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零加法性 Lusin 定理による Fuzzy neural network の近似について

(Approximation of fuzzy neural networks by using Lusin's theorem)

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概要

この論文は近似理論として議論されるレギュラー・ファジィ・ニューラル・ネットワーク (RFNN) について、Sugeno 積分を導入したノルム評価の結果を述べる。対象とする多重 RFNN は 4 層であり、従来扱われた一様収束としての評価に代えたものである。任意のファジィ値関数をファジィ値の単純関数と可測関数の場合の 2 通りを考える。

In this note, we study an approximation property of regular fuzzy neural network(RFNN). It is shown that any fuzzy-valued measurable function can be approximated by the four-layer RFNN in the sense of fuzzy integral norm for the finite sub-additive fuzzy measure on $\mathbb{R}$.

Keywords: Fuzzy measure; Lusin's theorem; Approximation; Regular fuzzy neural network

1 Introduction

In neural network theory, the learning ability of a neural network is closely related to its approximating capabilities, so it is important and interesting to study the approximation properties of neural networks. The studies on this matter were undertaken by many authors and a great number of important results were obtained ([1, 4, 13] etc). The similar approximation problems in fuzzy environment were investigated by Buckley [2, 3], P. Liu [7, 8] and other authors. In [8] Liu proved that continuous fuzzy-valued function can be closely approximated by a class of regular fuzzy neural networks (RFNNs) with real input and fuzzy-valued output. In this note, by using Lusin's theorem on

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fuzzy measure space, we show that such RFNNs is pan-approximator for fuzzy-valued measurable function. That is, any fuzzy-valued measurable function can be approximated by the four-layer RFNNs in the sense of fuzzy integral norm for the finite sub-additive measure on $\mathbb{R}$.

2 Preliminaries

We suppose that (X, ρ) is a metric space, and that $\mathcal{O}$ and $\mathcal{C}$ are the classes of all open and closed sets in (X, ρ), respectively, and $\mathcal{B}$ is Borel σ-algebra on X, i.e., it is the smallest σ-algebra containing $\mathcal{O}$.

A set function $\mu : \mathcal{B} \to [0, +\infty]$ is called ***a fuzzy measure***([11]), if it satisfies the following properties: (FM1) $\mu(\emptyset) = 0$; (FM2) $A \subset B$ implies $\mu(A) \leq \mu(B)$. A fuzzy measure μ is called ***null-additive*** ([15]), if for any $E, F \in \mathcal{B}$ and $\mu(F) = 0$ imply $\mu(E \cup F) = \mu(E)$; ***sub-additive*** ([12]), if for any $E, F \in \mathcal{B}$ we have $\mu(E \cup F) \leq \mu(E) + \mu(F)$.

In this paper, we always assume that μ is a finite, sub-additive and continuous fuzzy measure on $\mathcal{B}$.

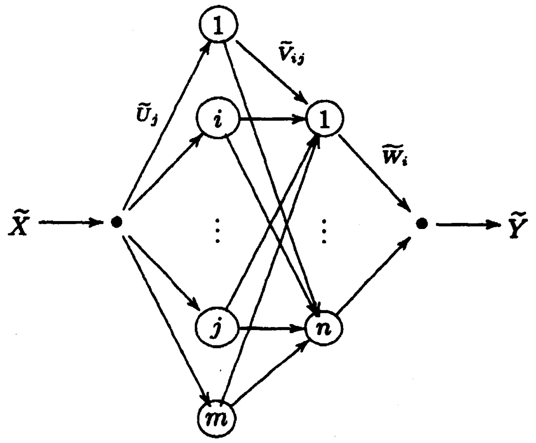
Consider a nonnegative real-valued measurable function f on A and the ***fuzzy integral*** of f on A with respect to μ, which is denoted by

$$(S)\int_A f\,d\mu \triangleq \bigvee_{0 \leq \alpha < +\infty} [\alpha \wedge \mu(\{x : f(x) \geq \alpha\} \cap A)]$$

Theorem 2.1 (Lusin's theorem cf. [6, 14]) *Let (X, ρ) be metric space and μ be null additive fuzzy measure on $\mathcal{B}$. If f is a real-valued measurable function on $E \in \mathcal{B}$, then, for every $\epsilon > 0$, there exists a closed subset $F_\epsilon \in \mathcal{B}$ such that f is continuous on F_ϵ and $\mu(E - F_\epsilon) < \epsilon$.*

3 Approximation in fuzzy mean by regular fuzzy neural networks

In this section, we study an approximation property of the four-layer RFNNs to fuzzy-valued measurable function in the sense of fuzzy integral norm for fuzzy measure on $\mathbb{R}$.



Let $\mathcal{F}_0(\mathbb{R})$ be the set of all bounded fuzzy numbers, i.e., for $\tilde{A} \in \mathcal{F}_0(\mathbb{R})$, the following conditions hold:

(i) $\forall\, \alpha \in (0, 1], \tilde{A}_\alpha \triangleq \{x \in \mathbb{R} \mid \tilde{A}(x) \geq \alpha\}$ is the closed interval of $\mathbb{R}$;

(ii) The support $\mathrm{Supp}(\tilde{A}) \triangleq \mathrm{cl}\{x \in \mathbb{R} \mid \tilde{A}(x) > 0\} \subset$ is a bounded set;

(iii) $\{x \in \mathbb{R} \mid \tilde{A}(x) = 1\} \neq \emptyset$.

For simplicity, $\mathrm{supp}(\tilde{A})$ is also written as $\tilde{A}_0$. Obviously, $\tilde{A}_0$ is a bounded and closed interval of $\mathbb{R}$. For $\tilde{A} \in \mathcal{F}_0(\mathbb{R})$, let $\tilde{A}_\alpha = [a_\alpha^-, a_\alpha^+]$ for each $\alpha \in [0, 1]$ and we denote

$$|\tilde{A}| \triangleq \bigvee_{\alpha \in [0,1]} (|a_\alpha^-| \vee |a_\alpha^+|).$$

For $\tilde{A}, \tilde{B} \in \mathcal{F}_0(\mathbb{R})$, define metric $d(\tilde{A}, \tilde{B})$ between

$\tilde{A}$ and $\tilde{B}$ by

$$d(\tilde{A},\tilde{B}) \triangleq \bigvee_{\alpha\in[0,1]} d_H(\tilde{A}_\alpha, \tilde{B}_\alpha)$$

where d_H means Hausdorff metric: for $A, B \subset \mathbb{R}$,

$$d_H(A,B)$$
$$\triangleq \max\left\{\sup_{x\in A}\inf_{y\in B}(|x-y|), \sup_{y\in B}\inf_{x\in A}(|x-y|)\right\}.$$

It is known that $(\mathcal{F}_0(\mathbb{R}), d)$ is a completely separable metric space ([5]). Also we note that the next assertion which is used in later.

Proposition 3.1 ([8]) *Assume* $\tilde{A}, \tilde{A}_1, \tilde{A}_2 \in \mathcal{F}_0(\mathbb{R})$, *and* $\tilde{W}_i, \tilde{V}_i \in \mathcal{F}_0(\mathbb{R})(i = 1, 2, \cdots, n)$. *Then*

(1) $d(\tilde{A}\cdot\tilde{A}_1, \tilde{A}\cdot\tilde{A}_2) \le |\tilde{A}|\cdot d(\tilde{A}_1, \tilde{A}_2)$,

(2) $d(\sum_{i=1}^n \tilde{W}_i, \sum_{i=1}^n \tilde{V}_i) \le \sum_{i=1}^n d(\tilde{W}_i, \tilde{V}_i)$.

By the well known extension principle, each function $f : \mathbb{R}^n \to \mathbb{R}$ may be extended to one $\mathcal{F}_0(\mathbb{R})^n \to \mathcal{F}(\mathbb{R})$ and, for each fuzzy number, the addition, the multiplication and the multiplication by a scalar are defined by the extension princile ([8]).

Let T be a measurable set in $\mathbb{R}^n$, $(T, \mathcal{B}\cap T, \mu)$ finite fuzzy measure space. Let $\mathcal{L}(T)$ denote the set of all fuzzy-valued measurable function

$$\tilde{F} : T \to \mathcal{F}_0(\mathbb{R}).$$

For any $\tilde{F}_1, \tilde{F}_2 \in \mathcal{L}(T)$, $d(\tilde{F}_1, \tilde{F}_2)$ is measurable function on $(T, \mathcal{B}\cap T)$, we will write a fuzzy integral norm as

$$\Delta_S(\tilde{F}_1, \tilde{F}_2) \triangleq (S)\int_T d(\tilde{F}_1, \tilde{F}_2)d\mu.$$

Proposition 3.2 *Let* $\tilde{F}_1, \tilde{F}_2, \tilde{F}_3 \in \mathcal{L}(T)$, *then*

$$\Delta_S(\tilde{F}_1, \tilde{F}_3) \le 2(\Delta_S(\tilde{F}_1, \tilde{F}_2) + \Delta_S(\tilde{F}_2, \tilde{F}_3)).$$

Proof. From subadditivity of μ, we have

$$\begin{aligned}
\Delta_S(\tilde{F}_1, \tilde{F}_3) &= (S)\int_T d(\tilde{F}_1, \tilde{F}_3)d\mu \\
&= \bigvee_{\alpha\in[0,\infty)} \{\alpha \wedge \mu(T \cap (d(\tilde{F}_1, \tilde{F}_3))_\alpha\} \\
&\le \bigvee_{\alpha\in[0,\infty)} \{\alpha \wedge \mu(T \cap \\
&\quad (d(\tilde{F}_1, \tilde{F}_2)_{\frac{\alpha}{2}} \cup d(\tilde{F}_2, \tilde{F}_3)_{\frac{\alpha}{2}}))\} \\
&\le \bigvee_{\alpha\in[0,\infty)} \{\alpha \wedge [\mu(T \cap d(\tilde{F}_1, \tilde{F}_2)_{\frac{\alpha}{2}}) \\
&\quad +\mu(T \cap d(\tilde{F}_2, \tilde{F}_3)_{\frac{\alpha}{2}})]\}.
\end{aligned}$$

Because of the elementary inequality: $a \wedge (b + c) \le (a \wedge b) + (a \wedge c)$ where $a, b, c \ge 0$, we have

$$\begin{aligned}
&\Delta_S(\tilde{F}_1, \tilde{F}_3) \\
\le &\bigvee_{\alpha\in[0,\infty)} \{\alpha \wedge \mu(T \cap d(\tilde{F}_1, \tilde{F}_2)_{\frac{\alpha}{2}}) \\
&+\alpha \wedge \mu(T \cap d(\tilde{F}_2, \tilde{F}_3)_{\frac{\alpha}{2}})\} \\
\le &\bigvee_{\alpha\in[0,\infty)} [\alpha \wedge \mu(T \cap d(\tilde{F}_1, \tilde{F}_2)_{\frac{\alpha}{2}})] \\
&+ \bigvee_{\alpha\in[0,\infty)} [\alpha \wedge \mu(T \cap d(\tilde{F}_2, \tilde{F}_3)_{\frac{\alpha}{2}})] \\
\le &\bigvee_{\alpha\in[0,\infty)} \Big[\frac{\alpha}{2} \wedge \mu(T \cap d(\tilde{F}_1, \tilde{F}_2)_{\frac{\alpha}{2}}) \\
&+ \frac{\alpha}{2} \wedge \mu(T \cap d(\tilde{F}_1, \tilde{F}_2)_{\frac{\alpha}{2}})\Big] \\
&+ \bigvee_{\alpha\in[0,\infty)} \Big[\frac{\alpha}{2} \wedge \mu(T \cap d(\tilde{F}_2, \tilde{F}_3)_{\frac{\alpha}{2}}) \\
&+ \frac{\alpha}{2} \wedge \mu(T \cap d(\tilde{F}_2, \tilde{F}_3)_{\frac{\alpha}{2}})\Big] \\
\le &\bigvee_{\frac{\alpha}{2}\in[0,\infty)} \Big[\frac{\alpha}{2} \wedge \mu(T \cap d(\tilde{F}_1, \tilde{F}_2)_{\frac{\alpha}{2}})\Big] \\
&+ \bigvee_{\frac{\alpha}{2}\in[0,\infty)} \Big[\frac{\alpha}{2} \wedge \mu(T \cap d(\tilde{F}_1, \tilde{F}_2)_{\frac{\alpha}{2}})\Big]
\end{aligned}$$

$$+ \bigvee_{\frac{\alpha}{2} \in [0,\infty)} \left[\frac{\alpha}{2} \wedge \mu(T \cap d(\tilde{F}_2, \tilde{F}_3)_{\frac{\alpha}{2}})\right]$$
$$+ \bigvee_{\frac{\alpha}{2} \in [0,\infty)} \left[\frac{\alpha}{2} \wedge \mu(T \cap d(\tilde{F}_2, \tilde{F}_3)_{\frac{\alpha}{2}})\right]$$
$$= 2\left(\triangle_S(\tilde{F}_1, \tilde{F}_2) + \triangle_S(\tilde{F}_2, \tilde{F}_3)\right).$$

Definition 3.1 ([8]) A fuzzy-valued function $\tilde{\Phi} : T \to \mathcal{F}_0(\mathbb{R})$ is called a fuzzy-valued simple function, if there exist $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_m \in \mathcal{F}_0(\mathbb{R})$, such that $\forall\, x \in T$,

$$\tilde{\Phi}(x) = \sum_{k=1}^{m} \tilde{A}_k \cdot \chi_{T_k}(x)$$

where $T_k \in \mathcal{B} \cap T$ $(k = 1, 2, \ldots, m)$, $T_i \cap T_j = \emptyset$ $(i \neq j)$ and $T = \bigcup_{k=1}^{m} T_k$.

Immediately, if $\mathcal{S}(T)$ denotes the set of all fuzzy-valued simple functions, then $\mathcal{S}(T) \subset \mathcal{L}(T)$.

Similar to the proof of Proposition 3.2 and by using subadditivity of μ, we can obtain the following proposition.

Proposition 3.3 *Let μ be a finite, sub-additive and continuous fuzzy measure on $\mathbb{R}$. If $\tilde{F} \in \mathcal{L}(T)$, then for every $\epsilon > 0$, there exists $\tilde{\Phi}_\epsilon \in \mathcal{S}(T)$ such that*

$$\triangle_S(\tilde{F}, \tilde{\Phi}_\epsilon) < \epsilon.$$

Define

$$\mathcal{H}[\sigma] \triangleq \left\{ \tilde{H} \;\middle|\; \tilde{H}(x) = \sum_{i=1}^{n} \tilde{W}_i\, V_i[\sigma] \right\}$$

where

$$V_i[\sigma] \triangleq \sum_{j=1}^{m} \tilde{V}_{ij} \cdot \sigma(x \cdot \tilde{U}_j + \tilde{\Theta}_j)$$

and σ, by the same notation, is a given extended function of $\sigma : \mathbb{R} \to \mathbb{R}$ (bounded, continuous and nonconstant), and $x \in \mathbb{R}, \tilde{W}_i, \tilde{V}_{ij}, \tilde{U}_j, \tilde{\Theta}_j \in \mathcal{F}_0(\mathbb{R})$.

For any $\tilde{H} \in \mathcal{H}[\sigma]$, $\tilde{H}$ is a four-layer feedforward RFNN with activation function σ, threshold vector $(\tilde{\Theta}_1, \ldots, \tilde{\Theta}_m)$ in the first hidden layer(cf. [8]).

Restricting fuzzy numbers $\tilde{V}_{ij}, \tilde{U}_j, \tilde{\Theta}_j \in \mathcal{F}_0(\mathbb{R})$, respectively, to be real numbers $v_{ij}, u_j, \theta_j \in \mathbb{R}$, we obtain the subset $\mathcal{H}_0[\sigma]$ of $\mathcal{H}[\sigma]$:

$$\mathcal{H}_0[\sigma] \triangleq \left\{ \tilde{H} \;\middle|\; \tilde{H}(x) = \sum_{i=1}^{n} \tilde{W}_i\, v_i[\sigma] \right\}.$$

where

$$v_i[\sigma] \triangleq \sum_{j=1}^{m} v_{ij} \cdot \sigma(x \cdot u_j + \theta_j).$$

Let define two classes of pan-approximation which is fundamental to our results.

Definition 3.2 (1) *$\mathcal{H}_0[\sigma]$ is call the pan-approximator of $\mathcal{S}(T)$ in the sense of $\triangle_S$, if for $\forall\, \tilde{\Phi} \in \mathcal{S}(T)$, $\forall\, \epsilon > 0$, there exists $\tilde{H}_\epsilon \in \mathcal{H}_0[\sigma]$ such that $\triangle_S(\tilde{\Phi}, \tilde{H}_\epsilon) < \epsilon$.*

(2) *For $\tilde{F} \in \mathcal{L}(T)$, $\mathcal{H}[\sigma]$ is call the pan-approximator for $\tilde{F}$ in the sense of $\triangle_S$, if $\forall\, \epsilon > 0$, there exists $\tilde{H}_\epsilon \in \mathcal{H}[\sigma]$ such that $\triangle_S(\tilde{F}, \tilde{H}_\epsilon) < \epsilon$.*

By using Lusin's theorem (Theorem 2.1), Proposition 3.2 and 3.3 we can obtain the main result in this paper, which is stated in the following.

Theorem 3.1 *Let $(T, \mathcal{B} \cap T, \mu)$ be fuzzy measure space and μ be finite, sub-additive and continuous. Then,*

(1) *$\mathcal{H}_0[\sigma]$ is the pan-approximator of $\mathcal{S}(T)$ in the sense of $\triangle_S$.*

(2) *$\mathcal{H}[\sigma]$ is the pan-approximator for $\tilde{F}$ in the sense of $\triangle_S$.*

Proof. By using the conclusion of (1) and Proposition 3.3 we can obtain (2). Now we only prove (1). Suppose that $\tilde{\Phi}(x)$ is a fuzzy-valued simple function, i.e.,

$$\tilde{\Phi}(x) = \sum_{k=1}^{m} \chi_{T_k}(x) \cdot \tilde{A}_k \quad (x \in T).$$

For arbitrarily given $\epsilon > 0$, applying Theorem 2.1 (Lusin's theorem) to each real measurable function $\chi_{T_k}(x)$, for every fixed k $(1 \le k \le m)$, there exists closed set $F_k \in \mathcal{B} \cap T$ such that

$$F_k \subset L_k \quad \text{and} \quad \mu(L_k - F_k) < \frac{\epsilon}{2m}$$

and $\chi_{T_k}(x)$ is continuous on F_k. Therefore, for every k there exist a Tauber-Wiener function σ and $p_k \in N, v'_{k1}, v'_{k2}, \cdots, v'_{kp_k}, \theta'_{k1}, \theta'_{k2}, \cdots, \theta'_{kp_k} \in \mathbb{R}$, and $\mathbf{w}'_{k1}, \mathbf{w}'_{k2}, \cdots, \mathbf{w}'_{kp_k} \in \mathbb{R}^n$ such that

$$\left| \chi_{T_k}(x) - \sum_{j=1}^{p_k} v'_{kj} \cdot \sigma(\langle \mathbf{w}'_{kj}, x \rangle + \theta'_{kj}) \right| < \frac{\epsilon}{2\sum_{k=1}^{m} |\tilde{A}_k|}$$

for $x \in L_k$. Note that we can assume $\sum_{k=1}^{m} |\tilde{A}_k| \neq 0$, without any loss of generality. Denote $L = \bigcap_{k=1}^{m} L_k$, then $T = L \cup (T - L)$. By the subadditivity of μ, we have

$$\begin{aligned} \mu(T - L) &= \mu(\textstyle\bigcup_{k=1}^{m}(T - L_k)) \\ &\le \textstyle\sum_{k=1}^{m} \mu(T - L_k) < \frac{\epsilon}{2}. \end{aligned}$$

We take $\beta_1 = 0$, $\beta_k = \sum_{i=1}^{k-1} p_i, k = 2, \cdots, m$, and $p = \sum_{k=1}^{m} p_k$. For $k = 1, 2, \cdots, m, j = 1, 2, \cdots, p$, we denote

$$v_{kj} = \begin{cases} v'_{k(j-\beta_k)}, & \text{if } \beta_k < j \le \beta_{k+1}, \\ 0 & \text{otherwise}, \end{cases}$$

$$\theta_{kj} = \begin{cases} \theta'_{k(j-\beta_k)}, & \text{if } \beta_k < j \le \beta_{k+1}, \\ 0 & \text{otherwise}, \end{cases}$$

$$\mathbf{w}_{kj} = \begin{cases} \mathbf{w}'_{k(j-\beta_k)}, & \text{if } \beta_k < j \le \beta_{k+1}, \\ 0 & \text{otherwise}, \end{cases}$$

then, for any $k \in \{1, 2, \cdots, m\}$, we have

$$\begin{aligned} &\textstyle\sum_{j=1}^{p} v_{ij} \cdot \sigma(\langle \mathbf{w}_{kj}, x \rangle + \theta_{kj}) \\ &= \textstyle\sum_{j=1}^{p_k} v'_{ij} \cdot \sigma(\langle \mathbf{w}'_{kj}, x \rangle + \theta'_{kj}). \end{aligned}$$

Now denote that

$$\tilde{H}(x) = \sum_{k=1}^{m} \tilde{A}_k \cdot \left(\sum_{j=1}^{p} v_{kj} \cdot \sigma(\langle \mathbf{w}_{kj}, x \rangle + \theta_{kj}) \right),$$

then $\tilde{H} \in \mathcal{H}_0[\sigma]$.

In the reminder part of this section we will prove $\triangle_S(\tilde{H}, \tilde{\Phi}) < \epsilon$. Denote

$$B_{kj} = v_{kj} \cdot \sigma(\langle \mathbf{w}_{kj}, x \rangle + \theta_{kj})$$

and

$$B'_{kj} = v'_{ij} \cdot \sigma(\langle \mathbf{w}'_{kj}, x \rangle + \theta'_{kj}).$$

By using Proposition 3.1 and noting $\mu(T - L) < \epsilon/2$, we have

$$\begin{aligned} &\triangle_S(\tilde{H}, \tilde{\Phi}) \\ = \;& (S) \int_T d(\tilde{H}, \tilde{\Phi}) d\mu \\ = \;& \bigvee_{0 \le \alpha < +\infty} \left[\alpha \wedge \mu(T \cap (d(\tilde{H}, \tilde{\Phi}))_\alpha) \right]. \end{aligned}$$

Since

$$\begin{aligned} &\mu(T \cap d(\tilde{H}, \tilde{\Phi})_\alpha) \\ = \;& \mu\left(T \cap d\left(\sum_{k=1}^{m} \tilde{A}_k \cdot \sum_{j=1}^{p} B_{kj}, \right.\right. \\ & \qquad \left.\left. \sum_{k=1}^{m} \chi_{T_k}(x) \cdot \tilde{A}_k \right)_\alpha \right) \\ \le \;& \mu\left((L \cup (T - L)) \cap (C_{mp})_\alpha \right) \\ \le \;& \mu(L \cap (C_{mp})_\alpha) + \mu\left((T - L) \cap (C_{mp})_\alpha \right) \end{aligned}$$

where the notation of set C_{mp} is assigned as

$$C_{mp} = \sum_{k=1}^{m} |\tilde{A}_k| \cdot d\left(\sum_{k=1}^{p} B_{kj}, \chi_{T_k}(x) \right).$$

Hence $\triangle_S(\tilde{H}, \tilde{\Phi})$ is dominated by

$$\bigvee_{0 \le \alpha < +\infty} \left[\alpha \wedge \mu\left(L \cap (D_{mp})_\alpha \right) \right]$$

$$+ \bigvee_{0\le\alpha<+\infty} [\alpha \wedge \mu((T-L))]$$

$$\le \bigvee_{0\le\alpha<+\infty} \left[\alpha \wedge \mu\left(L \cap (D'_{mp})_\alpha\right)\right] + \frac{\epsilon}{2}$$

Here, for simplicity, we use the notations of

$$D_{mp} = \sum_{k=1}^{m} |\tilde{A}_k| \cdot \left|\sum_{j=1}^{p} B_{kj} - \chi_{T_k}(x)\right|$$

and

$$D'_{mp} = \sum_{k=1}^{m} |\tilde{A}_k| \cdot d\left(\sum_{j=1}^{p_k} B'_{kj} - \chi_{T_k}(x)\right).$$

Now we estimate the first part in the above formula. If $x \in L$, then for every $k = 1, 2, \cdots, m$, we have $x \in L_k$, hence

$$\left| \chi_{T_k}(x) - \sum_{j=1}^{p_k} v'_{kj} \cdot \sigma(\langle \mathbf{w}'_{kj}, x\rangle + \theta'_{kj}) \right| < \frac{\epsilon}{2\sum_{k=1}^{m} |\tilde{A}_k|},$$

for every $k = 1, 2, \cdots, m$. That is, for $x \in L$,

$$D'_{mp} = \sum_{k=1}^{m} |\tilde{A}_k| \cdot d\left(\sum_{j=1}^{p_k} B'_{kj} - \chi_{T_k}(x)\right) < \frac{\epsilon}{2}.$$

Therefore,

$$\begin{aligned} & \bigvee_{0\le\alpha<+\infty} \left[\alpha \wedge \mu\left(L \cap (D'_{mp})_\alpha\right)\right] \\ = & \bigvee_{\alpha\in[0,\frac{\epsilon}{2}]} \left[\alpha \wedge \mu\left(L \cap (D'_{mp})_\alpha\right)\right] \\ & + \bigvee_{\alpha\in[\frac{\epsilon}{2},\infty)} \left[\alpha \wedge \mu\left(L \cap (D'_{mp})_\alpha\right)\right] \\ = & \bigvee_{\alpha\in[0,\frac{\epsilon}{2}]} \left[\alpha \wedge \mu\left(L \cap (D'_{mp})_\alpha\right)\right] \\ \le & \frac{\epsilon}{2}. \end{aligned}$$

Thus, combining with the previous evaluation, we obtain

$$\begin{aligned} & \Delta_S(\tilde{H}, \tilde{\Phi}) \\ \le & \bigvee\nolimits_{0\le\alpha<+\infty} \left[\alpha \wedge \mu\left(L \cap (D'_{mp})_\alpha\right)\right] + \frac{\epsilon}{2} \\ < & \epsilon. \end{aligned}$$

The proof of (1) is now completed. □

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