数理解析研究所講究録
第1562巻 2007年 87-91

The Gauss-Bonnet theorem for PL manifolds
~Banchoff's theorem and Homma's theorem~

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There is the Gauss-Bonnet theorem not only for smooth surfaces but also for polyhedra. It can be generalized to the theorem for higher dimensional PL mfds, in two methods of Banchoff's and Homma's. The purpose of this article is to consider the relation of them. First of all, the case of dimension 2 has to be described.

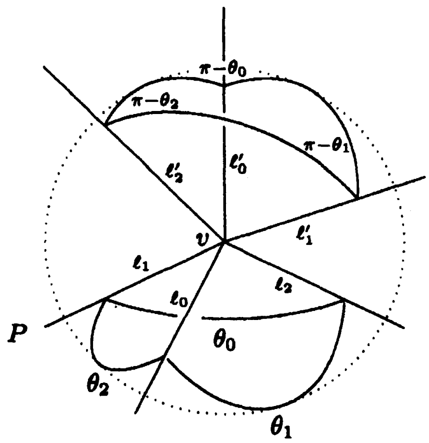
2-dim. Gauss' Theorema egregium.
For the boundary ∂P of a convex polyhedra $P = P^3$ of $\mathbb{R}^3$, let

$$\kappa(\{v\}) \underset{\text{def}}{=} 2 \cdot \frac{\sum_{i=0}^{m-1}(\pi - \theta_i) - (m-2)\pi}{4\pi} \tag{G}$$
$$\left(= \lim \frac{2}{4\pi} \int_{\text{small n.b.d. at } v \text{ of } \partial P} K dV \right).$$

Then

$$\kappa(\{v\}) = 1 - \frac{1}{2} \cdot m + \sum_{i=0}^{m-1} \frac{\pi - \theta_i}{2\pi} \tag{B}$$
$$= 1 - \sum_{i=0}^{m-1} \frac{\theta_i}{2\pi}. \tag{H}$$

See the figure below.

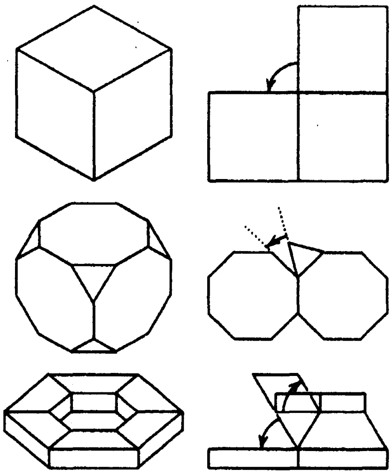


$\ell_i \perp \ell'_j$ if $i \neq j$

The value $\kappa(\{v\})$ depends only ∂P, so, for a general PL-mfd $M = M^2$, we can define the curvature κ by (B) = (H).

This is an abstract and the details will be published elsewhere. The title in Japanese is "PL 多様体の Gauss-Bonnet の定理 ~Banchoff の定理と本間の定理~".

2-dim. Gauss-Bonnet theorem.
We have $\sum_{v \in M} \kappa(\{v\}) = \chi(M)$.

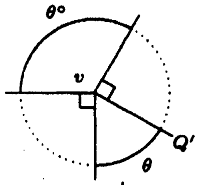


$\frac{1}{4} \cdot 8 = 2 = \chi(\mathbf{S}^2)$

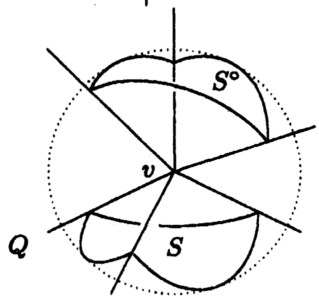
$\frac{1}{12} \cdot 24 = 2 = \chi(\mathbf{S}^2)$

$(-\frac{1}{6}) \cdot 12 + \frac{1}{6} \cdot 12 = 0 = \chi(\mathbf{T}^2)$

To generalize them for the higher dimension, the inner and outer angles for vertices are confirmed by the following figure.



$$\alpha(\{v\}, Q') = \frac{\theta}{2\pi}$$
$$\alpha^\circ(\{v\}, Q') = \frac{\theta^\circ}{2\pi}$$



$$\alpha(\{v\}, Q) = \frac{S}{4\pi}$$
$$\alpha^\circ(\{v\}, Q) = \frac{S^\circ}{4\pi}$$

As follows, Theorema egregium and the Gauss-Bonnet thorem are extended.

Gauss' Theorema egregium.
For the boundary ∂P of a convex polyhedra $P = P^n$ of $\mathbb{R}^n$, let

$$\kappa_B(\{v\}) \underset{\text{def}}{=} (1 + (-1)^{n-1})\alpha^\circ(\{v\}, P) \tag{G}$$
$$\left(= \lim \frac{1 + (-1)^{n-1}}{\omega_{n-1}} \int_{\text{small n.b.d. at } v \text{ of } \partial P} K dV \right).$$

Then

$$\kappa_B(\{v\}) = \sum_{Q \subsetneqq \partial P}^{v \in Q} (-1)^{|Q|} \alpha^\circ(\{v\}, Q).$$

So, for a general PL-mfd $M = M^{n-1}$, let

$$\kappa_B(\{v\}) \underset{\text{def}}{=} \sum_{Q \subseteqq M}^{v \in Q} (-1)^{|Q|} \alpha^\circ(\{v\}, Q). \tag{B}$$

Gauss-Bonnet theorem (Banchoff).

$$\sum_{v\in M}\kappa_B(\{v\})=\chi(M).$$

We have another generalization.

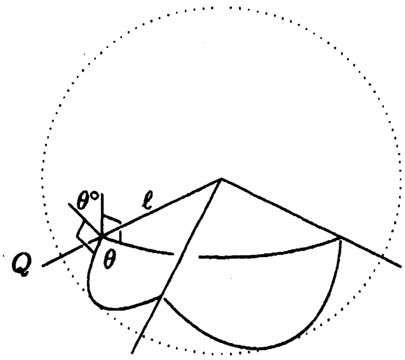
Gauss-Bonnet theorem (Homma).
For each face Q' of M, let

$$\kappa_H(Q')\underset{\text{def}}{=}1-\sum_{\substack{|Q|=n-1\\ Q'\subsetneqq Q\\ Q\subseteqq M}}\alpha(Q',Q). \tag{H}$$

Then

$$\sum_{Q'\subseteqq M}(-1)^{|Q'|}\kappa_H(Q')=\chi(M).$$

We also have to confirm the definition of angles for faces. See the figure below.



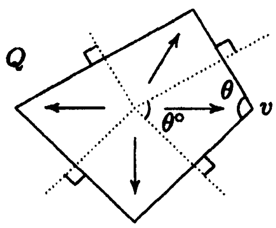
$$\alpha(\ell,Q)=\frac{\theta}{2\pi}$$
$$\alpha^\circ(\ell,Q)=\frac{\theta^\circ}{2\pi}$$

Notice that $\alpha(Q',Q)=\alpha^\circ(Q',Q)=\frac{1}{2}$ if $|Q|-|Q'|=1$, and $\alpha(Q,Q)=\alpha^\circ(Q,Q)=1$. The following is the main result of this article.

Relation of Banchoff and Homma (S.).
For each $v\in M$, we have

$$\kappa_B(\{v\})=\sum_{\substack{v\in Q\\ Q\subseteqq M}}(-1)^{|Q|}\kappa_H(Q)\cdot\alpha^\circ(\{v\},Q).$$

Remark that $\sum_{v\in Q}\alpha^\circ(\{v\},Q)=1$ for ${}^\forall Q\subseteqq M$.



$$\alpha(\{v\},Q)=\frac{\theta}{2\pi}$$
$$\alpha^\circ(\{v\},Q)=\frac{\theta^\circ}{2\pi}$$

Extend κ_B for all faces (and extend κ_B and κ_H for $P = P^n$) by

$$\kappa_B(Q'') = \begin{cases} \underset{\text{def}}{=} \sum_{Q' \subseteqq M} \overline{\alpha^\circ}(Q'', Q') - \delta_{|Q'|, n-1} & \text{if } Q'' \subseteqq M \\ \left(\underset{\text{def}}{=} 0 \right. & \left. \text{if } Q'' = P \right) \end{cases}$$
$$\left(= \alpha^\circ(Q'', P) - \overline{\alpha^\circ}(Q'', P) - \sum_{Q^{n-1} \subseteqq M} \delta(Q'', Q) \right),$$

$$\kappa_H(Q') = \begin{cases} = 1 - \sum_{Q^{n-1} \subseteqq M} \alpha(Q', Q) & \text{if } Q' \subseteqq M \\ \left(\underset{\text{def}}{=} 0 \right. & \left. \text{if } Q' = P \right) \end{cases}$$
$$\left(= \zeta(Q', P) - \delta(Q', P) - \sum_{Q^{n-1} \subseteqq M} \alpha(Q', Q) \right),$$

where $\delta(Q', Q') = 1$; $\delta(Q'', Q') = 0$ if $Q'' \subsetneqq Q'$; $\zeta(Q'', Q') = 1$ if $Q'' \subseteqq Q'$; $\overline{\beta}(Q'', Q') = (-1)^{|Q'|-|Q''|}\beta(Q'', Q')$ for $\beta = \alpha$, α°, ζ, and δ; and $\beta(Q'', Q') = 0$ if $Q'' \nsubseteqq Q'$ for each β.

Corollary.
From the generalization of the main result

$$\kappa_B = \overline{\alpha^\circ} \circ \kappa_H \qquad \left(\text{i.e., } \kappa_B(Q'') = \sum_{Q'} \overline{\alpha^\circ}(Q'', Q')\kappa_H(Q') \right)$$

and

$$\alpha \circ \overline{\alpha^\circ} = \delta \qquad \left(\text{i.e., } \sum_{Q'} \alpha(Q'', Q')\overline{\alpha^\circ}(Q', Q) = \delta(Q'', Q) \right),$$

we have

$$\alpha \circ \kappa_B = \kappa_H \qquad \left(\text{i.e., } \sum_{Q} \alpha(Q', Q)\kappa_B(Q) = \kappa_H(Q') \right).$$

The following content was not described in the presentation.

The theorem of Homma's curvature.
Let

$$\begin{aligned} \tau(Q', Q) &\underset{\text{def}}{=} \sum_{Q'' \subseteqq Q'} (-1)^{|Q'|-|Q''|}\alpha(Q'', Q) \\ &= \sum_{Q''} \overline{\zeta}(Q'', Q')\alpha(Q'', Q) \\ &= \sum_{Q''} \overline{\zeta}^T(Q', Q'')\alpha(Q'', Q). \end{aligned}$$

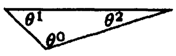
Then, for a simplex Q^m and a face $Q' \subsetneqq Q$,

$$\tau(Q', Q) = \frac{\mu_{m-1}(S(\overrightarrow{Q'Q''}))}{\omega_{m-1}},$$

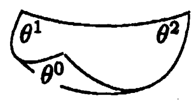
where $S(\overrightarrow{Q'Q''}) = \{\frac{\vec{q}''-\vec{q}'}{|\vec{q}''-\vec{q}'|} \in \mathbf{S}^{m-1} : \vec{q}' \in Q', \vec{q}'' \in Q''\}$, $\omega_{m-1} = \mu_{m-1}(\mathbf{S}^{m-1})$, and the face $Q'' \subsetneqq Q$ is such that the join of Q' and Q'' is Q, and

$$\tau(Q, Q) = \begin{cases} 1 & (\text{if } m = 0), \\ 0 & (\text{if } m \geqq 1). \end{cases}$$

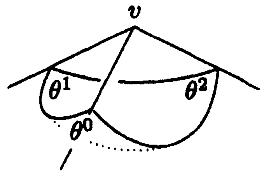
Remark.
The function $\overline{\zeta}^T$ does not satisfy "$\overline{\zeta}^T(Q'',Q')=0$ if $Q''\not\subseteq Q'$", so we cannot use the theorem above for spherical simplices and convex cones (we can only use it for Euclidean simplices).



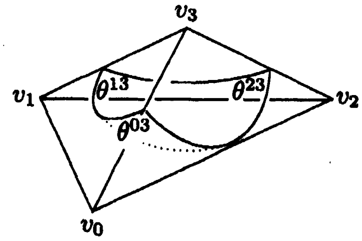
$$\sum_{i=0}^{2}\frac{\theta^i}{2\pi}-\sum_i\frac{1}{2}+1=0$$



$$\sum_{i=0}^{2}\frac{\theta^i}{2\pi}-\sum_i\frac{1}{2}+1\neq 0$$



$$-\frac{\sum_{i=0}^{2}\theta^i-\pi}{4\pi}+\sum_i\frac{\theta^i}{2\pi}-\sum_i\frac{1}{2}+1\neq 0$$



$$-\sum_{j=0}^{3}\frac{\sum_{i=0}^{3\ (i\neq j)}\theta^{ij}-\pi}{4\pi}+\sum_{i,j}^{i<j}\frac{\theta^{ij}}{2\pi}-\sum_j\frac{1}{2}+1=0$$

Conjecture for Homma's curvature.
For a polyhedra Q^m and a face $Q'\subsetneqq Q$, can we have

$$\tau(Q',Q)\underset{?}{=}\frac{\displaystyle\sum_{Q''\subsetneqq Q}^{Q'\cap Q''=\emptyset}(-1)^{|Q'|+|Q''|+1-m}\mu_{m-1}(S(\overrightarrow{Q'Q''}))}{\omega_{m-1}},$$

and

$$\tau(Q,Q)\underset{?}{=}\begin{cases}1 & (\textit{if } m=0),\\ 0 & (\textit{if } m\geqq 1).\end{cases}$$

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