambda_0^*\rho = \lambda|0\rangle\langle 0| + (1-\lambda)|\alpha\theta\rangle\langle\alpha\theta| \tag{26}$$

The eigenvalues of $\Lambda_0^*\rho$ are given by

$$\begin{aligned}\lambda_j' &= \|\Lambda_0^*\rho\| \\ &= \frac{1}{2}\left\{1+(-1)^j\sqrt{1-4\lambda(1-\lambda)\left(1-\exp\left(-|\alpha|^2|\theta|^2\right)\right)}\right\} \qquad (j=0,1).\end{aligned} \tag{27}$$

Then the Ohya mutual entropy with respect to the input state $\rho = \lambda|0\rangle\langle 0| + (1-\lambda)|\theta\rangle\langle\theta|$ and the attenuation channel Λ_0^* with the transmission rate η is rigorously calculated such as

$$I(\rho;\Lambda_0^*) = S(\Lambda_0^*\rho) - \sum_{j=0}^{1}\lambda_j S\left(\Lambda_0^* E_j^{0,\theta}\right), \tag{28}$$

where

$$S\left(\Lambda^* E_j^{0,\theta}\right) = -\sum_{i=0}^{1}\bar{\lambda}_{ji}^{\bar{x}_{j0},\bar{x}_{j1}}\log\bar{\lambda}_{ji}^{\bar{x}_{j0},\bar{x}_{j1}} \qquad (j=0,1) \tag{29}$$

$$\bar{\lambda}_{ji}^{\bar{x}_{j0},\bar{x}_{j1}} = \frac{1}{2}\left\{1+(-1)^j\sqrt{1-4\overline{\lambda_j}\left(1-\overline{\lambda_j}\right)\left(1-\exp\left(-|\xi_j|^2\right)\right)}\right\}, \tag{30}$$

$$\xi_j = \frac{\tau_{j,0,\theta} - 1}{\sqrt{\left(\tau_{j,0,\theta}^2 + 1\right)^2 - 4\exp\left(-|\alpha|^2|\theta|^2\right)\tau_{j,0,\theta}^2}} \neq 0, \tag{31}$$

$$\begin{aligned}\overline{\lambda_j} &= \frac{1}{2}\left\{1 + (-1)^j \exp\left(-\frac{1}{2}(1-\eta)|\theta|^2\right)\right\} \times \\ &\times \frac{\tau_{j,0,\theta}^2 + 2\exp\left(-\frac{1}{2}|\alpha|^2|\theta|^2\right)\tau_{j,0,\theta} + 1}{\tau_{j,0,\theta}^2 + 2\exp\left(-\frac{1}{2}|\theta|^2\right)\tau_{j,0,\theta} + 1} \quad (j = 0, 1)\end{aligned} \tag{32}$$

For the attenuation channel Λ_0^*, we have the following results [17]:

Lemma 5 ***Lemma 3.*** *For the attenuation channel Λ_0^* and the input state$\rho = \lambda|0\rangle\langle 0| + (1-\lambda)|\theta\rangle\langle\theta|$, there exists a unitary operator U such that*

$$UWU^* = \lambda|0\rangle\langle 0| + (1-\lambda)\left|-\bar{\beta}\theta\right\rangle\left\langle -\bar{\beta}\theta\right|. \tag{33}$$

Theorem 6 ***Therem 4.*** *For the attenuation channel Λ_0^* and the input state$\rho = \lambda|0\rangle\langle 0| + (1-\lambda)|\theta\rangle\langle\theta|$, the entropy exchange is obtained by*

$$S_e(\rho, \Lambda_0^*) = -trW\log W = -\sum_{j=0}^{1}\mu_j\log\mu_j, \tag{34}$$

where

$$\mu_j = \frac{1}{2}\left\{1 + (-1)^j\sqrt{1 - 4\lambda(1-\lambda)\left(1 - \exp\left(-|\beta|^2|\theta|^2\right)\right)}\right\} \quad (j = 0, 1). \tag{35}$$

Theorem 7 *For the attenuation channel Λ_0^* and the input state$\rho = \lambda|0\rangle\langle 0| +$ $(1-\lambda)|\theta\rangle\langle\theta|$, we have*
(1) $0 \leq I(\rho; \Lambda_0^*) \leq \min\{S(\rho), S(\Lambda_0^*\rho)\}$ *(Ohya mutual entropy),*
(2) $-S(\rho) \leq I_C(\rho; \Lambda_0^*) \leq S(\rho)$ *(coherent entropy),*
(3) $0 \leq I_L(\rho; \Lambda_0^*) \leq 2S(\rho)$ *(Lindblad entropy).*

This therem means that the coherent entropy $I_C(\rho; \Lambda_0^*)$ takes a minus value for $|\alpha|^2 < |\beta|^2$ and the Lindblad entropy $I_L(\rho; \Lambda_0^*)$ is bigger than the von Neumann entropy of the input state ρ for $|\alpha|^2 > |\beta|^2$.

From these theorems, Ohya mutual entropy $I(\rho; \Lambda^*)$ supports the inequality satisfied in classical communication processes. Therefore one can conclude that the Ohya mutual entropy is one of the most suitable measures to discuss the information transmission in quantum communication processes instead of the classical mutual entropy.

References

[1] Accardi, L., and Ohya, M., Compond channnels, transition expectation and liftings, Appl. Math, Optim., 39, 33-59 (1999).

[2] Barnum, H., Nielsen, M.A., and Schumacher, B.W., Information transmission through a noisy quantum channel, Physical Review A, 57, No.6, 4153-4175 (1998).

[3] Bennett, C.H., Shor, P.W., Smolin, J.A., and Thapliyalz, A.V., Entanglement-Assisted Capacity of a Quantum Channel and the Reverse Shannon Theorem, quant-ph/0106052.

[4] Fichtner, K.H., Freudenberg, W., and Liebscher, V., Beam splittings and time evolutions of Boson systems, Fakultat fur Mathematik und Informatik, Math/ Inf/96/ 39, Jena, 105 (1996).

[5] Holevo, A.S., Some estimates for the amount of information transmittable by a quantum communication channel (in Russian)}, Problemy Peredachi Informacii, 9, 3-11 (1973).

[6] Ingarden, R.S., Kossakowski, A., and Ohya, M., Information Dynamics and Open Systems, Kluwer, 1997.

[7] von Neumann, J., Die Mathematischen Grundlagen der Quantenmechanik, Springer-Berlin, 1932.

[8] Ohya, M., On compound state and mutual information in quantum information theory, IEEE Trans. Information Theory, 29, 770-774 (1983).

[9] Ohya, M., Some aspects of quantum information theory and their applications to irreversible processes, Rep. Math. Phys., 27, 19-47 (1989).

[10] Ohya, M., and Petz, D., Quantum Entropy and its Use, Springer, Berlin, 1993.

[11] Ohya, M., Petz, D., and Watanabe, N., On capacity of quantum channels, Probability and Mathematical Statistics, 17, 179-196 (1997).

[12] Ohya, M., Petz, D., and Watanabe, N., Numerical computation of quantum capacity, International Journal of Theoretical Physics, 37, No.1, 507-510 (1998).

[13] Ohya, M., and Watanabe, N., Quantum capacity of noisy quantum channel, Quantum Communication and Measurement, 3, 213-220 (1997).

[14] Ohya, M., and Watanabe, N., Foundatin of Quantum Communication Theory (in Japanese), Makino Pub. Co., 1998.

[15] Ohya, M., and Watanabe, N., Construction and analysis of a mathematical model in quantum communication processes, Electronics and Communications in Japan, Part 1, 68, No.2, 29-34 (1985).

[16] Ohya, M., and Watanabe, N., Comparison of mutual entropy - type measures, TUS preprint, 2003.

[17] Ohya, M., and Watanabe, N., Comparison of mutual entropy - type measures II, TUS preprint, 2006.

[18] Schatten, R., Norm Ideals of Completely Continuous Operators, Springer-Verlag, 1970.

[19] Schumacher, B.W., Sending entanglement through noisy quantum channels, Physical Review A, 54, 2614 (1996).

[20] Schumacher, B.W., and Nielsen, M.A., Quantum data processing and error correction, Physical Review A, 54, 2629 (1996).

[21] Shor, P., The quantum channel capacity and coherent information, Lecture Notes, MSRI Workshop on Quantum Computation, 2002.

[22] Umegaki, H., Conditional expectations in an operator algebra IV (entropy and information), Kodai Math. Sem. Rep., 14, 59-85 (1962).