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Some constructions of Lie superalgebras from triple systems and Extended Dynkin diagrams [1]

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Abstract Our aim is to give a charaterization of many mathematical and physical fields by means of concept of triple systems (here, triple systems mean a vector space equipped with a triple product $< xyz >$).

§1. Preliminaries and Examples

In this section, we will give the definition and some results for a certain triple system in order to make this paper as self-containd as possible.

Throughout this paper, we shall be concerned with algebras and triple systems over a field Φ that is characteristic not 2 and do not assume that our algebras and triple systems are finite dimensional, unless otherwise specified.

For $\varepsilon = \pm 1$ and $\delta = \pm 1$, a vector space $U(\varepsilon, \delta)$ over Φ with the triple product $< -, -, - >$ is called a (ε, δ)*–Freudenthal Kantor triple system if*

$$[L(a,b), L(c,d)] = L(< abc >, d) + \varepsilon L(c, < bad >) \qquad (L1)$$

$$K(< abc >, d) + K(c, < abd >) + \delta K(a, K(c,d)b) = 0, \qquad (K1)$$

where $L(a,b)c = < abc >, K(a,b)c = < acb > -\delta < bca >, [A,B] = AB - BA$.

The triple products are generally denoted by

$$< xyz >, \quad \{xyz\}, \quad (xyz), \quad and \ [xyz]$$

also, the bilinear forms are denoted by $< x|y >$ and $B(x,y)$, as is our convention.

Remark We note that $S(a,b) := L(a,b) + \varepsilon L(b,a)$ and $A(a,b) := L(a,b) - \varepsilon L(b,a)$ are a derivation and an anti-derivation of $U(\varepsilon, \delta)$,respectively.

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[1] This is an announcement and the details will be published elsewhere.

Example 1.1 Let V be a vector space equipped with a bilinear form $< x|y >= \varepsilon < y|x >$. Then V is a $(\varepsilon, \varepsilon)$-Freudenthal-Kantor triple systyem with respect to the product

$$< xyz > := < x|z > y + < y|z > x.$$

Example 1.2 Let V be a Jordan triple system. Then this triple system is a special case of the (-1,1)-Freudenthal-Kantor triple system, because the identity $K(a,b)c \equiv 0$ (identically zero) implies that $< acb >=< bca >$, and the identity (L1) implies that $< ab < cde >>=<< abc > de > - < c < bad > e > + < cd < abe >>$.

If its product satisfies the following; $< abc >= - < cba >$ and $< ab < cde >>=<< abc > de > + < c < bad > e > + < cd < abe >>$, then this triple system is called an anti-Jordan triple system, that is, we have the case of $\varepsilon = 1, \delta = -1$ in (L1) and $K(a,c)b =< abc > + < cba >\equiv 0$ (identically zero). That is, this is a special case of (-1,-1)-Freudenthal-Kantor triple system.

Definition A (ε, δ)–Freudenthal–Kantor triple system over Φ is said to be *balanced* if there exists a bilinear form $< \mid >$ such that $K(x,y) =< x|y > Id$, where $< x|y >\in \Phi^*$.

Definition. For $\delta = \pm 1$, a triple system over Φ is said to be δ *-Lie triple system* if the following are satisfied

$$[abc] = -\delta[bac], \qquad (LT1)$$

$$[abc] + [bca] + [cab] = 0, \qquad (LT2)$$

$$[ab[cde]] = [[abc]de] + [c[bad]e] + [cd[abe]] \qquad (LT3).$$

For the δ-Lie triple systems associated with (ε, δ) -Freudenthal-Kantor triple systems, we have the following.

Proposition 1.1. ([K-O.1],[K-O.5],[K.4]) *Let $U(\varepsilon,\delta)$ be a (ε,δ)-Freudenthal-Kantor triple system. If P is a linear transformation of $U(\varepsilon,\delta)$ such that $P < xyz >=< PxPyPz >$ and $P^2 = -\varepsilon\delta Id$, then $(U(\varepsilon,\delta), [-,-,-])$ is a Lie triple system for the case of $\delta = 1$ and an anti-Lie triple system for the case of $\delta = -1$ with respect to the product*

$$[xyz] := < xPyz > -\delta < yPxz > +\delta < xPzy > - < yPzx > .$$

Corollary. *Let $U(\varepsilon,\delta)$ be a (ε,δ)-Freudenthal-Kantor triple system. Then the vector space $T(\varepsilon,\delta) := U(\varepsilon,\delta) \oplus U(\varepsilon,\delta)$ becomes a Lie triple system for* *the case of $\delta = 1$ and an anti–Lie triple system for the case of $\delta = -1$ with respect to the triple product defined by*

$$[\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix}] = \begin{pmatrix} L(a,d) - \delta L(c,b) & \delta K(a,c) \\ -\varepsilon K(b,d) & \varepsilon(L(d,a) - \delta L(b,c)) \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix}.$$

Proposition 1.2. *Let V be an anti-Jordan triple system (that is, it satisfies the condition (L1) with $\varepsilon = 1$ and $L(x,y)z = -L(z,y)x$). Then, $V \oplus V$ becomes an anti-Lie triple system with respect to the product defined by*

$$[\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \begin{pmatrix} e \\ e \end{pmatrix}] = \begin{pmatrix} L(a,d) + L(c,b) & 0 \\ 0 & L(d,a) + L(b,c) \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix}.$$

From these results, it follows that the vector space

$$L(V) := \text{Inn Der } T \oplus T \ (= L(T,T) \oplus T),$$

where T is a δ- Lie triple system and Inn Der $T := \{L(X,Y) | X, Y \in T\}_{span}$, makes a Lie algebra ($\delta = 1$) or Lie superalgebra ($\delta = -1$) by

$$[D + X, D' + X'] = [D, D'] + L(X, X') + DX' - D'X.$$

We denote by $L(\varepsilon, \delta)$ the Lie algebras or Lie superalgebras obtained from these constructions associated with $U(\varepsilon, \delta))$ and call these algebras a *canonical standard embedding.* A (ε, δ)-Freudenthal-Kantor triple system $U(\varepsilon, \delta))$ is said to be *unitary* if the linear span $\mathbf{k}$ of the set $\{K(a,b) | a, b \in U(\varepsilon, \delta)\}$ contains the identity endomorphism Id. We note that the balanced property is unitary.

Proposition 1.3 ([K.3],[K.5]) *For a unitary (ε, δ) -Freudenthal-Kantor triple system $U(\varepsilon, \delta)$ over Φ, let $T(\varepsilon, \delta)$ be the Lie or anti-Lie triple system and $L(\varepsilon, \delta)$ be the standard embedding Lie algebra or superalgebra associated with $U(\varepsilon, \delta)$. The following are equivalent:*

a) $U(\varepsilon, \delta)$ is simple,
b) $T(\varepsilon, \delta)$ is simple,
c) $L(\varepsilon, \delta)$ is simple.

For these standard embedding Lie algebras or superalgebras $L(\varepsilon, \delta)$, we have the following 5 grading subspaces:

$$L(\varepsilon, \delta) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$$

where $U(\varepsilon, \delta) = L_{-1}, T(\varepsilon, \delta) = L_{-1} \oplus L_1$, $\mathbf{k} = \{K(a,b)\}_{span} = L_{-2}$.

Example 1.3 *Let V be a $2n$-dimensional vector space with an anti-symmetric nondegenerate bilinear form $< x|y >$. Then $(V, [xyz])$ is an anti-Lie triple system with respect to the triple product defined by*

$$[xyz] := < y|z > x + < x|z > y.$$

Thus the canonical standard embedding Lie superalgebra is type $osp(1, 2n)$.

In the end of this section, we give the following.

Proposition 1.4 *Let $(V, < xyz >)$ be an anti-Jordan triple system. Then $(V, [xyz])$ is an anti-Lie triple system with respect to the new product defined by*

$$[xyz] := < xyz > + < yxz > .$$

In particular, we can obtain the simple Lie superalgebras $spl(n, m)$, $P(n)$ and $Q(n)$ from anti-Jordan triple systems by means of the canonical standard embedding Lie superalgebra associated with V. ([K.O-4])

§ 2 Examples of (-1,-1) Freudenthal-Kantor triple systems

In this section, we will consider the standard embedding Lie superalgebras of the $B(m, n)$and $D(m, n)$ types associated with an anti-Lie triple system and a (-1,-1) Freudenthal-Kantor triple system.

From now on, assume that, the fieldΦ is an algebraically closed field of characteristic 0.

We will describe more precisely the situation in the subsequent theorems.

Theorem 2.1 *Let U be a vector space of $Mat(k, n; \Phi)$. Then the space U is a unitary (-1,-1) Freudenthal-Kantor triple system with respect to the triple product*

$$< xyz >= z\ {}^t yx + y\ {}^t xz - x\ {}^t yz.$$

where ${}^t x$ denotes the transpose matrix.

For this triple system, by straightforward calculations, from the results in section 1, we have the following;

$(i) k = 2m, (m \geq 2)$

$$L(U) \cong D(m, n) \ \ type's \ Lie \ superalgebra$$

and

$$dim L(U) = 2(n + m)^2 - m + n,$$

$(ii) k = 2m + 1, (m \geq 0)$

$$L(U) \cong B(m,n) \ \ \textit{type's Lie superalgebra}$$

and

$$dimL(U) = 2(n+m)^2 + 3n + m,$$

That is, sumarizimg these, we have the following.

Theorem 2.2 *Let U be the triple system of same as described in Theorem 2.1 and L(U) be the standard embedding Lie superalgebras associated with* $U = Mat(k,n;\Phi)$. *Then L(U) are Lie superalgebras of type D(m,n) or B(m,n) if k=2m or k= 2m+1,respectively.*

§3. Lie superalgebras – $D(2,1;\alpha), G(3)$ *and* $F(4)$ –

These constructions are considered in Proc. Edinburgh Math Soc ([K-O.2](2003)) or Glasgow Math.J.([E-K-O.1](2003)) of our papers.

But breifly describing, we have the following.

(i) Let V be a quartenion algebra over the complex number. Then V be a balanced (-1,-1) Freudenthal-Kantor triple system with respect to certain triple product. And the standard embedding Lie superalgebra $L(U)$ *is* $D(2,1;\alpha)$ *type's with* $dimL(V) = 17$.

(ii) Let V be a octonion algebra over the complex number. Then V be a balanced (-1,-1) Freudenthal-Kantor triple system with respect to certain triple product. And the standard embedding Lie superalgebra $L(U)$ *is* $F(4)$ *type's with* $dimL(V) = 40$.

(iii) Let V be a Im O (= the imagenary part of octonion algebra) . Then V be a balanced (-1,-1) Freudenthal-Kantor triple system with respect to certain triple product. And the standard embedding Lie superalgebra $L(U)$ *is* $G(3)$ *type's with* $dimL(V) = 31$.

§4. Extended Dynkin diagrams and triple systems

We will consider the Dynkin diagrams of simple Lie superalgebras as well as that Lie algebras. In this section, we will only describe about distinguished Extended Dynkin diagram of their canonical Lie superalgebras associated with (-1,-1) Freudenthal-Kantor triple systems B(m,n) and F(4) types, because for the other cases we may deal with the explain by means of same methods.

(i) For B(m,n) type's distinguished Extended Dynkin diagram and usualy Dynkin diagram, we have the following([F-S-S]);

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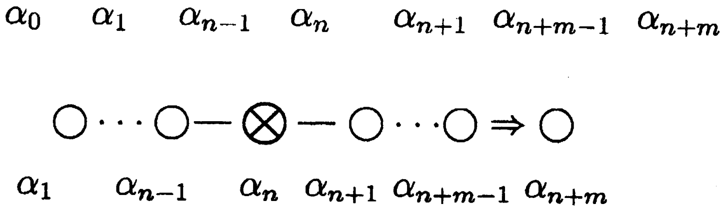


We recall the following product (cf. Section 2),

$$< xyz >= z^t yx + y^t xz - x^t yz$$

where $x, y, z \in U = Mat(2m+1, n; \Phi)$.

Let U: $=L_{-1}$ be (-1,-1) F-K.t.s. defined by above triple product.
$L(U) :=$ the standard embedding Lie superalgebra associated with U.

Then we can easily see to have the structure as follows;

$$L(U)/(L_{-2} \oplus L_0 \oplus L_2) \cong L_{-1} \oplus L_{-1} := T \text{ (as anti} - \text{Lie triple system)},$$

$$InnDerT \cong L_{-2} \oplus L_0 \oplus L_2 = C_n \oplus B_m$$
$$= \{\begin{pmatrix} L(a,b) & -K(c,d) \\ K(e,f) & -L(b,a) \end{pmatrix}\}_{span}$$

= *distinguished Extended Dynkin diagram with omitted* $\otimes$

And $L(U) = Inn\ Der\ T \oplus T$, equipped with
$L_0 = \{L(x,y)\}_{span} = \lambda I \oplus$ *Dynkin diagram with omitted* $\otimes$
$= \lambda I \oplus A_{n-1} \oplus B_m$

In particular, for the case of $n = 1$, we get

$$< xyz >=< x|y > z+ < x|z > y- < y|z > x,$$

where $< x|y >=^t xy \in \Phi$, dim $U = 2m+1$.

Thus this implies the case of balanced, and so that $L_{-2} \oplus L_0 \oplus L_2 = A_1 \oplus B_m$, and $A_1 \cong sl(2)$, with $dim\ L_{-2} = dim\ L_2 = 1$.

On the other hand, we have another decomposition,,

$$L(U)/L_0 = (L_{-2} \oplus L_{-1} \oplus L_1 \oplus L_2) := A$$

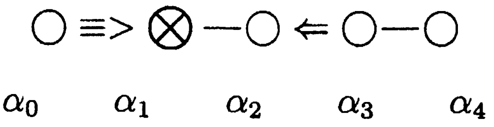
(as generalized structurable superalgebra)

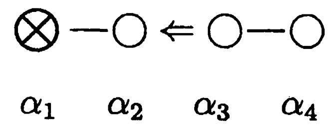
$$L_0 = \lambda I \oplus A_{n-1} \oplus B_m = \{\begin{pmatrix} L(a,b) & 0 \\ 0 & -L(b,a) \end{pmatrix}\}_{span}.$$

$DerA = L_0 \cong \lambda I \oplus$ *Dynkin diagram with omitted* $\otimes$.

From this fact, it seems that there exists a version of Lie superalgebras as well as the reductive space L/L_0 of Lie algebras.

(ii) For F(4) type's distinguished Extended Dynkin diagram and usual Dynkin diagram ([F-K-K]), we have the following;





$U = L_{-1}$ = (-1,-1)-balanced Freudenthal-Kantor triple system with dim U =8 (cf. Section 4).

$L(U)$ = the standard embedding Lie superalgebra associated with U and dim $L(U) = 40$, dimL_{-2} =dim $L_2 = 1$.

Then we can easily see to have the structure as follows;

$L(U)/(L_{-2} \oplus L_0 \oplus L_2) \cong L_{-1} \oplus L_1 := T$ (as anti-Lie triple system),

$$Inn\ DerT \cong L_{-2} \oplus L_0 \oplus L_2 = A_1 \oplus B_3$$

$$= distinguished\ Extended\ Dynkin\ diagram\ with\ omitted \otimes$$

$$= \{\begin{pmatrix} L(a,b) & -K(c,d) \\ K(e,f) & -L(b,a) \end{pmatrix}\}_{span}$$

$$L_0 = \lambda I \oplus B_3 = \{\begin{pmatrix} L(a,b) & 0 \\ 0 & -L(b,a) \end{pmatrix}\}_{span} = \{L(a,b)\}_{span},$$

of cause, $L(a,b) = S(a,b) + A(a,b)$,where $S(a,b)$ is a inner derivation of U, $K(a,b) = A(a,b) =< .|. > Id$, is an anti-derivation of U.

Furthermore, these imply

$$A_1 \cong \{\begin{pmatrix} 0 & Id \\ 0 & 0 \end{pmatrix}\}_{span} \oplus \{\begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix}\}_{span} \oplus \{\begin{pmatrix} 0 & 0 \\ Id & 0 \end{pmatrix}\}_{span}$$

$$= L_{-2} \oplus \{A(a,b)\}_{span} \oplus L_2.$$

$$InnDerU = \{S(a,b)\}_{span} \cong B_3 = Dynkin\ diagram\ with\ omitted \otimes$$

In the end of this section, we note the following (c.f. [K3],[K.5]). For a subspace $A = L_{-2} \oplus L_{-1} \oplus L_1 \oplus L_2$ of the standard embedding Lie superalgebra $L(U) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ associated with U,we set $L(U) := g_{\bar{0}} \oplus g_{\bar{1}}$ where $g_{\bar{0}} = L_{-2} \oplus L_0 \oplus L_2$ and $g_{\bar{1}} = L_{-1} \oplus L_1$. Then A is a generalized structurable superalgebra with respect to

$$X \circ Y := [x_{-2} + x_{-1} + x_1 + x_2, y_{-2} + y_{-1} + x_1 + y_2]$$

$$= [x_{-2}, y_1] + [x_{-1}, y_{-1}] + [x_{-1}, y_2] + [x_2, y_{-1}] + [x_1, y_{-2}] + [x_1, y_1]$$

$$D(X,Y) := ad([x_{-1},y_1]+[x_{-2},y_2]+[x_1,y_1]+[x_2,y_{-2}])$$

for all $X, Y \in A$

That is, these satisfy the following relations;
a) $X \circ Y = (-1)^{degX degY} Y \circ X$
b) $D(X,Y)$ is a superderivation of A.
c) $(-1)^{degX degZ} D(X \circ Y, Z) + (-1)^{degY degX} D(Y \circ Z, X) + (-1)^{degZ degY} D(Z \circ X, Y) = 0$, for all $X, Y, Z \in A$.

Remark Let $(A, (d_0, d_1, d_2))$ be a normal generalized symmetric algebra ([O.3]). Then $(A, D(x,y))$ is a generalized structurable algebra([K.5]) with respect to the new derivation

$$D(x,y) := d_0 + d_1 + d_2, \quad \textit{for all } x, y \in A.$$

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