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Gröbner bases on projective bimodules and the Hochschild cohomology *

Part II. Critical pairs

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This article is a continuation of the previous paper [5]. We develop the theory of Gröbner bases on an algebra F based on a well-ordered semigroup over a commutative ring K. We consider Gröbner bases on the algebra F as well as Gröbner bases on projective F-(bi)modules. Our framework is considered to be fairly general and unify the existing Gröbner basis theories on several types of algebras ([3, 4, 6]).

In this part we discuss critical pairs and give so-called critical pair theorems. We need to consider z-elements as well as usual critical pairs come from overlapping of rules.

5 Well-ordered reflexive semigroups and factors

Let $S = B \cup \{0\}$ be a semigroup with 0. S is *well-ordered* if B has a well-order $\succ$, which is compatible in the following sense:

(i) $a \succ b, ca \neq 0, cb \neq 0 \;\Rightarrow\; ca \succ cb$,
(ii) $a \succ b, ac \neq 0, bc \neq 0 \;\Rightarrow\; ac \succ bc$,
(iii) $a \succ b, c \succ d, ac \neq 0, bd \neq 0 \;\Rightarrow\; ac \succ bd$.

S is called *reflexive* if for any $a \in B$ there are $e, f \in B$ such that $a = eaf$.

In the rest of this section $S = B \cup 0$ is a well-ordered reflexive semigroup with 0. The following two lemmata were given in [2] (see also [1]).

Lemma 5.1. *For any $a \in B$, there is a unique pair (e, f) of idempotents such that $a = eaf$.*

In the above lemma, e (resp. f) is called the *source* (resp. *terminal*) of a and denoted by $\sigma(a)$ (resp. $\tau(a)$). Let $E(B)$ be the set of idempotents in B.

Lemma 5.2. $ef = 0$ *for any distinct* $e, f \in E(B)$.

*This is a preliminary report and the details will appear elsewhere.

The following lemma follows from the assumption that B is well-ordered.

Lemma 5.3. *Any $x \in B$ has only a finite number of left (right) factors.*

Corollary 5.4. *The set of triples (x_1, x_2, x_3) such that $x = x_1x_2x_3$ is finite for any $x \in B$.*

A factor of an idempotent in B is called an *identic* and ID(B) denotes the set of identic elements of B. An element of B that is not identic is *nonidentic* and NID(B) denotes the set of nonidentic elements; $\mathrm{NID}(B) = B \setminus \mathrm{ID}(B)$. An element $x \in B$ is *prime* if it is nonidentic and is not a product of two nonidentic elements.

Proposition 5.5. *Any element in* NE(B) *is a product of finite number of primes.*

Let U be a subset of B. If an element $x \in B$ is decomposed as $x = yuz$ with $y, z \in B$ and $u \in U$, the triple (y, u, z) is called *appearance* of U in x. For two appearances (y_1, u_1, z_1) and (y_2, u_2, z_2) of U in x, we order them as

$$(y_1, u_1, z_1) \succ (y_2, u_2, z_2) \;\Leftrightarrow\; y_1 \succ y_2 \text{ or } (y_1 = y_2 \text{ and } z_2 \succ z_1).$$

Proposition 5.6. *For any $x \in B$ and $U \subset B$, the set of all appearances of U in x forms a finite chain.*

Let

$$(y_1, u_1, z_1) \succ (y_2, u_2, z_2) \succ \cdots \succ (y_n, u_n, z_n)$$

be the chain of appearances of U in x. The first (y_1, u_1, z_1) is the *rightmost appearance*, and (y_i, u_i, z_i) appears *at the right of* $(y_{i+1}, u_{i+1}, z_{i+1})$. The *leftmost appearance* is defined dually.

Two appearances (y, u, z) and (y', u', z') of U in x is *disjoint* if $y = y'u'z''$ for some left factor z'' of z' or $y' = yuz''$ for some left factor z'' of z.

6 Gröbner bases on algebras and critical pairs

Let $F = K \cdot B$ be the algebra based on a well-ordered reflexive semigroup $S = B \cup \{0\}$ over a commutative ring K with 1. F is the K-algebra with the product induced from the semigroup operation of S.

Let R be a rewriting system on F. Consider two rules $u_1 \to v_1$ and $u_2 \to v_2$ in R. Let $x \in B$ and suppose that u_1 and u_2 in R appears in x, that is,

$$x = x_1u_1y_1 = x_2u_2y_2 \tag{1}$$

for some $x_1, x_2, y_1, y_2 \in B$. This situation is called *critical*, if the appearances are not disjoint, (x_1, u_1, y_1) is at the right of (x_2, u_2, y_2), x_1 and x_2 have no common nonidentic left factor, and y_1 and y_2 have no common nonidentic right factor. For the appearances in (1) of u_1 and u_2 in x, we have two reductions

$x \to_R x_1v_1y_1$ and $x \to_R x_2v_2y_2$. The pair $(x_1v_1y_2, x_2v_2y_2)$ is a *critical pair* if the situation is critical. The pair is *resolvable* if $x_1v_1y_1 \downarrow_R x_2v_2y_2$ holds.

A rule $u \to v$ is *normal* if $xuy = 0$ implies $xvy = 0$ for any $x, y \in B$ ([5]). A system R is normal if all the rules are normal. A set G of monic elements of F is normal if the associated system R_G is normal. A critical pair for R_G is a *critical pair for* G.

Theorem 6.1. *A normal rewriting system on F is complete if and only if all the critical pairs are resolvable. A set of monic uniform normal elements of F is a Gröbner basis if and only if all the critical pairs are resolvable.*

Let $f, \overline{f} \in F$. We say that f is *uniquely reduced to* $\overline{f}$ (with respect to R) if $\overline{f}$ is R-irreducible and any reduction sequence from f to an R-irreducible element ends in $\overline{f}$, that is, $\overline{f}$ is a unique normal form of f.

Lemma 6.2. *Suppose that $f \in F$ is uniquely reduced to $\overline{f} \in F$. If $g \to_R^* g'$ for $g, g' \in F$ and g is R-irreducible, then $f + g \to_R^* \overline{f} + g'$.*

If a rule $u \to v \in R$ or an element $u - v \in G$ is not normal, that is, $xuy = 0$ but $xvy \neq 0$, the element xvy is a *z-element*, and $Z(R)$ (or $Z(G)$) denotes the set of z-elements together with 0 ([5]). A z-element z is *resolvable* if $xvy \to_R^* 0$ (or $xvy \to_G^* 0$). It is *uniquely resolvable* if it is uniquely reduced to 0.

Lemma 6.3. *Suppose that all the elements in $Z(R)$ are uniquely resolvable. If $f \downarrow_R g$, then $xfy \downarrow_R xgy$ for any $x, y \in B$.*

Theorem 6.4. *A set G of monic uniform elements of F is a Gröbner basis if and only if all the critical pairs are resolvable and all the z-elements are uniquely resolvable.*

7 Critical pairs on left modules

In this and the next sections G is a reduced Gröbner basis on $F = K \cdot B$ of ideal I. Let $A = F/I$ be the quotient algebra of F by I. Let X be a left edged set so that the source $\sigma(\xi) \in E(B)$ is assigned for each $\xi \in X$. Let

$$F \cdot X = \bigoplus_{\xi \in X} F\sigma(\xi) \cdot \xi$$

is the projective left F-module generated by X.

Let T be a rewriting system on $F \cdot X$. Let $w\xi \to t$ and $w'\xi \to t'$ be two rules in T with $\xi \in X$, $w, w' \in B\sigma(\xi)$ and $t, t' \in F \cdot X$, and let $x \in B$. Suppose that $x = yw = y'w'$ for some $x, x' \in B$. Then, we have two reductions $yw\xi \to_R yt$ and $y'w'\xi \to y't'$. If this is a critical situation, that is, the appearance $(y, w, 1)$ is at the right of the appearance $(y', w', 1)$ among this type of appearances (appearances as right factors) and y and y' have no nonidentic common left factor in B,

$$(yt, y't')$$

is called a *critical pair of the first kind* for T. Let $u - v \in G$, and suppose that $x = yw = zuz'$ for some $y, z, z' \in B$. Then, we have two reductions $yw\xi \to yt$ and $zuz'\xi \to zvz'\xi$. If the situation is critical, that is, (z, u, z') is the rightmost appearance of u in x, $(y, w, 1)$ is the rightmost appearance of w in x as a right factor, they are disjoint, and x and y have no nonidentic common left factor, then

$$(yt, zvz'\xi)$$

is a *critical pair of the second kind* for T and G. The critical pair (f, g) is *resolvable* if $f \downarrow_{T,G} g$.

T is *normal* if each rule $s \to t$ in T is normal, that is, $xs = 0$ implies $xt = 0$ for any $x \in B$.

Theorem 7.1. *A normal system T on $F \cdot X$ is complete modulo G if and only if all the critical pairs (of the first and the second kinds) are resolvable. A set of monic uniform normal elements of $F \cdot X$ is a Gröbner basis if and only if all the critical pairs are resolvable.*

If $xs = 0$ but $xt \neq 0$ for $s \to t \in T$ and $x \in B$, xt is a *z-element*. It is *resolvable* if it is reduced to 0 with respect to $\to_{T,G}$ It is uniquely resolvable if 0 is the unique normal form of it with respect to $\to_{T,G}$.

Similar results to Lemmata 6.2 and 6.3 hold for rewriting systems on $F \cdot X$, and we have

Theorem 7.2. *A set of monic uniform elements is a Gröbner basis if and only if all the critical pairs are resolvable and all z-elements are uniquely resolvable.*

8 Critical pairs on bimodules

Let $S = B \cup \{0\}$ be a well-ordered reflexive semigroup. Define an operation on the set $S^e = (B \times B) \cup \{0\}$ by

$$(x, y) \cdot (x', y') = \begin{cases} (xy, y'x') & \text{if } xy \neq 0 \text{ and } y'x' \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

for $x, y, x', y' \in B$. Moreover, we define an order $\succ$ on $B \times B$ by

$$(x, y) \succ (x', y') \Leftrightarrow x \succ x' \text{ or } (x = x' \text{ and } y \succ y').$$

Proposition 8.1. *With the definition above, S^e is a well-ordered reflexive semigroup and the enveloping algebra $F^e = F \otimes_K F^o$ is an algebra based on S^e.*

For a subset G of F, define

$$G^e = \{g \otimes 1,\ 1 \otimes g \mid g \in G\}.$$

Proposition 8.2. *If G is a Gröbner basis on F of an ideal I of F, then G^e is a Gröbner basis of $I^e = I \otimes F + F \otimes I$ on F^e. Moreover, the quotient F^e/I^e is isomorphic to the enveloping algebra $A^e = A \otimes_K A^e$ of $A = F/I$.*

A F-bimodule (resp. A-bimodule) is naturally a left F^e-module (resp. A^e-module). Let X be an edged set and

$$F \cdot X \cdot F = \bigoplus_{\xi \in X} F\sigma(\xi) \times \tau(\xi) F$$

be the projective F-bimodule generated by X.

An element $x \otimes y \in B \times B$ acts upon $x'\xi y' \in B \cdot X \cdot B$ as

$$(x \otimes y) \cdot x'\xi y' = xx'\xi y'y.$$

A rewriting system T on the bimodule $F \cdot X \cdot F$ is considered to be a rewriting system on it as a left F^e-module. A rule $w\xi w' \to t$ in T, where $w, w' \in B$, $\xi \in X$ and $t \in F \cdot X \cdot F$, is applied to $f \in F \cdot X \cdot F$, if f has a term $k \cdot xw\xi w'x'$ with $k \in K, x, x' \in B$. In this case,

$$f \to_T f - k \cdot x(w\xi w' - t)x'.$$

For $g = u - v \in G$, the rule $u \otimes 1 \to v \otimes 1$ of G^e is applied to f, if f has a term $k \cdot xux'\xi x''$ with $k \in K$, $x, x', x'' \in B$ and $\xi \in X$, as

$$f \to_G f - k \cdot x(u - v)x'\xi x''.$$

Similarly, the rule $1 \otimes u \to 1 \otimes v \in G^e$ is applied to f with a term $k \cdot x\xi x'ux''$, as

$$f \to_G f - k \cdot x\xi x'(u - v)x''.$$

A *critical pair* for T modulo G is a critical pair for T modulo G^e in the sense of Section 7. So, we have three kinds of critical pairs. Let $w\xi w' \to t, z\xi z' \to t' \in T$ and suppose $xw = yz \neq 0$ and $w'x' = z'y' \neq 0$ for some $x, y, x', y' \in B$, then we have two reductions $xw\xi w'x' \to_T xtx'$ and $yz\xi z'y' \to_T yt'y'$. If the situation is critical of the first kind of in the sense of Section 7, we have a critical pair

$$(xtx', yt'y').$$

Let $u - v \in G$ and suppose that $xw = yuy' \neq 0$ for some $x, y, y' \in B$. Then, we have two reductions $xw\xi w' \to_T xt$ and $yuy'\xi w' \to_G yvy'\xi w'$. If the situation is critical of the second kind, we have a critical pair

$$(xt, yvy'\xi w').$$

Similarly, if $w'x = y'uy$ for some $x, y, y' \in B$, we have two reductions $w\xi w'x \to_T tx$ and $w\xi y'uy \to_G w\xi y'vy$ and a critical pair

$$(tx, w\xi y'vy)$$

in a critical situation.

A critical pair (f, g) is resolvable if $f \downarrow_{T,G} g$. T is *normal*, if $xsy = 0$ implies $xty = 0$ for any $s \to t \in T$ and $x, y \in B$.

Theorem 8.3. *A normal rewriting system T on $F \cdot X \cdot F$ is complete modulo G if and only if all the critical pairs are resolvable.*

If $xsy = 0$ but $xty \neq 0$ for $s \to t \in T$ and $x, y \in B$, xty is a z-element with respect to T. It is *(uniquely) resolvable* if it is (uniquely) reduced to 0 modulo $\to_{T,G}$.

Theorem 8.4. *A set H of monic uniform elements of $F \cdot X \cdot F$ is a Gröbner basis modulo G, if and only if all the critical pairs are resolvable and all the z-elements are uniquely resolvable.*

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