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On primitive numerical semigroups of genus 10 [1]

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Abstract

We investigate whether a primitive numerical semigroup of genus 10 is Weierstrass. There are 89 primitive numerical semigroups of genus 10. We prove that 75 semigroups among them are Weierstrass.

§1. A characterization of a primitive numerical semigroup

Let $\mathbb{N}_0$ be the additive semigroup of non-negative integers. A subsemigroup H of $\mathbb{N}_0$ is called a *numerical semigroup* if whose complement $\mathbb{N}_0\backslash H$ in $\mathbb{N}_0$ is a finite set. The cardinality of the set $\mathbb{N}_0\backslash H$ is called the *genus* of H, which is denoted by $g(H)$. Let H be a numerical semigroup. We set $\gamma(H)=\max\{\gamma\in\mathbb{N}_0 \mid \gamma\notin H\}$ and $a(H)=\min\{h>0\mid h\in H\}$. A numerical semigroup H is said to be *primitive* if $\gamma(H)<2a(H)$. A numerical semigroup is called an *n-semigroup* if $a(H)=n$. For $a_1,\ldots,a_m\in\mathbb{N}_0$ $\langle a_1,\ldots,a_m\rangle$ denotes the semigroup generatd by $a_1,\ldots,a_m$.

Example. Let $H=\langle 7,8,11,17,20\rangle$. Since $\mathbb{N}_0\backslash H=\{1\longrightarrow 6,9,10,12,13\}$, H is a primitive 7-semigroup H of genus 10.

We state some properties of primitive numerical semigroups without proof.

Remark. If H is a primitive n-semigroup of genus g, then we have $\frac{g}{2}+1\leqq n\leqq g+1$.

Remark. For a subset S of $\mathbb{N}_0$ the following are equivalent:
i) S is a primitive n-semigroup.
ii) $\{1,\ldots,n-1\}\subseteqq\mathbb{N}_0\backslash S$ and $\mathbb{N}_0\backslash S\subseteqq\{1,\ldots,n-1\}\cup\{n+1,\ldots,2n-1\}$.

We want to investigate the number of the primitive numerical semigroups of genus g.

[1]This is an abstract and the details will be published elsewhere.

Proposition. *Let* $\frac{g}{2}+1 \leqq n \leqq g+1$. *Then the number* $\mathcal{P}_n(g)$ *of the primitive* n*-semigroups of genus* g *is* ${}_{n-1}C_{g-(n-1)}$

Proposition. *Let* $\mathcal{P}(g)$ *be the number of the primitive numerical semigroups of genus* g. *Then we have* $\sum_{m=[\frac{g+1}{2}]}^{g} {}_mC_{g-m}$.

Example. The number of the primitive numerical semigroups of genus 10 is 89, because $\mathcal{P}(10) = \sum_{n=6}^{11} \mathcal{P}_n(10) = \sum_{m=5}^{10} {}_mC_{10-m} = 1+15+35+28+9+1 = 89$.

§2. On the weight of a primitive numerical semigroup

We are interested in primitive numerical semigroups of genus 10. We want to explain its reason. In this paper a *curve* C means a complete non-singular irreducible curve over an algebraically closed field k of characteristic 0. For a point P of C we set

$$H(P) = \{n \in \mathbb{N}_0 \mid \exists f \in k(C) \textit{ such that } (f)_\infty = nP\}$$

where $k(C)$ denotes the field of rational functions on C and $(f)_\infty$ is the polar divisor of the function f.

Definition. A numerical semigroup H is said to be *Weierstrass* if there is a pointed curve (C, P) such that $H = H(P)$.

Fact. Every primitive numerical semigroup of genus $g \leqq 9$ is Weierstrass ($g = 4$: [9], $5 \leqq g \leqq 8$: [5], $g = 9$: [6]).

We know that a *generic* primitive numerical semigroup is Weierstrass. We want to explain the meaning of *generic*. Let C be a curve of genus g. For every point P of C except the Weierstrass points we have $H(P) = \langle g+1, \ldots, 2g+1\rangle$, i.e., $\mathbb{N}_0 \backslash H(P) = \{1, \ldots, g\}$. We note that the number of the Weierstrass points is less than or equal to $(g-1)g(g+1)$. We introduce the notion of the *weight* $w(H)$ of a numerical semigroup H of genus g to indicate the difference between the semigroup $\langle g+1, \ldots, 2g+1\rangle$ and H as follows: Let $\mathbb{N}_0 \backslash H = \{\gamma_1 < \cdots < \gamma_g\}$. Then we set $w(H) = \sum_{i=1}^{g}(\gamma_i - i)$. So, the smaller is the weight $w(H)$, the more generic is H.

Fact. If H is a primitive numerical semigroup with $w(H) \leqq g(H) - 1$, then it is Weierstass ($w(H) \leqq g(H) - 2$: [2], $w(H) = g(H) - 1$: [4]).

Thus, it is sufficient to study primitive numerical semigroups H with $w(H) \geqq g(H)$ when we investigate whether a primitive numerical semigroup is Weierstrass or not. Here we give the table of the number of primitive numerical semigroups H of genus 10 classified by the minimum positive integer n in H and the weight of H.

n	$\mathcal{P}_n(10)$	$w(H) \leqq 9$	$w(H) \geqq 10$
6	1	1	0
7	15	11	4
8	35	20	15
9	28	19	9
10	9	9	0
11	1	1	0
total	89	61	28

Moreover, we give all primitive numerical semigroups of genus 10 with $w(H) \geqq 10$.

H	$\mathbb{N}_0 \backslash H$	$w(H)$
$\langle 7,8,9,19,20 \rangle$	$\{1 \longrightarrow 6, 10, 11, 12, 13\}$	12
$\langle 7,8,10,19 \rangle$	$\{1 \longrightarrow 6, 9, 11, 12, 13\}$	11
$\langle 7,8,11,17,20 \rangle$	$\{1 \longrightarrow 6, 9, 10, 12, 13\}$	10
$\langle 7,9,10,15 \rangle$	$\{1 \longrightarrow 6, 8, 11, 12, 13\}$	10
$\langle 8,9,10,11,12 \rangle$	$\{1 \longrightarrow 7, 13, 14, 15\}$	15
$\langle 8,9,10,11,13 \rangle$	$\{1 \longrightarrow 7, 12, 14, 15\}$	14
$\langle 8,9,10,11,14 \rangle$	$\{1 \longrightarrow 7, 12, 13, 15\}$	13
$\langle 8,9,10,11,15 \rangle$	$\{1 \longrightarrow 7, 12, 13, 14\}$	12
$\langle 8,9,10,12,13 \rangle$	$\{1 \longrightarrow 7, 11, 14, 15\}$	13
$\langle 8,9,10,12,14 \rangle$	$\{1 \longrightarrow 7, 11, 13, 15\}$	12
$\langle 8,9,10,12,15 \rangle$	$\{1 \longrightarrow 7, 11, 13, 14\}$	11
$\langle 8,9,10,13,14 \rangle$	$\{1 \longrightarrow 7, 11, 12, 15\}$	11
$\langle 8,9,10,13,15 \rangle$	$\{1 \longrightarrow 7, 11, 12, 14\}$	10
$\langle 8,9,11,12,13 \rangle$	$\{1 \longrightarrow 7, 10, 14, 15\}$	12
$\langle 8,9,11,12,14 \rangle$	$\{1 \longrightarrow 7, 10, 13, 15\}$	11
$\langle 8,9,11,12,15 \rangle$	$\{1 \longrightarrow 7, 10, 13, 14\}$	10
$\langle 8,9,11,13,14 \rangle$	$\{1 \longrightarrow 7, 10, 12, 15\}$	10

H	$\mathbb{N}_0\backslash H$	$w(H)$
$\langle 8,10,11,12,13,17\rangle$	$\{1 \longrightarrow 7,9,14,15\}$	11
$\langle 8,10,11,12,14,17\rangle$	$\{1 \longrightarrow 7,9,13,15\}$	10
$\langle 9,10,11,12,13,14,15\rangle$	$\{1 \longrightarrow 8,16,17\}$	14
$\langle 9,10,11,12,13,14,16\rangle$	$\{1 \longrightarrow 8,15,17\}$	13
$\langle 9,10,11,12,13,14,17\rangle$	$\{1 \longrightarrow 8,15,16\}$	12
$\langle 9,10,11,12,13,15,16\rangle$	$\{1 \longrightarrow 8,14,17\}$	12
$\langle 9,10,11,12,13,15,17\rangle$	$\{1 \longrightarrow 8,14,16\}$	11
$\langle 9,10,11,12,13,16,17\rangle$	$\{1 \longrightarrow 8,14,15\}$	10
$\langle 9,10,11,12,14,15,16\rangle$	$\{1 \longrightarrow 8,13,17\}$	11
$\langle 9,10,11,12,14,15,17\rangle$	$\{1 \longrightarrow 8,13,16\}$	10
$\langle 9,10,11,13,14,15,16\rangle$	$\{1 \longrightarrow 8,12,17\}$	10

§3. On the Schubert index of a primitive numerical semigroup

We introduce a new notion for describing primitive numerical semigroups. Let $0 \leqq \alpha_1 \leqq \cdots \leqq \alpha_g \leqq g-1$ be integers. $\alpha = (\alpha_1, \ldots, \alpha_g)$ is called a *Schubert index of genus g*. The number $\sum_{i=1}^{g} \alpha_i$ is called the *weight* of α, which is denoted by $w(\alpha)$. We set $H(\alpha) = \mathbb{N}_0\backslash G(\alpha)$ where $G(\alpha) = \{\alpha_1+1, \alpha_2+2, \ldots, \alpha_g+g\}$. The Schubert index α is *primitive* if $H(\alpha)$ is a primitive numerical semigroup. Let H be a numerical semigroup of genus g. Set $\mathbb{N}_0\backslash H = \{\gamma_1, \ldots, \gamma_g\}$ and $\alpha(H) = (\gamma_1 - 1, \ldots, \gamma_g - g)$. Then $\alpha(H)$ is a Schubert ndex of genus g. Moreover, $w(H) = w(\alpha(H))$.

Remark. If H is a primitive numerical semigroup, then $\alpha(H)$ is a primitive Schubert index.

Example. For an ordinary point P, i.e., a non-Weierstrass point, of a curve C of genus g we have $\alpha(H(P)) = (0, \ldots, 0) = (0^g)$, because $H(P) = \langle g+1, g+2, \ldots, 2g+1\rangle$ implies that $\mathbb{N}_0\backslash H(P) = \{1, 2, \ldots, g\}$.

We want to define a partial order on the set of primitive Schubert indices.

Definition. Let $\alpha = (\alpha_1, \ldots, \alpha_{g-1})$ and $\beta = (\beta_1, \ldots \beta_g)$ be primitive Schubert indices of genus $g-1$ and g respectively. We define $\alpha \Longrightarrow \beta$ if one of the following holds.
i) $\alpha_i = \beta_{i+1}$ for any $1 \leqq i \leqq g-1$,
ii) $1 \leqq \exists j \leqq g-1$ such that $\alpha_j = \beta_{j+1} - 1$ and $\alpha_i = \beta_{i+1}$ for $\forall i \neq j$.

Example. i) $(0^7, 6, 6) \Longrightarrow (0^8, 6, 6)$. ii) $(0^6, 2, 4, 4) \Longrightarrow (0^7, 2, 4, 5)$.
iii) $(0^6, 3, 3, 3) \Longrightarrow (0^6, 1, 3, 3, 3)$.

Definition. Let α and β be primitive Schubert indices. We define $\alpha \leqq \beta$ if $\alpha = \beta$ or $\exists$ a sequence $\alpha \Longrightarrow \alpha^{(n)} \Longrightarrow \cdots \Longrightarrow \alpha^{(1)} \Longrightarrow \beta$.

Example. i) $(0^5, 2, 3, 3) \leqq (0^6, 1, 3, 3, 3)$, because

$$(0^5, 2, 3, 3) \Longrightarrow (0^6, 3, 3, 3) \Longrightarrow (0^6, 1, 3, 3, 3).$$

ii) $(0^4, 3, 3) \leqq (0^8, 4, 6)$, because

$$(0^4, 3, 3) \Longrightarrow (0^5, 3, 4) \Longrightarrow (0^6, 3, 5) \Longrightarrow (0^7, 4, 5) \Longrightarrow (0^8, 4, 6).$$

Definition. A primitive Schubert index β is *minimal* if $\nexists \alpha$ such that $\alpha \leqq \beta$ and $\beta \neq \alpha$.

Example. $(0^4, 3, 3)$, $(0^5, 2, 2, 2, 2)$ and $(0^6, 2, 2, 3, 3)$ are minimal.

Remark. Let α be a primitive Shubert index of genus g.
i) If $w(\alpha) \leqq g - 2$, then $(0) \leqq \alpha$.
ii) If $w(\alpha) = g - 1$, then $\exists$ an odd h such that $(0^{\frac{h+1}{2}}, 2^{\frac{h-1}{2}}) \leqq \alpha$.

We give the Schubert indices and the properties of all primitive numerical semigroups H of genus 10 with $w(H) \geqq 10$.

H	$\alpha(H)$	*Property*	$w(\alpha(H))$
$\langle 7, 8, 9, 19, 20 \rangle$	$(0^6, 3^4)$	*minimal*	12
$\langle 7, 8, 10, 19 \rangle$	$(0^6, 2, 3^3)$	*minimal*	11
$\langle 7, 8, 11, 17, 20 \rangle$	$(0^6, 2^2, 3^2)$	*minimal*	10
$\langle 7, 9, 10, 15 \rangle$	$(0^6, 1, 3^3)$	$\geqq (0^5, 2, 3^2)$	10
$\langle 8, 9, 10, 11, 12 \rangle$	$(0^7, 5^3)$	*minimal*	15
$\langle 8, 9, 10, 11, 13 \rangle$	$(0^7, 4, 5^2)$	*minimal*	14
$\langle 8, 9, 10, 11, 14 \rangle$	$(0^7, 4^2, 5)$	$\geqq (0^6, 4^3)$	13
$\langle 8, 9, 10, 11, 15 \rangle$	$(0^7, 4^3)$	$\geqq (0^6, 3, 4^2)$	12
$\langle 8, 9, 10, 12, 13 \rangle$	$(0^7, 3, 5^2)$	*minimal*	13
$\langle 8, 9, 10, 12, 14 \rangle$	$(0^7, 3, 4, 5)$	$\geqq (0^6, 3, 4^2)$	12
$\langle 8, 9, 10, 12, 15 \rangle$	$(0^7, 3, 4^2)$	$\geqq (0^5, 3^3)$	11
$\langle 8, 9, 10, 13, 14 \rangle$	$(0^7, 3^2, 5)$	$\geqq (0^5, 3^3)$	11
$\langle 8, 9, 10, 13, 15 \rangle$	$(0^7, 3^2, 4)$	$\geqq (0^5, 2, 3^2)$	10

H	$\alpha(H)$	*Property*	$w(\alpha(H))$
$\langle 8,9,11,12,13\rangle$	$(0^7,2,5^2)$	*minimal*	12
$\langle 8,9,11,12,14\rangle$	$(0^7,2,4,5)$	$\geqq (0^6,2,4^2)$	11
$\langle 8,9,11,12,15\rangle$	$(0^7,2,4,4)$	$\geqq (0^4,3^2)$	10
$\langle 8,9,11,13,14\rangle$	$(0^7,2,3,5)$	$\geqq (0^5,2,3^2)$	10
$\langle 8,10,11,12,13,17\rangle$	$(0^7,1,5^2)$	$\geqq (0^5,4^2)$	11
$\langle 8,10,11,12,14,17\rangle$	$(0^7,1,4,5)$	$\geqq (0^4,3^2)$	10
$\langle 9,10,11,12,13,14,15\rangle$	$(0^8,7^2)$	*minimal*	14
$\langle 9,10,11,12,13,14,16\rangle$	$(0^8,6,7)$	$\geqq (0^7,6^2)$	13
$\langle 9,10,11,12,13,14,17\rangle$	$(0^8,6^2)$	$\geqq (0^6,5^2)$	12
$\langle 9,10,11,12,13,15,16\rangle$	$(0^8,5,7)$	$\geqq (0^6,5^2)$	12
$\langle 9,10,11,12,13,15,17\rangle$	$(0^8,5,6)$	$\geqq (0^5,4^2)$	11
$\langle 9,10,11,12,13,16,17\rangle$	$(0^8,5^2)$	$\geqq (0^4,3^2)$	10
$\langle 9,10,11,12,14,15,16\rangle$	$(0^8,4,7)$	$\geqq (0^5,4^2)$	11
$\langle 9,10,11,12,14,15,17\rangle$	$(0^8,4,6)$	$\geqq (0^4,3^2)$	10
$\langle 9,10,11,13,14,15,16\rangle$	$(0^8,3,7)$	$\geqq (0^4,3^2)$	10

§4. On the moduli space $\mathcal{M}_H$

Definition. Let $\mathcal{M}_{g,1}$ be the moduli space of pointed curves of genus g. Let α be a primitive Schubert index of genus g. We set

$$\mathcal{C}_\alpha = \{(C,P) \in \mathcal{M}_{g,1} \mid H(P) = H(\alpha)\}.$$

We say that α is *dimensionally proper* if there is an open subset U of $\mathcal{M}_{g,1}$ such that $\mathcal{C}_\alpha \cap U$ is a non-empty set with codimension $w(\alpha)$ in U.

Fact. Let $\alpha \leqq \beta$ be primitive Schubert indices. If α is dimensionally proper, so is β ([2]).

Remark. The primitive Schubert index (0^g) of genus g is dimensionally proper. Hence, if α is a primitive Schubert index of genus g with $w(\alpha) \leqq g-2$, then it is dimensionally proper, which implies that every primitive numerical semigroup of genus g with $w(H) \leqq g-2$ is Weierstrass.

Fact. i) For any odd h the index $(0^{\frac{h+1}{2}}, 2^{\frac{h-1}{2}})$ is dimensionally proper. Hence, every primitive numerical semigroup H of genus g with $w(H) = g-1$ is Weierstrass ([4]).
ii) $(0^4,3^2)$ is dimensionally proper ([10],[5]).

iii) $(0^5, 4^2)$ is dimensionally proper ([10]).
iv) $(0^6, 5^2)$ is dimensionally proper ([6]).

Fact. If H is a numerical semigroup of genus g with $\alpha(H) = (0^n, l^{g-n})$, then it is Weierstrass ([3]).

Using the above facts we get the following table which shows that certain semigroups are Weierstrass. In the table below △ means that the semigroup will be proved to be Weierstrass in the forthcoming paper.

H	$\alpha(H)$	*Property*	$w(\alpha(H))$	*Weierstrass*
$\langle 7,8,9,19,20\rangle$	$(0^6,3^4)$	*minimal*	12	○
$\langle 7,8,10,19\rangle$	$(0^6,2,3^3)$	*minimal*	11	?
$\langle 7,8,11,17,20\rangle$	$(0^6,2^2,3^2)$	*minimal*	10	?
$\langle 7,9,10,15\rangle$	$(0^6,1,3^3)$	$\geqq (0^5,2,3^2)$	10	?
$\langle 8,9,10,11,12\rangle$	$(0^7,5^3)$	*minimal*	15	○
$\langle 8,9,10,11,13\rangle$	$(0^7,4,5^2)$	*minimal*	14	△
$\langle 8,9,10,11,14\rangle$	$(0^7,4^2,5)$	$\geqq (0^6,4^3)$	13	△
$\langle 8,9,10,11,15\rangle$	$(0^7,4^3)$	$\geqq (0^6,3,4^2)$	12	○
$\langle 8,9,10,12,13\rangle$	$(0^7,3,5^2)$	*minimal*	13	?
$\langle 8,9,10,12,14\rangle$	$(0^7,3,4,5)$	$\geqq (0^6,3,4^2)$	12	?
$\langle 8,9,10,12,15\rangle$	$(0^7,3,4^2)$	$\geqq (0^5,3^3)$	11	△
$\langle 8,9,10,13,14\rangle$	$(0^7,3^2,5)$	$\geqq (0^5,3^3)$	11	?
$\langle 8,9,10,13,15\rangle$	$(0^7,3^2,4)$	$\geqq (0^5,2,3^2)$	10	?
$\langle 8,9,11,12,13\rangle$	$(0^7,2,5^2)$	*minimal*	12	?
$\langle 8,9,11,12,14\rangle$	$(0^7,2,4,5)$	$\geqq (0^6,2,4^2)$	11	?
$\langle 8,9,11,12,15\rangle$	$(0^7,2,4,4)$	$\geqq (0^4,3^2)$	10	○
$\langle 8,9,11,13,14\rangle$	$(0^7,2,3,5)$	$\geqq (0^5,2,3^2)$	10	△
$\langle 8,10,11,12,13,17\rangle$	$(0^7,1,5^2)$	$\geqq (0^5,4^2)$	11	○
$\langle 8,10,11,12,14,17\rangle$	$(0^7,1,4,5)$	$\geqq (0^4,3^2)$	10	○
$\langle 9,10,11,12,13,14,15\rangle$	$(0^8,7^2)$	*minimal*	14	○
$\langle 9,10,11,12,13,14,16\rangle$	$(0^8,6,7)$	$\geqq (0^7,6^2)$	13	△
$\langle 9,10,11,12,13,14,17\rangle$	$(0^8,6^2)$	$\geqq (0^6,5^2)$	12	○
$\langle 9,10,11,12,13,15,16\rangle$	$(0^8,5,7)$	$\geqq (0^6,5^2)$	12	○
$\langle 9,10,11,12,13,15,17\rangle$	$(0^8,5,6)$	$\geqq (0^5,4^2)$	11	○
$\langle 9,10,11,12,13,16,17\rangle$	$(0^8,5^2)$	$\geqq (0^4,3^2)$	10	○
$\langle 9,10,11,12,14,15,16\rangle$	$(0^8,4,7)$	$\geqq (0^5,4^2)$	11	○

H	$\alpha(H)$	*Property*	$w(\alpha(H))$	*Weierstrass*
$\langle 9,10,11,12,14,15,17\rangle$	$(0^8,4,6)$	$\geqq (0^4,3^2)$	10	○
$\langle 9,10,11,13,14,15,16\rangle$	$(0^8,3,7)$	$\geqq (0^4,3^2)$	10	○

Let H_1 and H_2 be primitive numerical semigroups. We define $H_1 \leqq H_2$ by $\alpha(H_1) \leqq \alpha(H_2)$.

Problem 1. Let H_0 be a Weierstrass primitive semigroup. If H is a primitive numerical semigroup with $H_0 \leqq H$, then is it Weierstrass ? i.e., $\forall H_0 =$ "Wei" $\leqq H =$ "Wei" ?

Definition. Let H be a numerical semigroup of genus g. For any $m \geqq 2$ we set

$$L_m(H) = \{\gamma_1 + \cdots + \gamma_m \mid \gamma_i \in \mathbb{N}_0 \backslash H \ , \forall i\}.$$

We say that H is *quasi-Weierstrass* if $\sharp L_m(H) \leqq (2m-1)(g-1)$ for $\forall m \geqq 2$.

Fact. If a numerical semigroup H is Weierstrass, then it is quasi-Weierstrass ([1]).

Example A. There exists a primitive numerical semigroup H which is not quasi-Weierstrass such that $H_0 \leqq H$ for some quasi-Weierstrass primitive numerical semigroup H_0, i.e., $\exists H_0 =$ "quasi-Wei" $\leqq \exists H =$ "not quasi-Wei".

Example B. There are non-Weierstrass semigroups H which are quasi-Weierstrass ([11]). Up to now all such semigroups H are not primitive.

Problem 2. Every quasi-Weierstrass primitive semigroup is Weierstrass ?

Either Problem 1 or Problem 2 is at false, because of Example A. Finally we give the table which shows the situation on our problem ([5], [8], [6], [7]).

Genus	*Weierstrass*	*Unknown*
$g \leqq 8$	*All*	*None*
$g = 9 \wedge$ *primitive*	*All*(55)	*None*
$g = 9 \wedge$ *non-primitive*	61	2
$g = 10 \wedge$ *primitive*	75	14

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