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Logico-Algebraic Structures for Information Integration in the Brain

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Abstract In the study of the brain mechanisms responsible for consciousness, the most mysterious, and probably the most important for mathematical modeling is the phenomenal unity of conscious awareness. The author has proposed in his earlier work an approach to explain this unity in terms of integration of information, where information is understood as identification of a variety, and its integration as transformation of the selective manifestation of information into structural one. Also in the earlier study, the general mathematical model of integration has been exemplified using the process of color discrimination, and a hypothetical interpretation of the unity of consciousness has been presented in terms of the irreducibility of the algebraic structures involved in modeling of integration.

The present paper is devoted to the issue of the relationship between partially ordered sets and transitive closure operators which seem the core concepts of the model. The main question is about the way how the closure operation can be selected from the class of closure operators compatible with the partial order. It has been shown that the structure of orthocomplementation (or more generally of strong orthogonality relation) build over the partial order gives a unique selection of the closure operator. Introducing a generalized form of logic into the process of integration turns out to be equivalent to the selection of the unique closure operator compatible with the logical structure.

1. Introduction

The present paper is a continuation of the earlier work on a mathematical model for information integration. [1] As before, the ultimate goal is to provide a mathematical model of the brain mechanisms responsible for consciousness. Since the most outstanding, if not defining characteristic of consciousness is its phenomenal unity, such mechanisms must involve processes of information integration, and this is the reason why they are in the center of our interest.

Before we can discuss information integration, it is necessary to recollect how in this and our earlier studies information is understood, since information has diverse and frequently fallacious conceptualizations. Information can be defined in the framework of the philosophically fertile theme of the "one-many" relationship as the identification of a variety.[2] The identification can be understood as any unifying aspect of the variety, such as selection of the one out of many, or as unification of the many into one. The distinction of the two mentioned modes of identification gives forth two fundamental manifestations of information, the selective and the structural. However, they are only different manifestations of the uniform phenomenon being in an ever present dual relationship. The selection of the one out of many requires some structural characteristics distinguishing selected element, on the other hand the structure unifying the many into a whole is a selection of the one out of many ways of unification.

Integration of information is understood as a process of information transformation in which its selective manifestation is replaced by structural. What should be emphasized here is the fact that integration of information cannot be reduced to its accumulation or its quantitative increase, but must involve some form of qualitative change. It is natural to expect that the outcome of such a process will be characterized in terms of the unity or wholeness of some variety. This is why we can expect that it is information integration which is responsible for the unity of conscious experience into which a large variety of multi-modal perceptions is integrated.

The history of the analysis of this unity in modern psychology, as well as the account of the attempts to explain it, has been presented elsewhere.[1] For the purpose of making the present paper self-contained, it will be sufficient to recall that the only approach free from the "homunculus fallacy" arising in all attempts to model consciousness without taking into account an essential transformation of information in the cognitive processes, was based on the interpretation of the unity of consciousness as a result of the quantum entanglement (coherence) of the processing units in the brain. However, "the possibility that the totality of microtubules [...] in our brains may well take part in global quantum coherence – or at least that there is sufficient quantum entanglement between the states of different microtubules across the brain..." [3] considered by Roger Penrose as an opportunity for involving the quantum mechanical description in the study of brain mechanisms, seems as unrealistic now, as it has been fifteen years ago.

This is why in the earlier paper the idea of searching for quantum-type coherence has been initiated as a promising direction of inquiry, but without incorporating all formalism of quantum mechanics, which seems to be not suitable for description of the brain as a physical system. For someone familiar only with the standard Hilbert space formalism of quantum mechanics in which quantum coherence is simply superposition of wave functions, this idea may seem as incomprehensible as an attempt to contemplate a smile in the absence of the face. However, in a more abstract formalism of quantum mechanics, the so called quantum logic, quantum coherence has a very simple, yet fundamental algebraic interpretation in terms of the irreducibility of the lattice of closed subsets of the Hilbert space, or more generally, of the lattice of quantum logical propositions.[1, 4]

Thus, we can explore the possibility that the unity of consciousness is a result of irreducibility of the lattice, or if there is need for increased generality, of the partially ordered set, of some basic elements which have equally fundamental function in the description of brain mechanisms as the basic yes-no experiments in quantum mechanics. Our original idea was based on the heuristic speculative argument that the most likely structure of this type could be a complete lattice of all closed subsets of the set of neurons, or other functional units in the brain, with respect to some unidentified yet transitive closure operation. It has to be stressed that the reference to closure operators and complete lattices of the closed subsets is not necessary for such a model utilizing the concept of algebraic irreducibility, as the irreducibility of partially ordered sets can be considered instead of the irreducibility of lattices. The heuristic argument for searching among closure operators for the formalism of information integration was their omnipresence in mathematics (logic, topology, geometry, probability, etc.) and the intuitive association of the process of uniting the variety of all subsets into the restricted Moore family of closed subsets with the process of integration of the variety of perceptions into the uniform objects of conscious awareness.

In the case of quantum mechanics, the lattice of closed subsets of the Hilbert space (here too we have an instance of a closure operation,) has a function of the empirical logic for physical characteristics of the system. However, it would be an error to conclude that if a concept of logic is involved, we should follow the track of the calculus of logical operations and look for a fundamental partial order structure in the computational models of the brain. The computational metaphor for the brain is the main source of the homunculi fallacies "populating" the domain of Artificial Intelligence. Also, the logic of computation is based on the Boolean algebra, which is an extreme case of completely de-coherent structure, or more precisely of a structure which can be completely reduced into the direct product of simple two element substructures. Thus, if we want to take advantage of the uniting characteristic of the quantum coherence, we should look for the formalism somewhere else.

In our earlier paper another partially ordered set has been considered. Its simple instance is identified in the simplified model of integration of information in the process of color vision which can be described as follows.

The model has the form of a Venn diagram for three sets (for that reason it was called a Venn gate) with the two sets of arrows. The eight arrows on the left side terminating in each of the eight regions of the Venn diagram represent the variety of eight basic colors of the rainbow (including white and black). Each of them can activate appropriate receptors represented by the circles of the Venn diagram, and these activations form the output marked by the three arrows on the right side. The selection of one of the variety of eight colors is transformed into structural configuration of activations in receptors. Each selection produces unique pattern of activations, if the input light is homogeneous.

For instance, the yellow light is uniquely represented as the activation of two receptors, with each of them representing another color (red and green respectively) when activated separately. However, when several different input lights are coming at the same time, the pattern may be the same for a combination of inputs as for a single input. For instance, the output pattern for the yellow color can appear when the input consists of the first and third arrows corresponding to the red and green light. The possibility of representing the yellow color as a combination of the representations for green and red makes it a greater element than the other two in the partial ordering induced on the input set.

What is critical for the understanding of the model (which otherwise would be completely trivial account of the tri-color vision) is the fact, that what makes the receptor a processing unit for the green color in perception is not its sensitivity to the light of particular length, but the way how it is "wired" into the brain mechanisms. If the same receptor is re-wired in place of the receptor processing red light, we would perceive grass as red. There is no reason to believe that the brain simply "knows" to what light its receptors are sensitive. Thus, color discrimination is not so much the matter of the chemistry of light sensitive substances, but of the internal organization of the brain and its peripherals such as the retina. Moreover, it suggests that the actual functional units in the color discrimination are not receptors, but some processing units (gates,) each involving three receptors (possibly more) organized as in the model above.

In our particular elementary model of integration of information in the simplified process of color discrimination, the induced partial order is a Boolean algebra, which is not of a great interest for us in our search for the source of the unity of conscious awareness. However, nothing prevents us from building models of similar "integrating gates" which induce quantum-like irreducible partial orders.

The present paper reports further exploration of this idea with the focus on the relationship between partial orders and transitive closure operators.

2. Transitive closure operators compatible with partial order

The simple model of integration of information involved in color discrimination can be easily generalized when we observe that the "processing gate" is essentially a function from the (unstructured) set of inputs to the structure built on the outputs, in our particular example a Boolean algebra $\mathbf{2}^3$. Each output itself is a subset of the set of atoms of the lattice on which this algebra is built, where an atom in a partially ordered set with the least element 0 is an element greater than 0, but not greater than any other element. The partial order induced on the inputs is generated by the inclusion of representing them subsets of atoms. We have here something which could be interpreted as a "logarithmic" set operation, as opposed to constructing "power sets". Thus, the question is whether the structure which we want to extract from the process and utilize for modeling information integration is a partial order, partial order with orthocomplementation (Boolean algebra is its special case,) or closure operator (in case of a Boolean algebra it is trivial one in which every subset is closed.)

To answer this question, we will study the relationship between these structures. But, we will have to start from establishing some notational conventions and from developing the conceptual framework of necessary definitions.

In addition to the notation commonly used in the literature of partially ordered sets (or posets) and lattices, [5] the following conventions and simple facts will be used hereafter.

If R is a binary relation on the set X, R* is its converse, and

$R^a(A) = \{x \in X: \forall y \in A, yRx\}$, $R^e(A) = \{x \in X: \exists y \in A, yRx\}$. We can simplify our notation for single element subsets: $R(x) = R^a(\{x\}) = R^e(\{x\})$. In the case of the partial order relation:

$\leq^a(A) = \{x \in X: \forall y \in A, y \leq x\}$,

$\leq^e(A) = \{x \in X: \exists y \in A, y \leq x\}$,

$\leq *^a(A) = \{x \in X: \forall y \in A, y \geq x\}$,

$\leq *^e(A) = \{x \in X: \exists y \in A, y \geq x\}$.

Obviously, $R^a(A) = \cap\{R(x): x \in A\}$ and $R^e(A) = \cup\{R(x): x \in A\}$.

If R is a binary relation on set X, then the pair of functions from the power set of X to itself $A \to R^a(A)$ and $A \to R^{*a}(A)$ forms a Galois connection, and therefore both operations on subsets of X: $A \to R^a R^{*a}(A)$ and $A \to R^{*a} R^a(A)$ are transitive closure operations on X understood as functions f from the power set of X to itself such that for all A, B $\subseteq$ X:

1. $A \subseteq f(A)$,
2. $A \subseteq B \Rightarrow f(A) \subseteq f(B)$, and
3. $f(A) = f(f(A))$.

The third condition can be replaced by $A \subseteq f(B) \Rightarrow f(A) \subseteq f(B)$,

Every closure operator is uniquely defined by the Moore family of its closed sets f-Cl = $\{ A \subseteq X: A = f(A)\}$, and every Moore family $\Im$ of subsets of X, i.e. family of sets which includes X and is closed with respect to arbitrary intersections, is the family of closed sets for the closure operator defined by f(A) $= \{B \in \Im: A \subseteq B\}$. It is easy to see that for every closure operator its family of closed sets forms a complete lattice with respect to the set inclusion. Finally, there is a natural partial ordering on closure operators defined by:

$f \leq g$ if $\forall A \subseteq X: f(A) \subseteq g(A)$,

which is equivalent to the condition for the families of closed subsets:

g-Cl $\subseteq$ f-Cl.

The history of the study of the relationship between partially ordered sets and closure operators started from the work of Oystein Ore in which he observed that there is a bijective correspondence between finite partially ordered sets and finite T_0 topological spaces, i.e. finite spaces with closure operators f satisfying two additional conditions:

1. For all A, B $\subseteq$ X, $f(A) \cup f(B) = f(A \cup B)$ (the finite additivity of the closure operator distinguishing topological spaces).

2. For all x, y $\in$ g, $x \in g(\{y\}) \Rightarrow y \in f(\{x\})$ (T_0 topology). [6]

The correspondence is based on the relationship between the partial order and the topological closure on singleton sets: $x \leq y$ iff $x \in f(\{y\})$. This suffices to define partial ordering when the closure operation is given. Here, the role of T_0 condition becomes clear, as otherwise the relation would be only reflexive and transitive, i.e. a quasi-order (not necessarily anti-symmetric.)

Going the other direction, the extension of the closure operation from singletons to larger subsets can be achieved by:

$f(A) = \leq^{*e}(A) = \cup\{\leq^*(x): x \in A\} = \cup\{f(\{x\}): x \in A\}$.

The assumption that the posets and therefore topological spaces under consideration are finite comes from the fact that topological spaces are defined by the finite additivity condition (first condition above) which do not allow for the extension of the closure operation on singletons to infinite subsets. It is obvious, when we recall that in T_1 topological spaces each singleton set is closed, yet the closure is non trivial for infinite subsets.

It is quite obvious that the topological closure operation above is only one of many possible closure operations compatible with given partial order, i.e. satisfying the condition: $x \leq y$ iff $x \in f(\{y\})$. For our purpose the question what are these compatible closure operations, and what conditions for the ordering have to be added to identify the unique closure operation is of special interest.

Historically, the development of the inquiry was driven by different question. The relationship between partially ordered sets and closure spaces provided a way to embed the partially ordered set in a complete lattice of closed sets in such a way that all existing finite or infinite infima and suprema of the poset are preserved. These embeddings called completions by cuts are generalizations of Dedekind's construction of real numbers. Although, we are more interested in the relationship between partial orders and closure operators rather, than in the issue of embedding posets in complete lattices, we will study some more general forms of completion, which will give as a point of departure for our own inquiry. We have to recollect definitions of some of the concepts related to this subject. [5]

Definition 1

1. A nonvoid subset J of a poset $[P, \leq]$ is a *semi-ideal* if $\forall a \in J\ \forall x \in P,\ x \leq a \Rightarrow x \in J$.
2. A semi-ideal J is *principal* if there exists $a \in P$ such that $J = \leq^*(a)$.
3. A semi-ideal J is an *ideal* if $\forall a, b \in J$, $a \vee b$ exists in $P \Rightarrow a \vee b \in J$.
4. A semi-ideal J is a *complete-ideal* if $\forall A \subseteq J$, $\vee\{x\colon x \in A\}$exists in $P \Rightarrow \vee\{x\colon x \in A\} \in J$.
5. A subset J of a a poset $[P, \leq]$ is a *closed-ideal* if it contains all lower bounds to the set of its upper bounds, i.e. $\leq^{*a} \leq^a(J) \subseteq J$ (and therefore $\leq^{*a} \leq^a(J) = J$.)

It is obvious that every complete ideal is an ideal, every ideal is semi-ideal, and that P is a complete ideal. For finite posets there is no difference between complete-ideals and ideals. Similarly, it is obvious that the families of complete-ideals, ideals, and semi-ideals are closed with respect to arbitrary intersections. Thus they form Moore families, and they define transitive closure operators. Semi-ideals are closed subsets for the closure operator:

$f_p(A) = \leq^{*e}(A) = \{x \in X\colon \exists y \in A,\ y \geq x\}$,

ideals are closed subsets for the closure operator f_i and $f_p \leq f_i$, complete-ideals for operator f_{ci} and $f_i \leq f_{ci}$, and finally closed-ideals are closed sets for the closure operator: $f_c(A) = \leq^{*a} \leq^a(A)$.

Proposition 1

Let $[P, \leq]$ *be a poset and f be a closure operator compatible with its partial ordering, i.e.* $\forall x, y \in P,\ x \leq y$ *iff* $x \in f(\{y\})$. *Then* $\forall A \subseteq P,\ A = f(A) \Rightarrow A$ *is a semi-ideal.*

Proof: $\forall x \in A$, $\leq^*(x) = f(\{x\}) \subseteq f(A) = A$, so , $\leq^*(x) \subseteq A$, and therefore $\leq^{*e}(A) \subseteq A$, i.e. A is a semi-ideal.

Corollary

For every closure operator f compatible with the partial order, i.e. such that $\forall x,y \in P$, $x \leq y$ *iff* $x \in f(\{y\})$, *we have* $f_p \leq f$.

Thus, since $f_c(\{x\}) = \leq^{*a} \leq^{a}(\{x\}) = \leq^{*}(x)$, *every closed-ideal is a semi-ideal, and therefore we have* $f_p \leq f_c$.

Proposition 2

Let $[P, \leq]$ *be a poset and f be a closure operator compatible with its partial ordering, i.e.* $\forall x, y \in P$, $x \leq y$ *iff* $x \in f(\{y\})$. *Then* $f \leq f_c$, *and therefore we have* $f_p \leq f \leq f_c$.

Proof: First observe that for every binary relation R on set X, and for every $A \subseteq X$, $A = R^{*a}R^{a}(A)$ iff $\exists B \subseteq X$, $A = R^{*a}(B)$. (B is simply $R^{a}(A)$ for "$\Rightarrow$")

Now, $f_c(A) = A$ iff $\exists B \subseteq P$, $A = \leq^{*a}(B) = \{x \in P: \forall b \in B, x \in f(\{b\})\} = \bigcap\{f(\{b\}): b \in B\}$, and therefore as an intersection of f-closed sets $f_c(A)$ must be f-closed. Thus, f_c–Cl $\subseteq$ f-Cl, which is equivalent to $f \leq f_c$.

It can be easily shown that even when the poset $[P, \leq]$ is a complete lattice, in general not all inequalities in the sequence $f_p \leq f_i \leq f_{ci} \leq f_c$ can be replaced by equalities, although for obvious reason the middle inequality becomes an equality in finite posets. Simple example of the four element Boolean algebra ("the diamond") shows that to the set consisting of the two atoms (the middle elements) f_c closure operator assigns as its closure all poset, while f_p assigns the set of the three lower elements, so the first and the fourth closure operators can be different. The example of the complete lattice of all natural numbers ordered by divisibility, with the greatest element 0, the closure of the set of all nonzero numbers is all poset for f_c, but the set of all nonzero numbers is a semi-ideal and ideal, and therefore is closed for the first and second closure operators. Only third inequality can be replaced by the equality in complete lattices as the following proposition shows.

Proposition 3

If a poset $[P, \leq]$ *is a complete lattice, then* $f_{ci} = f_c$.

Proof: Let $J = f_{ci}(A)$ and $\vee\{x: x \in A\} = a$, and $a \in J$. But, by the definition of the supremum of A, $\leq^{a}(A) = \leq(c)$, and therefore $\leq^{*a} \leq^{a}(A) = \leq^{*}(c) \subseteq J$. The reverse inclusion is always true, which gives us f_{ci}-Cl = f_c-Cl.

We will return to the closure operators associated with posets which are complete lattices, but first we will focus on the poset completions, starting from the classical MacNeille theorem on the "completion by cuts."[5]

Proposition 4 (MacNeille)

Let $[P, \leq]$ *be a poset and* φ *a function from P to the complete lattice* L_c *of the* f_c *-closed subsets of P defined by* $\varphi(x) = \leq^{*a} \leq^{a}(\{x\})$. *Then* φ *is an injective, isotone and inverse-isotone function preserving all suprema and infima that happen to exist in the poset* $[P, \leq]$.

Definition 2

Let $[P, \leq]$ be a poset and φ be an injective, isotone and inverse-isotone function from P to a complete lattice L satisfying the condition of minimality, i.e. whose all complete-sublattices (i.e. substructures not only with respect to finite, but also infinite infima and suprema) K satisfy the condition:

$\varphi(P) \subseteq K \subseteq L \Rightarrow K = L$. Such a complete lattice will be called a *completion* of the poset P. [7]

Lemma 5

Let $[P, \leq]$ *be a poset and* f *be a transitive closure operator on subsets of* P. *Then the function* φ *from the poset* P *to the complete lattice* L_f *of the f-closed subsets of P given by* $\varphi(x)=f(\{x\})$ *is injective, isotone and inverse-isotone iff* $\forall x \in P$, $f(\{x\}) = f_p(\{x\})$.

Proof: We want to show that $[\forall x, y \in P, x \leq y \text{ iff } \varphi(x) \subseteq \varphi(y)]$ iff $[\forall x \in P, f(\{x\}) = f_p(\{x\})]$. One direction of the implication is obvious. The other can be shown when we recall that the three conditions for the transitive closure operator are equivalent to:

$\forall A, B \subseteq P, A \subseteq f(B) \text{ iff } f(A) \subseteq f(B)$.

Thus, $\forall x,y \in P$, $x \in f_p(\{y\})$ iff $x \leq y$ iff $f(\{x\}) \subseteq f(\{y\})$ iff $x \in f(\{y\})$.

Proposition 6

Let $[P, \leq]$ *be a poset and* f *be a transitive closure operator on subsets of* P *compatible with the partial order, i.e.* $\forall x, y \in P$, $x \leq y$ *iff* $x \in f(\{y\})$, *which is equivalent to the condition* $\forall x \in P$, $f(\{x\}) = f_p(\{x\})$. *Then the function* φ *from* P *to the complete lattice* L_f *of the f-closed subsets of* P *given by* $\varphi(x) = f(\{x\})$ *defines a completion of the poset* $[P, \leq]$.

Proof: By Lemma 5, we have to show only that L_f is minimal. Suppose that there exists a complete-sublattice K of L_f, such that $\varphi(P) \subseteq K \subseteq L_f$, but different from L_f. Then $\exists A \subseteq P$, $f(A) = A$, but $A \notin K$. However, $\forall a \in A$, $f(\{a\}) \in K$ and $f(\{a\}) \subseteq A$, and since A is f-closed, we have $\vee\{ f(\{a\}): a \in A\} = A$. But in K and L_f all suprema are identical, so $A \in K$, contradiction.

Thus, we can see that all lattices of closed subsets with respect to closure operators compatible with the partial order are completions of poset $[P, \leq]$. However, MacNeille's completion has the advantage that it preserves all existing infima and suprema of the poset. Also, it has been known for long time that MacNeille's completion of Boolean algebra is a Boolean algebra, while it does not have to be true for ideal completion defined by f_i. On the other hand, the ideal completions preserve distributivity and modularity of lattices, while MacNeille's completion not necessarily. [5]

The strongest argument for the superiority of MacNeille's completion comes from the fact that it is isomorphic to the original poset whenever it is already a complete lattice, while for example the semi-ideal completion defined by f_p is never isomorphic, unless the poset is a chain. The first fact follows easily from the following proposition. [5]

Proposition 7

A poset $[P, \leq]$ *is a complete lattice iff all its closed-ideals, i.e. subsets with respect to the closure operator* f_c *are principal ideals.*

The second fact, that no complete lattice is isomorphic to the lattice of its semi-ideals (subsets closed with respect to f_p), unless it is a chain follows from the simple fact that the union of any two principal semi-ideals $\leq^*(x)$ and $\leq^*(y)$ is a semi-ideal, and for the lattice to be isomorphic with its semi-ideal completion this semi-deal has to be principal semi-ideal $\leq^*(z)$. But this means that z has to belong to one or the other of $\leq^*(x)$ and $\leq^*(y)$. Therefore, z has to be either x or y, so either $x \leq y$ or $y \leq x$.

Thus, the only complete lattices isomorphic to their semi-ideal completions are complete chains, and for complete chains $f_p = f_c$.

Thus, we have reasons to believe that the closure operator f_c is a superior candidate among the closure operators compatible with the given partial order considered above for our purpose of modeling integration of information. However, there are many other possible closure operators compatible with the partial order which we did not consider yet. There is no convincing argument for the choice of closed-ideal closure operator.

3. Transitive closure operators compatible with structures built over the partial order

First, let's recall that the closure operator f_c has been constructed with the help of the Galois connection defined by the functions on the power set of the set P:

$A \to R^a(A)$ and $A \to R^{*a}(A)$. In this particular case the relation R was the partial order of the poset. We can ask how to identify those closure operators which are defined by some binary relation R. An answer considering symmetric relations was given by Oystein Ore in his early study of Galois connections. [8]

Definition 3

A function γ from a poset $[P, \leq]$ to poset $[Q, \leq]$ is called a *dual isomorphism*, if it is bijective, and satisfies the following two conditions:

i) $\forall x,y \in P,\ x \leq y \Rightarrow \gamma(y) \leq \gamma(x)$,

ii) $\forall x,y \in P,\ \gamma(x) \leq \gamma(y) \Rightarrow y \leq x$.

A dual isomorphism from a poset $[P, \leq]$ to itself is called an *involution*, if it is of order two, i.e. if $\forall x \in P,\ \gamma\gamma(x) = x$.

An orthocomplementation on a poset $[P, \leq]$ with the least element 0 and the greatest element 1 is an involutive dual isomorphism, i.e. a bijective mapping γ from a poset

$[P, \leq]$ to itself $x \to \gamma(x)$ such that $\forall x,y \in P$:

i) $\gamma\gamma(x) = x$,

ii) if $x \leq y$, then $\gamma(y) \leq \gamma(x)$,

iii) $x \wedge \gamma(x) = 0$ and $x \vee \gamma\ (a) = 1$, whenever the meet and join exist.

In usual notation $\gamma\ (x)$ is indicated by x'.

Frequently, the fact that $a \leq b'$ is written $a \perp b$ and is read "a is orthogonal to b." A poset with orthocomplementation is called an orthoposet, a lattice with an orthocomplementation is called an ortholattice.

The following theorem belonging to the earliest studies of Galois connections provides the condition for the closure operation to be a Galois closure operation, i.e. closure defined by a Galois connection. [8]

Proposition 8

Let f *be a transitive closure operator on set* P. *Then there is a binary, symmetric relation* R *on* P, *such that* f *is a Galois closure operator defined by* $A \to R^a R^a(A)$ *iff there exists an involution* γ *on the complete lattice of all f-closed subsets. For the given closure operator* f *and involution* γ, $\forall x \in P$, $R(x) = \gamma(f(\{x\}))$.

If the relation R *is anti-reflexive* (xRx *for no* x *in* P) *or satisfies the weaker condition:* $\forall x \in P$, $xRx \Rightarrow xRy$ *for all* y *in* P, *then the involution defines an orthocomplementation on the lattice of f-closed subsets.*

It turns out that the Galois closures for the partial order and orthogonality relation of the orthoposet coincide.

Proposition 9

Let $[P, \leq, x \to \gamma(x) = x']$ *be an orthoposet. Then* $\forall A \subseteq P$, $\leq^{*a} \leq^{a}(A) = \perp^{a}\perp^{a}(A)$.

Proof: $x \in \perp^{a}\perp^{a}(A)$ iff $\{\forall y \in P, [\forall a \in A, y \perp a] \Rightarrow x \perp y\}$ iff

$\{\forall y \in P, [\forall a \in A, y \leq a'] \Rightarrow x \leq y'\}$ iff

$\{\forall y \in P, [\forall a \in A, a \leq y'] \Rightarrow x \leq y'\}$ iff $x \in \leq^{*a} \leq^{a}(A)$.

Thus, when we have an orthoposet, instead of just a poset, the closure operator defined by closed-ideals associated with this structure becomes uniquely determined. What is interesting, that this way we are also closer to the quantum logic formalism of quantum mechanics based on the concept of an ortholattice satisfying additional conditions of being orthomodular [if $a \leq b$, then $b = a \vee (b \wedge a')$], complete, atomic and atomistic lattice with the atomic covering property and exchange property (orthomodular, complete, atomic, AC lattice if the redundant conditions have been eliminated). [1]

We can consider more general structure of a poset (not orthoposet) with so called strong orthogonality relation $\perp$ defined on P by the conditions:

1. The relation $\perp$ is symmetric,
2. $\forall x \in P$, $x \perp x \Rightarrow x \perp y$ for all y in P,
3. $\forall x, y \in P$, $x \leq y$ iff $\perp(y) \subseteq \perp(x)$.

The same symbol $\perp$ is used here, as every orthogonality relation defined by orthocomplementation ($a \perp b$ if $a \leq b'$) is a special instance of strong orthogonality.

Although the Galois closure defined by the strong orthogonality relation does not coincide in general with the closed-ideal closure operator, it is compatible with the partial order.

Proposition 10

Let $[P, \leq, \perp]$ *be a poset with strong orthogonality relation. Then the closure operator* f *defined on subsets* A *of* P *by* $f(A) = \perp^{a}\perp^{a}(A)$ *is compatible with the partial order, i.e.* $\forall x \in P$, $f(\{x\}) = f_p(\{x\})$.

Proof: $\forall x, y \in P$, $[y \in f_p(\{x\})$ iff $y \leq x$ iff

$\perp(x) \subseteq \perp(y) \Rightarrow \perp^{a}\perp^{a}(x) \subseteq \perp^{a}\perp^{a}(y)$ iff $y \in f(\{x\})]$.

Now we need to show only $\forall x \in P$, $f(\{x\}) \subseteq f_p(\{x\})$.

For every relation R and every subset A, we have $R^{a} R^{*a} R^{a}(A) = R^{a}(A)$, but the orthogonality is symmetric, so

$\perp^{a}\perp^{a}\perp^{a}(A) = \perp^{a}(A)$.

Thus, $\forall x, y \in P$, $[\, y \in f(\{x\})$ iff $y \in \perp^{a}\perp(x) \Rightarrow \perp(x) \subseteq \perp(y) \Rightarrow$

$y \leq x$ iff $y \in f_p(\{x\})$.

In either case we can use the orthogonality relation to determine selection of the closure operation out of those compatible with the partial order.

4. Conclusion

For given partially ordered sets, there are many different closure operators compatible with the order. There are some reasons why some of these closures may be more useful, but there is no objective criterion for the selection, even when we think in terms of the modeling information integration. As it has been shown, the structure of an orthoposet, or more generally of a poset with strong orthogonality relation on a set P, which can be interpreted as a generalized "logic" for information integration, can be used as a means to select a unique closure operation on P. As a result, we get a bijective correspondence between posets with strong orthogonality relation and closure operators.

This concludes the first stage of our inquiry. The abstract orthogonality relations defined on P by the conditions:

1. The relation $\perp$ is symmetric,
2. $\forall x \in P$, $x \perp x \Rightarrow x \perp y$ for all y in P,

correspond to weak tolerance relations, which are simply complement relations to orthogonality relations (xTy iff not $x \perp y$). [9] Since tolerance relations are mathematical formalizations of similarity, or the relation which is frequently invoked as Wittgenstein's "family resemblance," there is an interesting question about the function of this relation in the model of information integration. From the fact that equivalence relations are just transitive tolerances, we can expect some relevance of these relations for the process of abstraction of information.

The next step is to consider closure operations related to partially ordered sets, but defined not on the entire set on which order is defined, but on their subsets, such as sets of atoms, join-irreducible elements, etc.

The consecutive step is to search for the conditions for irreducibility of the structures describing information integration which is our ultimate goal. This can be done either in terms of partial order, or of closure operations.

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