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FIO-product representation of solutions to first-order symmetrizable hyperbolic systems

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Abstract

We consider the first-order Cauchy problem

$$\partial_z u + a(z, x, D_x)u = 0, \quad 0 < z \le Z,$$
$$u\,|_{z=0} = u_0,$$

with $Z > 0$ and $a(z, x, D_x)$ a $k \times k$ matrix of pseudodifferential operators of order one, whose principal part is assumed symmetrizable: there exists $L(z, x, \xi)$ of order 0, invertible, such that

$$a_1(z, x, \xi) = L(z, x, \xi)\,(-i\beta_1(z, x, \xi) + \gamma_1(z, x, \xi))\,(L(z, x, \xi))^{-1},$$

where β_1 and γ_1 are hermitian symmetric and $\gamma_1 \ge 0$. An approximation Ansatz for the operator solution, $U(z', z)$, is constructed as the composition of global Fourier integral operators with complex matrix phases. In the symmetric case, an estimate of the Sobolev operator norm in $L((H^{(s)}(\mathbf{R}^n))^k, (H^{(s)}(\mathbf{R}^n))^k)$ of these operators is provided, which yields a convergence result for the Ansatz to $U(z', z)$ in some Sobolev space as the number of operators in the composition goes to ∞, in both the symmetric and symmetrizable cases. We thus obtain a representation of the solution operator $U(z', z)$ as an infinite product of Fourier integral operators with matrix phases.

Keywords: Degenerate hyperbolic system; Symmetrizable system; Pseudodifferential initial value problem; Fourier integral operator; Matrix phase function; Global Sobolev norm estimate; Infinite product.

AMS 2000 subject classification: 35L40; 35L80; 35S10; 35S30.

Introduction

Let $k, n \in \mathbf{N}^*$. We consider the Cauchy problem

$$\partial_z u + a(z, x, D_x)u = 0, \quad 0 < z \le Z, \tag{1}$$
$$u\,|_{z=0} = u_0, \tag{2}$$

with $Z > 0$, $u(z, x) \in \mathbf{C}^k$, and $a(z, x, \xi)$ a $k \times k$ matrix with entries continuous with respect to (w.r.t.) z with values in $S^1(\mathbf{R}^n \times \mathbf{R}^n)$, with the usual notation $D_x = \frac{1}{i}\partial_x$. (Symbol spaces are precisely defined at the end of this introductive section.)

When $a(z, x, \xi)$ is scalar, $k = 1$, and independent of x and z it is natural to treat such a problem by means of Fourier transformation: $u(z, x') = \iint e^{i\langle x'-x|\xi\rangle - za(\xi)}\, u_0(x)\, đ\xi\, dx$, where $đ\xi := d\xi/(2\pi)^n$. Some assumption need to be made on the symbol $a(\xi)$ for this oscillatory integral to be well defined, e.g. non-negativity will be imposed on $a(\xi)$. When the symbol a depends on both x and z we can naively expect

$$u(z, x') \approx u_1(z, x') := \iint e^{i\langle x'-x|\xi\rangle - za(0,x',\xi)} u_0(x)\, đ\xi\, dx, \tag{3}$$

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for z small, and hence approximately solve the Cauchy problem (1)–(2) for $z \in [0, z^{(1)}]$ with $z^{(1)}$ small. If we want to progress in the z direction we have to solve the Cauchy problem

$$\partial_z u + a(z, x, D_x)u = 0, \quad z^{(1)} < z \le Z,$$
$$u(z, .)\,|_{z=z^{(1)}} = u_1(z^{(1)}, .),$$

which we again approximately solve by $u(z, x') \approx u_2(z, x') := \iint e^{i\langle x'-x|\xi\rangle-(z-z^{(1)})a(z^{(1)},x',\xi)}u_1(z^{(1)}, x)\, đ\xi\, dx$. This procedure can be iterated until we reach $z = Z$.

In the scalar case, $k = 1$, upon appropriate assumptions, this procedure converges [18, 19] and yields the solution operator to the Cauchy problem (1)–(2). The convergence is obtained in Sobolev spaces. We wish to extend this type of results to the case of a symmetrizable system, which then yields a representation of the solution operator as an infinite product of Fourier integral operators (FIO) of the form of (3). The extension is far from being straightforward mainly because we have to deal with matrix symbols and phases which do not commute in general and some simple algebraic operations in the scalar case become impossible. Here, we introduce classes of FIOs with *matrix* phase functions. Some care is required for them to be well defined and some assumptions will be made on the matrix symbol $a(z, x, \xi)$, which generalize those made in the scalar case in [18, 19].

We write $a(z, x, \xi) = a_1(z, x, \xi) + a_0(z, x, \xi)$, where a_1 is the principal part of a and a_0 is a matrix symbol with entries in $S^0(\mathbf{R}^n \times \mathbf{R}^n)$. The principal part is assumed homogeneous of degree one and symmetrizable in the sense that there exists a matrix $L(z, x, \xi)$ with entries in $S^0(\mathbf{R}^n \times \mathbf{R}^n)$, with z as a parameter, such that

$$a_1(z, x, \xi) = L(z, x, \xi)\, \alpha_1(z, x, \xi)\, (L(z, x, \xi))^{-1},$$

and $\alpha_1(z, x, \xi) = -i\beta_1(z, x, \xi) + \gamma_1(z, x, \xi)$, where β_1 and γ_1 are hermitian symmetric $k \times k$ matrices. The matrix $\gamma_1(z, x, \xi)$ is also assumed non-negative. For the precise statements of the assumptions we make on the symbol $a(z, x, \xi)$ refer to the subsequent sections.

Following [18], we define the so-called thin-slab propagator, $\mathcal{G}_{(z',z)}$, as the operator with (matrix) kernel

$$G_{(z',z)}(x', x) = \int e^{i\langle x'-x|\xi\rangle} e^{-(z'-z)a_0(z,x',\xi)} e^{-(z'-z)a_1(z,x',\xi)}\, đ\xi.$$

Note that $e^{-(z'-z)a_0(z,x',\xi)}$ and $e^{-(z'-z)a_1(z,x',\xi)}$ do not commute in general. Combining all iteration steps above involves composition of such operators: let $0 \le z^{(1)} \le \cdots \le z^{(k)} \le Z$, we then have

$$u_{k+1}(z, x) = \mathcal{G}_{(z,z^{(k)})} \circ \mathcal{G}_{(z^{(k)},z^{(k-1)})} \circ \cdots \circ \mathcal{G}_{(z^{(1)},0)}(u_0)(x),$$

if $z \ge z^{(k)}$. We then define the operator $\mathcal{W}_{\mathfrak{P},z}$ for a subdivision $\mathfrak{P}$ of the interval $[0, Z]$, $\mathfrak{P} = \{z^{(0)}, z^{(1)}, \ldots, z^{(N)}\}$, with $0 = z^{(0)} < z^{(1)} < \cdots < z^{(N)} = Z$,

$$\mathcal{W}_{\mathfrak{P},z} := \begin{cases} \mathcal{G}_{(z,0)} & \text{if } 0 \le z \le z^{(1)}, \\ \mathcal{G}_{(z,z^{(k)})} \prod_{i=k}^{1} \mathcal{G}_{(z^{(i)},z^{(i-1)})} & \text{if } z^{(k)} \le z \le z^{(k+1)}. \end{cases}$$

According to the procedure described above, $\mathcal{W}_{\mathfrak{P},z}$ yields an approximation Ansatz for the solution operator to the Cauchy problem (1)–(2) with step size $\Delta_{\mathfrak{P}} = \sup_{i=1,\ldots,N}(z_i - z_{i-1})$.

Note that a similar procedure can be used to show the existence of an evolution system by approximating it by composition of semigroup solutions of the Cauchy problem with z 'frozen' in $a(z, x, D_x)$ [7, 22]. It should be noticed that $\mathcal{W}_{\mathfrak{P},z}$ is however not the solution operator to problem (1)–(2) even in the case where the symbol a only depends on the transverse variable, x. For instance in the scalar case, $k = 1$, while singularities propagate along the bicharacteristics associated with $-\mathrm{Im}(a_1)$, we however observe that, with the form of the phase function, the operator $\mathcal{G}_{(z',z)}$ propagates singularities along straight lines. See [20] for further details. Furthermore, by composing the operators $\mathcal{G}_{(z'',z')}$ and $\mathcal{G}_{(z',z)}$, one convinces oneself that

$$\mathcal{G}_{(z'',z)} \neq \mathcal{G}_{(z'',z')} \circ \mathcal{G}_{(z',z)}$$

in general if $z'' \ge z' \ge z \in [0, Z]$ (use again that singularities propagate along straight lines). The family of operators $(\mathcal{G}_{(z',z)})_{(z',z)\in[0,Z]^2}$ is thus neither a semigroup nor an evolution system.

Under Hölder regularity of order α of $a(z, x, \xi)$ w.r.t. z, and Lipschitz regularity of $L(z, x, \xi)$ w.r.t. z in the symmetrizable case, we shall prove convergence of the Ansatz $\mathcal{W}_{\mathfrak{P},z}$ to the solution operator $U(z, 0)$ of the Cauchy problem (1)–(2):

$$\text{(4)} \qquad \|\mathcal{W}_{\mathfrak{P},z} - U(z,0)\|_{((H^{(s+1)})^k,(H^{(s+r)})^k)} \le \Delta_{\mathfrak{P}}^{(1-r)\alpha},$$

for $0 \le r < 1$ (Theorem 2.7 in the symmetric case, $L(z, x, \xi) = I_k$, and Theorem 3.13 in the symmetrizable case). We thus obtain a representation of the solution operator $U(z', z)$ as an infinite product of FIOs with matrix phases. As in the scalar case [18], such a result is achieved by first proving a precise estimate of the Sobolev operator norm of the thin-slab propagator $\mathcal{G}_{(z',z)}$ in the symmetric case, $L(z, x, \xi) = I_k$: for all $s \in \mathbf{R}$, there exists $M \ge 0$ such that

$$\text{(5)} \qquad \|\mathcal{G}_{(z',z)}\|_{((H^{(s)})^k,(H^{(s)})^k)} \le 1 + M\Delta,$$

for all $0 \le z \le z' \le Z$ with $\Delta = z' - z$ sufficiently small (Theorem 1.20). To prove (5), we assume that the symbols β_1 and γ_1 and commute and are diagonalizable with a diagonalizing symbol that is smooth w.r.t. x and ξ and only *bounded* w.r.t. z, i.e. the symbol $a(z, x, \xi)$ is assumed to be "geometrically regular". Note that this assumption allows for crossing smooth eigenvalues. Estimate (5) in the symmetric case is then used to treat the case of a symmetrizable system. We then further assume that the symmetrizing symbol $L(z, x, \xi)$ is smooth w.r.t. x and ξ and *Lipschitz continuous* w.r.t. z. In this case, solutions to (1)–(2) exist and are unique and we prove the convergence of the proposed Ansatz $\mathcal{W}_{\mathfrak{P},z}$ to $U(z, 0)$. An estimation of the form of (5) is however not obtained in the case of a symmetrizable system. Many proofs are omited in the present article. A complete version of this article can be found in [17].

Multi-composition of FIOs to approximate solutions of Cauchy problems were first proposed in the scalar case in [13] and [12]. In these articles, the exact solution operator of a first-order hyperbolic equation is approximated with a different Ansatz. The approximation is made up to a regularizing operator. The technique is based on the computation and the estimation of the phase functions and amplitudes of the FIO resulting from these multi-products, a result know as the Kumano-go-Taniguchi Theorem. It is based on the earlier work of H. Kumano-go in [10]. This approach is synthesized in Chapter 10 of [11].

The case of systems with constant multiplicities in non-diagonal form in also treated in [11, Section 10.4]. However, the system is diagonalized by the application of elliptic pseudodifferential operators and the solution is only recovered by the use of a parametrix which yields a solution operator up to a regularizing operator. In the present article, we aim at obtaining an exact representation of the solution operator. Hence we do not rely on such a diagonalization procedure of the system.

The multi-product technique was further applied to Schrödinger equations with specific symbols [9, 14]. In these latter works, the multi-product in also interpreted as an iterated *integral of Feynman's type* and convergence is studied in a weak sense. The time slicing approximation, closely related to our approach, allows to give a rigorous mathematical meaning to Feynman path integrals [15, 3]. In [9] a convergence result in L^2 is proven. This is the type of results sought here for first-order hyperbolic systems. We however do not use the apparatus of multi-phases and rather focus on estimating the Sobolev regularity of each term in the multi-product of FIOs in the proposed Ansatz $\mathcal{W}_{\mathfrak{P},z}$. While the resulting product is an FIO, we do not compute its phase and amplitude. The Sobolev regularity allows us to use an a priori energy estimate for the Cauchy problem (1)–(2) to prove convergence of the approximating Ansatz to the solution operator.

In this article, when the constant C is used, its value may change from one line to another. If we want to keep track of the value of a constant we shall use another letter. When we write that a function is bounded w.r.t. z and/or Δ we shall actually mean that z is to be taken in the interval $[0, Z]$ and Δ in some interval $[0, \Delta_{max}]$ unless otherwise stipulated. We shall generally write $X, X', X'', X^{(1)}, \ldots, X^{(N)}$ for $\mathbf{R}^n$, according to variables, e.g., $x, x', \ldots, x^{(N)}$. We shall sometimes use the variables y, t that belong to Y and T which will denote $\mathbf{R}^{n_y}, \mathbf{R}^{n_t}$ with possibly $n_y \ne n$ and $n_t \ne n$.

In a standard way, we set $\langle\xi\rangle := \sqrt{1 + |\xi|^2}$ for $\xi \in \mathbf{R}^p$. Throughout the article, we use spaces of global symbols; a function $a \in \mathscr{C}^\infty(\mathbf{R}^q \times \mathbf{R}^p)$ is in $S^m_{\rho,\delta}(\mathbf{R}^q \times \mathbf{R}^p)$, $0 < \rho \le 1$, $0 \le \delta < 1$, if for all multi-indices α, β there exists $C_{\alpha\beta} > 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \le C_{\alpha\beta} \langle\xi\rangle^{m - \rho|\beta| + \delta|\alpha|}, \quad x \in \mathbf{R}^q,\ \xi \in \mathbf{R}^p.$$

The best possible constants $C_{\alpha\beta}$, i.e.,

$$p_{\alpha\beta}(a) := \sup_{(x,\xi)\in\mathbf{R}^q\times\mathbf{R}^p} \langle\xi\rangle^{-m+\rho|\beta|-\delta|\alpha|}|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|,$$

define seminorms for a Fréchet space structure on $S^m_{\rho,\delta}(\mathbf{R}^q\times\mathbf{R}^p)$. As usual we write $S^m_\rho(\mathbf{R}^q\times\mathbf{R}^p)$ in the case $\rho = 1-\delta$, $\frac{1}{2}\le\rho<1$, and $S^m(\mathbf{R}^q\times\mathbf{R}^p)$ in the case $\rho=1$, $\delta=0$. We shall denote by $\mathcal{M}_kS^m_{\rho,\delta}(\mathbf{R}^q\times\mathbf{R}^p)$ the space of $k\times k$ matrices with entries in $S^m_{\rho,\delta}(\mathbf{R}^q\times\mathbf{R}^p)$. By $\mathcal{M}_k(\mathbb{C})$, we denote the space of $k\times k$ matrices with complex entries, furnished with some norm $\|.\|_{\mathcal{M}_k(\mathbb{C})}$. Seminorms on $\mathcal{M}_kS^m_{\rho,\delta}(\mathbf{R}^q\times\mathbf{R}^p)$ are naturally built from $\|.\|_{\mathcal{M}_k(\mathbb{C})}$ and the seminorms on $S^m_{\rho,\delta}(\mathbf{R}^q\times\mathbf{R}^p)$. It yields a Fréchet space structure on $\mathcal{M}_kS^m_{\rho,\delta}(\mathbf{R}^q\times\mathbf{R}^p)$. In the case of matrix symbols, we shall also use the notation simplifications given above in the case $\rho=1-\delta$, and the case $\rho=1,\delta=0$.

We shall use, in a standard way, the notation # for the composition of symbols of pseudodifferential operators (ψDO). In the case of matrix symbols, $a \# b$ will naturally denote the matrix symbol with entries $(a\#b)_{ij} = \sum_k a_{ik}\# b_{kj}$. When given an amplitude $p(x,y,\xi)\in\mathcal{M}_kS^m_{\rho,\delta}(X\times X\times\mathbf{R}^n)$, $\rho\ge\delta$, we shall also use the notation $\sigma\{p\}(x,\xi)$ for the symbol of the pseudodifferential operator with amplitude p.

For $r\in\mathbf{R}$ we let $E^{(r)}$ be the ψDO with symbol $\langle\xi\rangle^r$. The operator $E^{(r)}$ maps $H^{(s)}(X)$ onto $H^{(s-r)}(X)$ unitarily for all $s\in\mathbf{R}$ with $E^{(-r)}$ being the inverse map. We shall use the same notation for the diagonal operator that maps $(H^{(s)}(X))^k$ onto $(H^{(s-r)}(X))^k$ unitarily, $k\in\mathbf{N}^*$.

1 The thin-slab propagator for a symmetric system

1.1 The Cauchy problem

Let $k,n\in\mathbf{N}$. Let $s\in\mathbf{R}$ and $Z>0$. We consider the Cauchy problem

$$\partial_z u + a(z,x,D_x)u = 0,\quad 0<z\le Z, \tag{1}$$

$$u\,|_{z=0} = u_0\in(H^{(s+1)}(\mathbf{R}^n))^k, \tag{2}$$

where the matrix symbol $a(z,x,\xi)$ takes values in $\mathcal{M}_kS^1(X\times\mathbf{R}^n)$ with z as a parameter.

In this section, we focus on the symmetric case. More precisely, we shall make the following assumption.

Assumption 1.1. *The principal matrix symbol of a, $a_1(z,x,\xi) = -ib_1(z,x,\xi)+c_1(z,x,\xi)$, is such that both b_1 and c_1 are continuous w.r.t. z with values in $\mathcal{M}_kS^1(X\times\mathbf{R}^n)$ and homogeneous of degree one in ξ, for $|\xi|\ge 1$. Furthermore, they are hermitian symmetric and $c_1(z,x,\xi)$ is non-negative.*

We set the remaining part of the symbol $a(z,x,\xi)$ as $a_0(z,x,\xi)$, $a_0(z,x,\xi) := a(z,x,\xi)-a_1(z,x,\xi)$, which is assumed to be continuous w.r.t. z with values in $\mathcal{M}_kS^0(X\times\mathbf{R}^n)$.

Adapting the proof of Lemma 23.1.1 in [4] to systems (making use of the sharp Gårding inequality for positive first-order matrix symbols [16, Theorem 3.2], [25], [24, Chapter VII]) for any function in

$$V := \mathscr{C}^1([0,Z],(H^{(s)}(\mathbf{R}^n))^k)\cap\mathscr{C}^0([0,Z],(H^{(s+1)}(\mathbf{R}^n))^k),$$

we have the following energy estimate (see also [1, Theorem 6.4.3])

$$\sup_{z\in[0,Z]}\|u(z,.)\|_{(H^{(s)})^k}\le C\|u(0,.)\|_{(H^{(s)})^k} + C\int_0^Z\|\partial_z u + a(z,x,D_x)u\|_{(H^{(s)})^k}\,dz. \tag{3}$$

Then, there exists a unique solution in V to the Cauchy problem (1)–(2) ([4, Theorem 23.1.2], [1, Theorem 6.4.5]).

By Proposition 9.3 in [2, Chapter VI] the family of operators $(a(z,x,D_x))_{z\in[0,Z]}$ generates a strongly continuous evolution system. Let $U(z',z)$ denote the corresponding evolution system:

$$U(z'',z')\circ U(z',z) = U(z'',z),\quad Z\ge z''\ge z'\ge z\ge 0,$$

with

$$\partial_z U(z,z_0)(u_0) + a(z,x,D_x)U(z,z_0)(u_0) = 0,\quad 0\le z_0<z\le Z,$$
$$U(z_0,z_0)(u_0) = u_0\in(H^{(s+1)}(\mathbf{R}^n))^k,$$

while $U(z, z_0)(u_0) \in (H^{(s+1)}(\mathbb{R}^n))^k$ for all $z \in [z_0, Z]$. For the Cauchy problem (1)–(2) we take $z_0 = 0$.

We shall make the following further assumption on $a(z, x, \xi)$.

Assumption 1.2. *There exists $w(z, x, \xi)$ continuous w.r.t. z with values in $\mathcal{M}_k S^0(X \times \mathbb{R}^n)$, unitary, homogeneous of degree zero in ξ, for $|\xi| \geq 1$, such that*

$$b_1(z, x, \xi) = w(z, x, \xi)\, d_b(z, x, \xi)\, (w(z, x, \xi))^{-1},$$
$$c_1(z, x, \xi) = w(z, x, \xi)\, d_c(z, x, \xi)\, (w(z, x, \xi))^{-1},$$

where $d_b(z, x, \xi)$ and $d_c(z, x, \xi)$ are $k \times k$ diagonal matrices with entries continuous w.r.t. z with values in $S^1(\mathbb{R}^n \times \mathbb{R}^n)$ and homogeneous of degree one in ξ, for $|\xi| \geq 1$. The diagonal entries of $d_c(z, x, \xi)$ are non-negative since $c_1(z, x, \xi) \geq 0$.

This assumption is sometimes referred to as having a "geometrically regular" matrix symbol $a(z, x, \xi)$ (see e.g. [21, Definition 2.2 (ii)]).

Assumption 1.2 will be satisfied for instance if the eigenvalues of $b_1(z, x, \xi)$ have constant multiplicities[1] since b_1 is hermitian symmetric [8, Section II.4] and if the matrices $b_1(z, x, \xi)$ and $c_1(z, x, \xi)$ commute. However, Assumption 1.2 is much more general and allows for *crossing* smooth eigenvalues.

We set $v(z, x, \xi) := (w(z, x, \xi))^{-1} = (w(z, x, \xi))^t$ and $d(z, x, \xi) := -id_b(z, x, \xi) + d_c(z, x, \xi)$. We shall denote by $d_{b,l}(z, x, \xi)$ and $d_{c,l}(z, x, \xi)$, $1 \leq l \leq k$, the diagonal entries of the matrices $d_b(z, x, \xi)$ and $d_c(z, x, \xi)$.

Example 1.3. The Dirac operator $\sum_j = 1^3 \alpha_j D_{x_j} + m\beta$ has an hermitian symmetric principal symbol. Here the Dirac matrices are 4×4 matrices and are given by

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad j = 1, 2, 3, \quad \text{with } \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The two eigenvalues $\pm|\xi|$ both have constant multiplicity two. A norm convergence result of a Trotter-product formula for the Dirac operator can be found in [6].

1.2 The thin-slab propagator

Let $0 \leq z \leq z' \leq Z$. We set, for $\Delta = z' - z$,

$$g_{(z',z)}(x, \xi) := e^{-\Delta a_0(z,x,\xi)}. \tag{4}$$

The function $g_{(z',z)}(x, \xi)$ is bounded w.r.t. z and smooth w.r.t. Δ with values in $\mathcal{M}_k S^0(X \times \mathbb{R}^n)$.

Following the definition of the thin-slab propagator given in the scalar case [18], we define the following ($k \times k$ matrix) kernel

$$G_{(z',z)}(x', x) := \int e^{i(x'-x|\xi)} g_{(z',z)}(x', \xi) e^{-\Delta a_1(z,x',\xi)}\, đ\xi.$$

Such a kernel is well defined since we can write

$$G_{(z',z)}(x', x) = \int e^{i(x'-x|\xi)} g_{(z',z)}(x', \xi) w(z, x', \xi) e^{-\Delta d(z,x',\xi)} v(z, x', \xi)\, đ\xi,$$

and thus, each entry of the matrix kernel is a finite sum of scalar kernels of the form of

$$\int e^{i\phi_{(z',z)}(x',x,\xi)}\, \sigma_{(z',z)}(x', \xi)\, đ\xi, \tag{5}$$

with $\sigma_{(z',z)}(x', x, \xi)$ bounded w.r.t. z' and z with value in S^0 and $\phi_{(z',z)}(x', x, \xi)$ a complex phase. We can prove that each component of the associated operator is thus a finite sum of global FIOs for Δ sufficiently small (see [18]). We have the following regularity results.

[1] in particular if the eigenvalues are simple, i.e., in the strictly hyperbolic case.

Proposition 1.4. *Let $s \in \mathbb{R}$. For Δ sufficiently small, the operator $\mathcal{G}_{(z',z)}$ with distribution kernel $G_{(z',z)}(x',x)$, is a continuous map of $(\mathscr{S}(\mathbb{R}^n))^k$ into $(\mathscr{S}(\mathbb{R}^n))^k$, $(\mathscr{S}'(\mathbb{R}^n))^k$ into $(\mathscr{S}'(\mathbb{R}^n))^k$, and $(H^{(s)}(\mathbb{R}^n))^k$ into $(H^{(s)}(\mathbb{R}^n))^k$. In particular, there exists $C \geq 0$ such that*

$$\|\mathcal{G}_{(z',z)}\|_{((H^{(s)})^k,(H^{(s)})^k)} \leq C,$$

for all $z', z \in [0, Z]$, $\Delta = z' - z$.

Let $m \in \mathbb{R}$. If the matrix symbol $g_{(z',z)}$ is changed into a function bounded w.r.t. Δ and z with values in $\mathcal{M}_k S^m_{\rho'}(\mathbb{R}^n \times \mathbb{R}^n)$, $\rho' \in [\frac{1}{2}, 1]$, then the associated operator maps $(H^{(s)}(\mathbb{R}^n))^k$ into $(H^{(s-m)}(\mathbb{R}^n))^k$ continuously, with a uniform operator-norm estimate as above.

In the sequel, we shall say that operators of the form of $\mathcal{G}_{(z',z)}$ are FIOs with the complex *matrix* phase

$$\phi_{(z',z)}(x',x,\xi) = \langle x' - x|\xi\rangle + b_1(z,x',\xi) + ic_1(z,x',\xi). \tag{6}$$

We aim at giving a more precise estimate of the norm of the thin-slab propagator in $L((H^{(s)}(\mathbb{R}^n))^k, (H^{(s)}(\mathbb{R}^n))^k)$. We shall in fact obtain

$$\|\mathcal{G}_{(z',z)}\|_{((H^{(s)})^k,(H^{(s)})^k)} \leq 1 + M\Delta, \quad \Delta = z' - z,$$

for some $M \geq 0$ and for Δ sufficiently small (Theorem 1.20). To obtain such an estimate we need to understand the properties of the matrix symbol $e^{-\Delta c_1(x,\xi)}$ when Δ is small.

1.3 A class of symbols

Here, we follow the developments of [18]. We first introduce some definitions.

Definition 1.5. *Let $L \geq 2$. A (scalar) symbol $q(z,.)$ bounded w.r.t. z with values in $S^1(\mathbb{R}^p \times \mathbb{R}^r)$ is said to satisfy Property (P_L) if it is non-negative and satisfies*

$$(P_L) \qquad |\partial_y^\alpha \partial_\eta^\beta q(z,y,\eta)| \leq C\langle\eta\rangle^{-|\beta|+(|\alpha|+|\beta|)/L}(1+q(z,y,\eta))^{1-(|\alpha|+|\beta|)/L}, \quad z \in [0,Z],\ y \in \mathbb{R}^p,\ \eta \in \mathbb{R}^r.$$

We then set $\rho = 1 - 1/L$ and $\delta = 1/L$.

Examples of symbols with such a property with $L > 2$ are given in [23]. In fact we have the following lemma [18].

Lemma 1.6. *Let $q(z,y,\eta)$ be bounded w.r.t. z with values in $S^1(\mathbb{R}^p \times \mathbb{R}^r)$. If $q \geq 0$ then q satisfies Property (P_L) for $L = 2$.*

Remark 1.7. If the symbol $q(z,y,\eta)$ and $p(z,y,\eta)$ both satisfy Property (P_L) then the amplitude $q(z,y',\eta)+p(z,y,\eta)$ also satisfies Property (P_L) (with derivatives w.r.t. y, y' and η).

The following definition concerns matrix symbols.

Definition 1.8. *Let $L \geq 2$, $\rho = 1 - 1/L$ and $\delta = 1/L$. Let $\rho_\Delta(z,y,\eta)$ be a function in $\mathscr{C}^\infty(Y \times \mathbb{R}^r, \mathcal{M}_k(\mathbb{C}))$ depending on the parameters $\Delta \geq 0$ and $z \in [0,Z]$. We say that ρ_Δ satisfies Property (Q_L) if the following holds*

$$(Q_L) \qquad \partial_y^\alpha \partial_\eta^\beta(\rho_\Delta - \rho_\Delta|_{\Delta=0})(z,y,\eta) = \Delta^{m+\delta(|\alpha|+|\beta|)}\rho_\Delta^{m\alpha\beta}(z,y,\eta), \quad \text{for } |\alpha|+|\beta| \leq L, \quad 0 \leq m \leq 1 - \delta(|\alpha|+|\beta|),$$

where $\rho_\Delta^{m\alpha\beta}(z,y,\eta)$ is bounded w.r.t. Δ and z with values in $\mathcal{M}_k S_\rho^{m-\rho|\beta|+\delta|\alpha|}(Y \times \mathbb{R}^r)$. It follows that $\rho_\Delta(z,y,\eta) - \rho_\Delta|_{\Delta=0}(z,y,\eta)$ is itself bounded w.r.t. Δ and z with values in $\mathcal{M}_k S_\rho^0(Y \times \mathbb{R}^r)$.

In [18], the following three lemmas, are proven in the scalar case, $k = 1$.

Lemma 1.9. *Let $q(z,.)$ be bounded w.r.t. z with values in $S^1(Y \times \mathbb{R}^r)$ and satisfy Property (P_L). Define $\rho_\Delta(z,y,\eta) = e^{-\Delta q(z,y,\eta)}$. Then ρ_Δ satisfies Property (Q_L), $k = 1$, for $\Delta \in [0, \Delta_{max}]$ for any $\Delta_{max} > 0$. As $\rho_\Delta|_{\Delta=0} = 1$, ρ_Δ is itself bounded w.r.t. Δ and z with values in $S_\rho^0(Y \times \mathbb{R}^r)$.*

Lemma 1.10. *Let $f \in \mathscr{C}^\infty(\mathbb{R})$ and $\rho_\Delta(z,y,\eta)$ in $\mathscr{C}^\infty(Y \times \mathbb{R}^r)$ that satisfies Property (Q_L), $k = 1$, and such that $\rho_\Delta(z,.)|_{\Delta=0}$ is independent of y and η. Then $f(\rho_\Delta)(z,y,\eta)$ satisfies Property (Q_L).*

Lemma 1.11. *Let $\rho_\Delta(z,y,\eta) \in S^0_\rho(Y \times \mathbb{R}^r)$ satisfy Property (Q_L), $k = 1$, such that $\rho_\Delta|_{\Delta=0}$ is constant. Let $f_\Delta(z,y,\eta)$ be bounded w.r.t. z and Δ with values in $S^1(Y \times \mathbb{R}^r)$ be homogeneous of degree one in η for $|\eta| \geq 1$. Define $\tilde{\eta}(\Delta,z,y,\eta) := \eta - \Delta f_\Delta(z,y,\eta)$. Then*

$$\tilde{\rho}_\Delta(z,y,\eta) := \rho_\Delta(z,y,\tilde{\eta}(\Delta,z,y,\eta))$$

satisfies Property (Q_L), $k = 1$, for Δ sufficiently small.

Remark 1.12. These three lemmas naturally extend to diagonal matrix symbols and we shall use them in this form below.

Proposition 1.13. *Let $L \geq 2$. Let $\rho_\Delta(z,y,\eta)$ be bounded w.r.t. z and Δ with values in $\mathcal{M}_k S^0_\rho(Y \times \mathbb{R}^r)$ that satisfies Property (Q_L), $L \geq 2$, $\rho = 1 - 1/L$. Let $r(z,.)$ be bounded w.r.t. z with values in $\mathcal{M}_k S^0(Y \times \mathbb{R}^n)$. Then $(r\rho_\Delta)(z,y,\eta)$ and $(\rho_\Delta r)(z,y,\eta)$ both satisfy Property (Q_L).*

Corollary 1.14. *Assume that the entries of the diagonal matrix symbol $d_c(z,x,\xi)$ satisfy Property (P_L), for $L \geq 2$. Then the matrix symbol $e^{-\Delta c_1(z,x,\xi)}$ satisfies Property (Q_L) in $\mathcal{M}_k S^0_\rho(X \times \mathbb{R}^n)$.*

Recall that by Lemma 1.6, by default, the entries of the diagonal matrix symbol $d_c(z,x,\xi)$ satisfy Property (P_L), for $L = 2$.

Lemma 1.15. *Let the matrix amplitudes $\rho_\Delta(z,x,y,\xi)$ and $\mu_\Delta(z,x,y,\xi)$ both satisfy Property (Q_L) and be such that $\rho_\Delta(z,.)|_{\Delta=0}$ and $\mu_\Delta(z,.)|_{\Delta=0}$ are constant. Then the amplitude $\rho_\Delta(z,x,y,\xi)\mu_\Delta(z,x,t,\xi)$ satisfies Property (Q_L) (with differentials w.r.t. x, y, t, and ξ).*

Lemma 1.16. *Let $\rho_\Delta(z,x,y,\xi)$ be an amplitude in $\mathcal{M}_k S^0_\rho(\mathbb{R}^{2p} \times \mathbb{R}^p)$ depending on the parameters $\Delta \geq 0$ and $z \in [0,Z]$ that satisfies Property (Q_L) for $1 \leq |\alpha| + |\beta| \leq 2$ and such that $\rho_\Delta(z,.)|_{\Delta=0}$ is independent of (x,y,ξ). Let $r(x,\xi) \in \mathcal{M}_k S^s(\mathbb{R}^p \times \mathbb{R}^p)$ for some $s \in \mathbb{R}$. Then*

$$\sigma\{\rho_\Delta\, r\}(z,x,\xi) - \rho_\Delta(z,x,x,\xi)\, r(x,\xi) = \Delta^{m+2\delta}\lambda^m_\Delta(z,x,\xi),\ 0 \leq m \leq \rho - \delta,$$

where the function $\lambda^m_\Delta(z,x,\xi)$ is bounded w.r.t. Δ and z with values in $\mathcal{M}_k S^{m+s-(\rho-\delta)}_\rho(\mathbb{R}^p \times \mathbb{R}^p)$.

For a proof see the proof of Lemma 2.22 in [18], which can be directly adapted to the matrix case. We shall also need the following lemma which is a closely related results.

Lemma 1.17. *Let $q_\Delta(z,x,\xi)$ be an symbol in $\mathcal{M}_k S^0_\rho(\mathbb{R}^p \times \mathbb{R}^p)$ depending on the parameters $\Delta \geq 0$ and $z \in [0,Z]$ that satisfies Property (Q_L) for $|\alpha| = 1$ and such that $q_\Delta(z,.)|_{\Delta=0}$ is independent of x. Let $r_z(x,\xi)$ be bounded w.r.t. z with values in $\mathcal{M}_k S^s(\mathbb{R}^p \times \mathbb{R}^p)$ for some $s \in \mathbb{R}$. Then*

$$(r_z \# q_\Delta)(z,x,\xi) - r_z(x,\xi)\, q_\Delta(z,x,\xi) = \Delta^{m+\delta}\lambda^m_\Delta(z,x,\xi),\ 0 \leq m \leq \rho,$$

where the function $\lambda^m_\Delta(z,x,\xi)$ is bounded w.r.t. Δ and z with values in $\mathcal{M}_k S^{m+s-\rho}_\rho(\mathbb{R}^p \times \mathbb{R}^p)$.

For a proof see the proof of Proposition 2.5 in [19], which can be directly adapted to the matrix case.

We shall need the following result.

Theorem 1.18. *Let $\frac{1}{2} \leq \rho \leq 1$ and set $\delta = 1 - \rho$. Let $p(x,\xi)$ be a real non-negative $\mathscr{C}^\infty$ function that satisfies*

$$\|p(x,\xi)\|_{\mathcal{M}_k(\mathbb{C})} \leq C\langle\xi\rangle, \tag{7}$$

$$\|\partial^\alpha_x p(x,\xi)\|_{\mathcal{M}_k(\mathbb{C})} \leq C_\alpha\langle\xi\rangle,\quad |\alpha| = 1,\quad \|\partial^\beta_\xi p(x,\xi)\|_{\mathcal{M}_k(\mathbb{C})} \leq C_\beta,\quad |\beta| = 1, \tag{8}$$

and

$$\partial^\alpha_x\partial^\beta_\xi p(x,\xi) \in \mathcal{M}_k S^{\rho-\delta+\delta|\alpha|-\rho|\beta|}_\rho(X \times \mathbb{R}^n),\quad \text{for } |\alpha+\beta| = 2. \tag{9}$$

Then there exists a non-negative constant C such that

$$\mathrm{Re}(p(x,D)u,u)_{((L^2)^k,(L^2)^k)} \geq -C\|u\|^2_{(L^2)^k},\quad u \in (\mathscr{S}(\mathbb{R}^n))^k.$$

The constant C can be chosen uniformly if the symbol p remains in a set such that the constants in (7)–(8) are uniform and if $\partial^\alpha_x\partial^\beta_\xi p(x,\xi)$, $|\alpha+\beta| = 2$, remain in bounded domains of $\mathcal{M}_k S^{\rho-\delta+\delta|\alpha|-\rho|\beta|}_\rho(X \times \mathbb{R}^n)$ respectively.

In other words, for the partial differentiation of order zero and one the symbol p behaves like an element $\mathcal{M}_k S^{1}_{1,0}$ and like an element of $\mathcal{M}_k S^{\rho-\delta}_{\rho}$ for higher-order derivatives. The result we prove is of the form of the sharp Gårding inequality. Note that considering $p(x,\xi)$ to be in $\mathcal{M}_k S^{1}_{\rho}(X \times \mathbf{R}^n)$, we cannot directly apply the usual sharp Gårding inequality to obtain a lower L^2 bound when $\frac{1}{2} \leq \rho < 1$.

The proof of Theorem 1.18 can be done by revisiting the proof of the sharp Gårding inequality for instance as given in [11, Section 3.4].

The following result is at the heart of the precise Sobolev operator-norm estimation of the thin-slab propagator $\mathcal{G}_{(z',z)}$.

Theorem 1.19. *Let $k(z,x,\xi)$ be a $k \times k$ diagonal matrix symbol with non-negative entries, that satisfy Property (P_L), $L \geq 2$. Let $m(z,x,\xi)$ be a $k \times k$ matrix symbol, such that $m(z,x,\xi)$ and $(m(z,x,\xi))^{-1}$ are both bounded w.r.t. z with entries in $S^0(X \times \mathbf{R}^n)$. Set*

$$h(z,x,\xi) = m(z,x,\xi)k(z,x,\xi)(m(z,x,\xi))^{-1},$$

The matrix symbol $e^{-\Delta h(z,x,\xi)}$ is bounded w.r.t. Δ and z with values in $\mathcal{M}_k S^0_{\rho}(X \times \mathbf{R}^n)$ and the pseudodifferential operator $\mathcal{A}_{(z',z)} = e^{-\Delta h(z,x,D_x)}$ is continuous from $(L^2(\mathbf{R}^n))^k$ into $(L^2(\mathbf{R}^n))^k$. There exist $\Delta_2 > 0$ and $M \geq 0$ such that $\mathcal{A}_{(z',z)}$ satisfies

$$\|\mathcal{A}_{(z',z)}\|_{((L^2)^k,(L^2)^k)} \leq 1 + M\Delta,$$

for all $z', z \in [0, Z]$ such that $0 \leq \Delta = z' - z \leq \Delta_2$.

Proof. In the proof, we shall always assume that Δ is sufficiently small to apply the invoked properties and results. By Proposition 1.13, the symbols $\rho_\Delta(z,x,\xi) = e^{-\Delta k(z,x,\xi)}$ and $\mu_\Delta(z,x,\xi) = e^{-\Delta h(z,x,\xi)}$ both satisfy Property (Q_L).

She prove that $(\mathcal{A}_{(z',z)} \circ \mathcal{A}^*_{(z',z)} u, u)_{((L^2)^k,(L^2)^k)} \leq (1 + C\Delta)\|u\|^2_{(L^2)^k}$ for all $u \in (\mathcal{S}(\mathbf{R}^n))^k$. The pseudodifferential operator $\mathcal{A}_{(z',z)} \circ \mathcal{A}^*_{(z',z)}$ has the matrix amplitude

$$p_\Delta(z,x,y,\xi) = e^{-\Delta h(z,x,\xi)} e^{-\Delta h(z,y,\xi)},$$

which satisfies Property (Q_L) by Lemma 1.15. We then obtain

$$\sigma\{p_\Delta(z,x,y,\xi)\} - e^{-2\Delta h(z,x,\xi)} = \Delta\lambda_\Delta(z,x,\xi),$$

where $\lambda_\Delta(z,x,\xi)$ is bounded w.r.t. z and Δ with values in $\mathcal{M}_k S^0_{\rho}(X \times \mathbf{R}^n)$ by Lemma 1.16 (using $m = \rho - \delta$). By the Calderón-Vaillancourt Theorem (see [11, Chapter 7, Sections 1,2] or [24, Section XIII-2]), we shall obtain the desired estimate for $(\mathcal{A}_{(z',z)} \circ \mathcal{A}^*_{(z',z)} u, u)_{((L^2)^k,(L^2)^k)}$ if we prove $\mathrm{Re}(e^{-2\Delta h(z,x,D_x)} u, u)_{((L^2)^k,(L^2)^k)} \leq (1 + C\Delta)\|u\|^2_{(L^2)^k}$ for all $u \in (\mathcal{S}(\mathbf{R}^n))^k$.

We set $q_\Delta(z,x,\xi) = (1 - e^{-2\Delta h(z,x,\xi)})/\Delta$ for $\Delta > 0$ and observe that $q_\Delta(z,x,\xi)$ satisfies the conditions listed in Theorem 1.18 uniformly w.r.t. z and Δ. In fact, a first-order Taylor formula gives $\|q_\Delta(z,x,\xi)\|_{\mathcal{M}_k(\mathbf{C})} \leq C\langle\xi\rangle$. By Property (Q_L) we obtain

$$\|\partial_x^\alpha q_\Delta(z,x,\xi)\|_{\mathcal{M}_k(\mathbf{C})} \leq C\langle\xi\rangle, \quad |\alpha| = 1,$$
$$\|\partial_\xi^\beta q_\Delta(z,x,\xi)\|_{\mathcal{M}_k(\mathbf{C})} \leq C, \quad |\beta| = 1,$$

using $m = \rho$ in (Q_L) in both cases. Finally, if $|\alpha + \beta| = 2$, we obtain that $\partial_x^\alpha \partial_\xi^\beta q_\Delta(z,x,\xi)$ is bounded uniformly w.r.t. z and Δ with values in $\mathcal{M}_k S^{\rho-\delta+\delta|\alpha|-\rho|\beta|}_{\rho}(X \times \mathbf{R}^n)$ by choosing $m = \rho - \delta$ in (Q_L).

By Theorem 1.18 we thus obtain $\mathrm{Re}(q_\Delta(z,x,D_x)u, u)_{((L^2)^k,(L^2)^k)} \geq -C\|u\|^2_{(L^2)^k}$ for all $u \in (\mathcal{S}(\mathbf{R}^n))^k$ which yields

$$\|u\|^2_{(L^2)^k} - \mathrm{Re}(e^{-2\Delta h(z,x,D_x)}u, u)_{((L^2)^k,(L^2)^k)} \geq -C\Delta\|u\|^2_{(L^2)^k},$$

which concludes the proof. ∎

1.4 Sobolev space regularity estimate for the thin-slab propagator

We now state and prove the main theorem of this section, which will be essential to give a meaning to infinite products of operators of the form of $\mathcal{G}_{(z',z)}$ in Sections 2 and 3.

Theorem 1.20. *Let $s \in \mathbb{R}$. There exist $\Delta_3 > 0$ and $M \geq 0$ such that*

$$\|\mathcal{G}_{(z',z)}\|_{((H^{(s)})^k,(H^{(s)})^k)} \leq 1 + M\Delta,$$

for all $z', z \in [0, Z]$ such that $0 \leq \Delta = z' - z \leq \Delta_3$.

In the proof we assume that the diagonal entries of d_c satisfy Property (P_L) for some $L \geq 2$. We know that it is always true for $L = 2$ by Lemma 1.6 but special choices for c_1 can be made. As before we use $\rho = 1 - 1/L$ and $\delta = 1/L$ with $\rho > \delta$ for $L > 2$ and $\rho = \delta = \frac{1}{2}$ for $L = 2$.

Proof. We first observe that we can write, $g_{(z',z)}(x, \xi) = I_k + \Delta\tilde{g}_{(z',z)}(x, \xi)$, with $\tilde{g}_{(z',z)}$ bounded w.r.t. Δ and z with values in $\mathcal{M}_k S^0(\mathbb{R}^n \times \mathbb{R}^n)$, from Taylor's formula, and (1.1.9) in [5]. We thus obtain $\mathcal{G}_{(z',z)} = \mathcal{G}^{I_k}_{(z',z)} + \Delta\widetilde{\mathcal{G}}_{(z',z)}$, where the operator $\mathcal{G}^{I_k}_{(z',z)}$ is of the same form as $\mathcal{G}_{(z',z)}$ with the amplitude $g_{(z',z)}$ replaced by I_k. With the last point in Proposition 1.4, we have

$$\|\widetilde{\mathcal{G}}_{(z',z)}\|_{((H^{(s)})^k,(H^{(s)})^k)} \leq C,$$

for Δ sufficiently small. Without any loss of generality we can thus assume that $g_{(z',z)}(x, \xi) = I_k$, i.e., $a_0(z, x, \xi) = 0$.

Let $s \in \mathbb{R}$. Then the kernel of $\mathcal{B}_{(z',z)} := \mathcal{G}_{(z',z)} \circ E^{(-s)}$ is given by

$$B_{(z',z)}(x', x) = \int e^{i\langle x'-x|\xi\rangle} e^{-\Delta a_1(z,x',\xi)} \langle\xi\rangle^{-s}\, d\!\!\!{}^-\xi.$$

The kernel of the adjoint operator of $\mathcal{B}_{(z',z)}$ is given by

$$B^*_{(z',z)}(x', x) = \int e^{i\langle x'-x|\xi\rangle} e^{-i\Delta b_1(z,x,\xi)-\Delta c_1(z,x,\xi)} \langle\xi\rangle^{-s}\, d\!\!\!{}^-\xi,$$

since the matrix $b_1(z, x, \xi)$ and $c_1(z, x, \xi)$ are hermitian symmetric. Introducing $\mathcal{D}_{(z',z)} = \mathcal{B}_{(z',z)} \circ \mathcal{B}^*_{(z',z)}$, we find its kernel to be

$$D_{(z',z)}(x', x) = \int e^{i\langle x'-x|\xi\rangle} e^{-\Delta a_1(z,x',\xi)} e^{-i\Delta b_1(z,x,\xi)-\Delta c_1(z,x,\xi)} \langle\xi\rangle^{-2s}\, d\!\!\!{}^-\xi,$$

which we write

$$D_{(z',z)}(x', x) = \int e^{i\langle x'-x|\xi\rangle} \left(w(z,.)e^{-\Delta d(z,.)}v(z,.)\right)(x', \xi) \left(w(z,.)e^{-\Delta\bar{d}(z,.)}v(z,.)\right)(x, \xi) \langle\xi\rangle^{-2s}\, d\!\!\!{}^-\xi.$$

With Taylor's formula we write

$$v(z, x', \xi) = v(z, x, \xi) + \langle x' - x|\tilde{v}(z, x', x, \xi)\rangle,$$

with $\tilde{v}(z, x', x, \xi)$ bounded w.r.t. z in $(\mathcal{M}_k S^0(X' \times X \times \mathbb{R}^n))^n$ by (1.1.9) in [5]. This yields

$$D_{(z',z)}(x', x) = D_{(z',z),a}(x', x) + D_{(z',z),b}(x', x) + D_{(z',z),c}(x', x),$$

where

$$D_{(z',z),a}(x', x) = \int e^{i\langle x'-x|\xi\rangle} w(z, x', \xi) e^{-\Delta(d(z,x',\xi)+\bar{d}(z,x,\xi))} v(z, x', \xi) \langle\xi\rangle^{-2s}\, d\!\!\!{}^-\xi,$$

$$D_{(z',z),b}(x', x) = \int e^{i\langle x'-x|\xi\rangle} w(z, x', \xi) e^{-\Delta d(z,x',\xi)} \langle x' - x|\tilde{v}(z, x', x, \xi)\rangle\, w(z, x, \xi) e^{-\Delta\bar{d}(z,x,\xi)} v(z, x, \xi) \langle\xi\rangle^{-2s}\, d\!\!\!{}^-\xi,$$

and

$$D_{(z',z),c}(x', x) = -\int e^{i\langle x'-x|\xi\rangle} w(z, x', \xi) e^{-\Delta(d(z,x',\xi)+\bar{d}(z,x,\xi))} \langle x' - x|\tilde{v}(z, x', x, \xi)\rangle \langle\xi\rangle^{-2s}\, d\!\!\!{}^-\xi.$$

We can prove that the associated operators, namely $\mathcal{D}_{(z',z),a}$, $\mathcal{D}_{(z',z),b}$, and $\mathcal{D}_{(z',z),c}$, satisfy

$$\|E^{(s)} \circ \mathcal{D}_{(z',z),a} \circ E^{(s)}\|_{((L^2)^k,(L^2)^k)} \le 1 + C\Delta,$$

$$\|E^{(s)} \circ (\mathcal{D}_{(z',z),b} + \mathcal{D}_{(z',z),c}) \circ E^{(s)}\|_{((L^2)^k,(L^2)^k)} \le C\Delta,$$

for some $C \ge 0$, uniformly in $z \in [0, Z]$ and Δ, Δ sufficiently enough. The first estimate is obtained by using Theorem 1.19. ∎

2 Convergence properties of the Ansatz $\mathcal{W}_{\mathfrak{P},z}$ in the symmetric case

As in Section 1, the z-family of symbols $a_1(z,.)$ satisfies Assumptions 1.1 and 1.2. Let $\rho \in [\frac{1}{2}, 1]$. We assume that c_1 is chosen such that

$$p_\Delta(z,.) := e^{-\Delta c_1(z,.)} \tag{1}$$

takes values in $\mathcal{M}_k S^0_\rho(X \times \mathbf{R}^n)$ (see Lemma 1.9 and corollary 1.14).

We first define the Ansatz that approximates the solution operator to (1)–(2). The regularity properties of the thin-slab propagator $\mathcal{G}_{(z',z)}$ given in Proposition 1.4 allow to compose operators of the form of $\mathcal{G}_{(z',z)}$.

We chose to use a constant-step subdivision of the interval $[0, Z]$ but the method and results presented here can be naturally adapted to any subdivision of $[0, Z]$.

Definition 2.1. *Let $\mathfrak{P} = \{z^{(0)}, z^{(1)}, \ldots, z^{(N)}\}$ be a subdivision of $[0, Z]$ with $0 = z^{(0)} < z^{(1)} < \cdots < z^{(N)} = Z$ such that $z^{(i+1)} - z^{(i)} = \Delta_{\mathfrak{P}}$. The operator $\mathcal{W}_{\mathfrak{P},z}$ is defined as*

$$\mathcal{W}_{\mathfrak{P},z} := \begin{cases} \mathcal{G}_{(z,0)} & \text{if } 0 \le z \le z^{(1)}, \\ \mathcal{G}_{(z,z^{(k)})} \prod_{i=k}^{1} \mathcal{G}_{(z^{(i)},z^{(i-1)})} & \text{if } z^{(k)} \le z \le z^{(k+1)}. \end{cases}$$

Thanks to the estimate proven in Theorem 1.20 we can now obtain the following proposition.

Proposition 2.2. *Let $s \in \mathbf{R}$. There exists $K \ge 0$ such that for every subdivision $\mathfrak{P}$ of $[0, Z]$ and $\mathcal{W}_{\mathfrak{P},z}$ as defined in Definition 2.1 we have*

$$\forall z \in [0, Z], \quad \|\mathcal{W}_{\mathfrak{P},z}\|_{((H^{(s)})^k,(H^{(s)})^k)} \le K,$$

for $\Delta_{\mathfrak{P}}$ sufficiently small.

Proof. By Theorem 1.20, there exits $M \ge 0$ such that for $\Delta = z' - z$ small enough we have $\|\mathcal{G}_{(z',z)}\|_{((H^{(s)})^k,(H^{(s)})^k)} \le 1 + \Delta M$ for all $z \in [0, Z]$; we then obtain

$$\|\mathcal{W}_{\mathfrak{P},z}\|_{((H^{(s)})^k,(H^{(s)})^k)} \le (1 + \Delta_{\mathfrak{P}} M)^N = \left(1 + \frac{ZM}{N}\right)^N$$

which is bounded as it converges to $\exp(ZM)$ as N goes to ∞. ∎

As in [18], we have the following regularity result for the Ansatz $\mathcal{W}_{\mathfrak{P},z}$.

Proposition 2.3. *Let $s \in \mathbf{R}$, $\mathfrak{P}$ a subdivision of $[0, Z]$ as in Definition 2.1 and let $u_0 \in (H^{(s+1)}(X))^k$. Then the map $z \mapsto \mathcal{W}_{\mathfrak{P},z}(u_0)$ is in $\mathscr{C}^0([0, Z], (H^{(s+1)}(X))^k)$ and is piecewise $\mathscr{C}^1([0, Z], (H^{(s)}(X))^k)$ if $\mathfrak{P}$ is chosen such that $\Delta_{\mathfrak{P}}$ is small enough. The map $z \mapsto \mathcal{W}_{\mathfrak{P},z}(u_0)$ is in fact globally Lipschitz with $C \ge 0$ such that*

$$\|\mathcal{W}_{\mathfrak{P},z'}(u_0) - \mathcal{W}_{\mathfrak{P},z}(u_0)\|_{(H^{(s)})^k} \le C|z' - z| \, \|u_0\|_{(H^{(s+1)})^k},$$

where the constant C is uniform w.r.t. $\mathfrak{P}$, z' and z, if $\Delta_{\mathfrak{P}}$ is sufficiently small.

Before proceeding to estimating the approximation error made between the Ansatz $W_{\mathfrak{P},z}$ and the solution operator $U(z, 0)$ of (1)–(2), we need to establish a ψDO-FIO composition formula adapted to the case of matrix phase functions such as $\phi_{(z',z)}$ given in (6).

Theorem 2.4. *let $\rho', \rho'' \in [\frac{1}{2}, 1]$. Let $\mu(z,x,\xi)$ be bounded w.r.t. z with values in $\mathcal{M}_k S^{m}_{\rho'}(X \times \mathbf{R}^n)$, and the operator $\mathcal{A}_{(z',z)}$ defined by*

$$\mathcal{A}_{(z',z)}(u)(x') = \iint e^{i\langle x'-x|\xi\rangle} \sigma_{(z',z)}(x',\xi) e^{-\Delta a_1(z,x',x,\xi)} u(x)\, d\!\!{}^-\xi\, dx,$$

where $0 \le z \le z' \le Z$, $\Delta = z' - z$, with $\sigma_{(z',z)}(x',\xi)$ bounded w.r.t. z' and z with values in $\mathcal{M}_k S^{m'}_{\rho''}(X \times \mathbf{R}^n)$. Then, for Δ sufficiently small, we have

$$\mu(z,x,D_x) \circ \mathcal{A}_{(z',z)} = \mathcal{B}_{(z',z)} + \Delta\, \mathcal{R}_{(z',z)},$$

where for all $s \in \mathbf{R}$ there exists $C \ge 0$ such that

(2) $$\|\mathcal{R}_{(z',z)}\|_{((H^{(s)})^k,(H^{(s-m-m')})^k)} \le C, \quad 0 \le z \le z' \le Z,$$

and the operator $\mathcal{B}_{(z',z)}$ has for kernel

$$B_{(z',z)}(x',x) = \int e^{i\langle x'-x|\xi\rangle} q_{(z',z)}(x',\xi) e^{i\Delta b_1(z,x',\xi)}\, d\!\!{}^-\xi,$$

with $q_{(z',z)}(x',\xi)$ bounded w.r.t. to z' and z with values in $\mathcal{M}_k S^{m+m'}_{\min(\rho,\rho',\rho'')}(X \times \mathbf{R}^n)$ and given by the oscillatory integral representation

(3) $$q_{(z',z)}(x',\xi) = \iint e^{i\langle x'-y|\eta-\xi\rangle} \mu(z,x',\eta)\, \sigma_{(z',z)}(y,\xi) p_\Delta(z,y,\xi)\, w(z,y,\xi) e^{i\Delta(d_b(z,y,\xi)-d_b(z,x',\xi))} v(z,y,\xi)\, d\!\!{}^-\eta\, dy.$$

with $p_\Delta(z,y,\xi)$ given in (1).

To estimate the norm of $\mathcal{W}_{\mathfrak{P},z}u_0 - U(z,0)(u_0)$ in some Sobolev space, where $U(z,0)$ is the solution operator of (1)–(2), we first need to have an understanding of the infinitesimal error made by the use of the thin-slab propagator, i.e., find a bound for

$$(\partial_{z'} + a_{z'}(x,D_x)) \circ \mathcal{G}_{(z',z)}(u), \quad 0 \le z \le z' \le Z,\ u \in (H^{(s)})^k,$$

in some properly chosen norm when $\Delta = z' - z$ is small. For the next proposition we shall need the following assumption.

Assumption 2.5. *The matrix symbol $a(z,.)$ is in $\mathscr{C}^{0,\alpha}([0,Z], \mathcal{M}_k S^1(X \times \mathbf{R}^n))$, i.e., Hölder continuous w.r.t. z with values in $\mathcal{M}_k S^1(X \times \mathbf{R}^n)$, in the sense that,*

$$a(z',x,\xi) - a(z,x,\xi) = (z'-z)^\alpha \tilde{a}(z',z,x,\xi), \quad 0 \le z \le z' \le Z$$

with $\tilde{a}(z',z,x,\xi)$ bounded w.r.t. z' and z with values in $\mathcal{M}_k S^1(\mathbf{R}^n \times \mathbf{R}^n)$.

It should be noted that Assumption 2.5 concerns the full symbol $a(z,.)$ and not simply its principal symbol.

Proposition 2.6. *Let $s \in \mathbf{R}$. There exist $\Delta_4 > 0$ and $C \ge 0$ such that for $z' - z = \Delta$, $\Delta \in [0, \Delta_4]$,*

(4) $$\|(\partial_{z'} + a_{z'}(x,D_x)) \circ \mathcal{G}_{(z',z)}\|_{((H^{(s)})^k,(H^{(s-1)})^k)} \le C\Delta^\alpha.$$

The proof is along the lines of that of Theorem 2.8 in [19] and uses the calculus result of Theorem 2.4 since in the present case phase functions are matrices.

Adapting the proof of energy estimate (3) to the case of piecewise $\mathscr{C}^1$ function yet globally Lipschitz functions like $\mathcal{W}_{\mathfrak{P},z}(u_0)$ (see Proposition 2.3) we find that

(5) $$\|U(z,0)(u_0) - \mathcal{W}_{\mathfrak{P},z}(u_0)\|_{(H^{(s)})^k} \le +C \int_0^Z \|(\partial_z + a(z,x,D_x))\mathcal{W}_{\mathfrak{P},z}(u_0)\|_{(H^{(s)})^k} dz,$$

with the constant C uniform w.r.t. z and and the subdivision $\mathfrak{P}$, for $u_0 \in (H^{(s+1)})^k$.

let $\mathfrak{P} = \{z^{(0)}, \ldots, z^{(N)}\}$. We take $z \in]z^{(j)}, z^{(j+1)}[$. Then

$$\begin{aligned}(\partial_z + a(z,x,D_x))\, \mathcal{W}_{\mathfrak{P},z}(u_0) &= (\partial_z + a(z,x,D_x)) \left(\mathcal{G}_{(z,z^{(j)})} \prod_{i=j}^{1} \mathcal{G}_{(z^{(i)},z^{(i-1)})}(u_0) \right) \\ &= (\partial_z + a(z,x,D_x)) \left(\mathcal{G}_{(z,z^{(j)})}(u_j) \right)\end{aligned}$$

with $u_j := \prod_{i=j}^{1} \mathcal{G}_{(z^{(i)},z^{(i-1)})}(u_0)$ which is in $(H^{(s+1)}(\mathbf{R}^n))^k$ by Theorem 1.20. By Proposition 2.2, the norm of u_j in $(H^{(s+1)}(\mathbf{R}^n))^k$ remains bounded even if $|\mathfrak{P}| = N$ becomes very large:

$$\exists K \geq 0,\ \|u_j\|_{(H^{(s+1)})^k} \leq K \|u_0\|_{(H^{(s+1)})^k},\ j \in \{0, \ldots, N\}, N = |\mathfrak{P}| \in \mathbf{N},\ u_0 \in (H^{(s+1)}(\mathbf{R}^n))^k,$$

if $\Delta_{\mathfrak{P}}$ is small enough. By Proposition 2.6, we thus obtain

(6) $$\|(\partial_z + a(z,x,D_x)) \mathcal{W}_{\mathfrak{P},z}(u_0)\|_{(H^{(s)})^k} \leq CK\Delta^{\alpha} \|u_0\|_{(H^{(s+1)})^k}, \quad z \in [0,Z] \setminus \mathfrak{P},$$

with the constants C and K uniform w.r.t. z and $\mathfrak{P}$.

An interpolation argument, as in [18] yields the main result of this Section.

Theorem 2.7. *Assume that the symbol $a(z,.)$ satisfies Assumptions 1.1 and 1.2, and is in $\mathscr{C}^{0,\alpha}([0,Z], \mathcal{M}_k S^1(\mathbf{R}^n \times \mathbf{R}^n))$, i.e. Hölder continuous w.r.t. z, with values in $\mathcal{M}_k S^1(\mathbf{R}^n \times \mathbf{R}^n)$, in the sense that, for some $0 < \alpha < 1$*

$$a(z', x, \xi) - a(z, x, \xi) = (z' - z)^{\alpha}\, \tilde{a}(z', z, x, \xi), \quad 0 \leq z \leq z' \leq Z,$$

or Lipschitz ($\alpha = 1$), with $\tilde{a}(z', z, x, \xi)$ bounded w.r.t. z' and z with values in $\mathcal{M}_k S^1(\mathbf{R}^n \times \mathbf{R}^n)$. Let $s \in \mathbf{R}$ and $0 \leq r < 1$. Then the approximation Ansatz $\mathcal{W}_{\mathfrak{P},z}$ converges to the solution operator $U(z,0)$ of the Cauchy problem (1)–(2) in $L((H^{(s+1)}(\mathbf{R}^n))^k, (H^{(s+r)}(\mathbf{R}^n))^k)$ uniformly w.r.t. z as $\Delta_{\mathfrak{P}}$ goes to 0 with a convergence rate of order $\alpha(1-r)$:

$$\|\mathcal{W}_{\mathfrak{P},z} - U(z,0)\|_{((H^{(s+1)})^k, (H^{(s+r)})^k)} \leq C \Delta_{\mathfrak{P}}^{\alpha(1-r)}, \quad z \in [0,Z].$$

Furthermore, the operator $\mathcal{W}_{\mathfrak{P},z}$ strongly converges to the solution operator $U(z,0)$, uniformly w.r.t. $z \in [0,Z]$, in $L((H^{(s+1)}(\mathbf{R}^n))^k, (H^{(s+1)}(\mathbf{R}^n))^k)$.

Proof. The case $r = 0$ is an immediate consequence of (5) and (6).

From energy estimate (3) for $s+1$ we have

(7) $$\|U(z,0)(u_0)\|_{(H^{(s+1)})^k} \leq C \|u_0\|_{(H^{(s+1)})^k}.$$

From Proposition 2.2 we obtain

(8) $$\|\mathcal{W}_{\mathfrak{P},z}(u_0)\|_{(H^{(s+1)})^k} \leq C \|u_0\|_{(H^{(s+1)})^k}$$

and thus

(9) $$\|\mathcal{W}_{\mathfrak{P},z}(u_0) - U(z,0)(u_0)\|_{(H^{(s+1)})^k} \leq C \|u_0\|_{(H^{(s+1)})^k},$$

uniformly w.r.t. $z \in [0,Z]$. The interpolation inequality

$$\|v\|_{(H^{(s+r)})^k} \leq C \|v\|_{(H^{(s)})^k}^{1-r} \|v\|_{(H^{(s+1)})^k}^{r}, \quad 0 \leq r \leq 1,$$

then yields

$$\|\mathcal{W}_{\mathfrak{P},z}(u_0) - U(z,0)(u_0)\|_{(H^{(s+r)})^k} \leq C \Delta_{\mathfrak{P}}^{\alpha(1-r)} \|u_0\|_{(H^{(s+1)})^k}, \quad 0 \leq r < 1,$$

uniformly w.r.t. $z \in [0,Z]$.

Let $u_0 \in (H^{(s+1)})^k$ and let $\varepsilon > 0$. For the strong convergence in $(H^{(s+1)})^k$ we choose $u_1 \in (H^{(s+2)})^k$ such that $\|u_0 - u_1\|_{(H^{(s+1)})^k} \leq \varepsilon$. We then write

$$\begin{aligned}\|\mathcal{W}_{\mathfrak{P},z}(u_0) - U(z,0)(u_0)\|_{(H^{(s+1)})^k} &\leq \|\mathcal{W}_{\mathfrak{P},z}(u_0 - u_1)\|_{(H^{(s+1)})^k} + \|\mathcal{W}_{\mathfrak{P},z}(u_1) - U(z,0)(u_1)\|_{(H^{(s+1)})^k} \\ &\quad + \|U(z,0)(u_0 - u_1)\|_{(H^{(s+1)})^k} \\ &\leq C\varepsilon + C\Delta_{\mathfrak{P}}^{\alpha} \|u_1\|_{(H^{(s+2)})^k}\end{aligned}$$

from estimates (7) and (8) and the case $r = 0$ of the first part of the Theorem. This last estimate is uniform w.r.t. $z \in [0,Z]$ and yields the result. ■

3 Symmetrizable systems

In this section, we consider the more general situation where the matrix symbol a_1 is symmetrizable. Namely we make the following assumption.

Assumption 3.1. *There exists a $k \times k$ invertible matrix $L(z,x,\xi)$ that is bounded w.r.t. z with values in $\mathcal{M}_k S^0(X \times \mathbf{R}^n)$, homogeneous of degree zero in ξ, $|\xi| \geq 1$, with $(L(z,x,\xi))^{-1}$ satisfying the same property, and such that*

$$a_1(z,x,\xi) = L(z,x,\xi)\, \alpha_1(z,x,\xi)\, (L(z,x,\xi))^{-1},$$

with $\alpha_1 = -i\beta_1 + \gamma_1$ satisfying Assumptions 1.1 and 1.2.

Note that this formulation is in fact equivalent to that in which we choose $L(z,x,\xi)$ to be itself hermitian symmetric or to the formulation given in [1]: we have

$$S(z,x,\xi)\, a(z,x,\xi) = ((L(z,x,\xi))^{-1})^t\, \alpha_1(z,x,\xi)\, (L(z,x,\xi))^{-1},$$

with $S(z,x,\xi) = ((L(z,x,\xi))^{-1})^t\, L(z,x,\xi))^{-1}$ which is hermitian symmetric.

We shall make the additional following assumption.

Assumption 3.2. *The matrix symbol $L(z,x,\xi)$ is Lipschitz continuous, in the sense that*

$$L(z',x,\xi) - L(z,x,\xi) = (z'-z)\widetilde{L}(z',z,x,\xi),$$

with $\widetilde{L}(z',z,x,\xi)$ bounded w.r.t. z' and z with values in $\mathcal{M}_k S^0(X \times \mathbf{R}^n)$.

The same property naturally follows for the matrix symbol $R(z,x,\xi) := (L(z,x,\xi))^{-1}$.

Example 3.3. The first-order system that describes linear anisotropic elastodynamic, written in terms of velocity and stress field, is smoothly symmetrizable if multiplicities remain constant. Similarly, Maxwell's equations are a possible application of the results of this article if multiplicities remain constants. Conical refraction in crystal optics is thus not considered here.

With the two assumptions made, the energy estimate (3) remains valid [1, Chapter VI] and there exists a unique solution to the Cauchy problem (1)–(2) in $\mathscr{C}([0,Z],(H^{(s+1)}(\mathbf{R}^n))^k) \cap \mathscr{C}^1([0,Z],(H^{(s)}(\mathbf{R}^n))^k)$.

The thin-slab propagator $\mathcal{G}_{(z',z)}$ is defined as in Section 1. We check that it satisfies the regularity properties of Proposition 1.4. The approximation Ansatz $\mathcal{W}_{\mathfrak{P},z}$ can be defined as in Section 2. As in the previous sections, we may assume that $e^{-\Delta d_\gamma(z,.)}$ and $e^{-\Delta\gamma_1(z,.)}$ take values in $\mathcal{M}_k S^0_\rho(X \times \mathbf{R}^n)$, $\rho \in [\frac{1}{2}, 1]$ (see Lemma 1.9 and corollary 1.14).

3.1 Composition of two thin-slab propagators

Because the matrix symbol $L(z,x,\xi)$ is not unitary we cannot proceed as in Section 1 and obtain an estimate for the Sobolev operator norm of $\mathcal{G}_{(z',z)}$ as in Theorem 1.20.

We instead investigate the product $\mathcal{G}_{(z'',z')} \circ \mathcal{G}_{(z',z)}$ with $0 \leq z \leq z' \leq z'' \leq Z$, as it appears in the definition of the Ansatz $\mathcal{W}_{\mathfrak{P},z}$ in Section 2.

We define the following matrix-phase FIOs

$$\mathcal{H}^l_{(z',z)}(u)(x') := \iint e^{i\langle x'-x|\xi\rangle} g_{(z',z)}(x',\xi) L(z,x',\xi) e^{-\Delta\alpha_1(z,x',\xi)} u(x)\, dx\, d\!\!{}^-\xi, \tag{1}$$

$$\mathcal{H}^r_{(z',z)}(u)(x') := \iint e^{i\langle x'-x|\xi\rangle} g_{(z',z)}(x',\xi) e^{-\Delta\alpha_1(z,x',\xi)} R(z,x',\xi)\, u(x)\, dx\, d\!\!{}^-\xi, \tag{2}$$

$$\mathcal{H}_{(z',z)}(u)(x') := \iint e^{i\langle x'-x|\xi\rangle} g_{(z',z)}(x',\xi) e^{-\Delta\alpha_1(z,x',\xi)}\, u(x)\, dx\, d\!\!{}^-\xi, \tag{3}$$

and

$$\mathcal{H}^{lr}_{(z',z)}(u)(x') := \mathcal{G}_{(z',z)}(u)(x'), \quad u \in (H^{(-\infty)}(\mathbf{R}^n))^k. \tag{4}$$

Proposition 3.4. *Let $s \in \mathbb{R}$. There exists an operator $\mathcal{K}_{(z'',z',z)}$ bounded from $(H^{(s)}(\mathbb{R}^n))^k$ into $(H^{(s)}(\mathbb{R}^n))^k$, uniformly w.r.t. z'', z' and z, such that*

$$\mathcal{H}^{lr}_{(z'',z')} \circ \mathcal{H}^{lr}_{(z',z)} = \mathcal{H}^{l}_{(z'',z')} \circ \mathcal{H}^{r}_{(z',z)} + \max(\Delta, \Delta')\mathcal{K}_{(z'',z',z)} + \mathcal{M}(z'', x, D_x),$$

for $\Delta = z' - z$ and $\Delta' = z'' - z'$ both sufficiently small, and where

$$\mathcal{M}(z, x, D_x) := I - L(z, x, D_x) \circ R(z, x, D_x). \tag{5}$$

In the sequel, we shall often write $\mathcal{M}_z$ in place of $\mathcal{M}(z, x, D_x)$ for concision.

Proposition 3.5. *Let $s \in \mathbb{R}$. There exists an operator $\mathcal{K}_{(z'',z',z)}$ bounded from $(H^{(s)}(\mathbb{R}^n))^k$ into $(H^{(s)}(\mathbb{R}^n))^k$ uniformly w.r.t. z'', z' and z such that*

$$\mathcal{H}^{lr}_{(z'',z')} \circ \mathcal{H}^{l}_{(z',z)} = \mathcal{H}^{l}_{(z'',z')} \circ \mathcal{H}_{(z',z)} + \max(\Delta', \Delta)\mathcal{K}_{(z'',z',z)},$$

for Δ' and Δ sufficiently small.

Finally, we shall use the following result.

Proposition 3.6. *Let $s \in \mathbb{R}$. There exists an operator $\mathcal{K}_{(z',z)}$ bounded from $(H^{(s)}(\mathbb{R}^n))^k$ into $(H^{(s)}(\mathbb{R}^n))^k$ uniformly w.r.t. z' and z such that*

$$\mathcal{H}^{lr}_{(z',z)} \circ \mathcal{M}(z, x, D_x) = \mathcal{M}(z', x, D_x) + \Delta\mathcal{K}_{(z',z)},$$

for Δ sufficiently small.

3.2 Stability of the Ansatz $\mathcal{W}_{\varphi,z}$ and conclusion

Let $s \in \mathbb{R}$. Let $K \geq 0$. We shall denote by $\mathcal{K}$ a generic operator continuous from $(H^{(s)}(\mathbb{R}^n))^k$ into $(H^{(s)}(\mathbb{R}^n))^k$ such that $\|\mathcal{K}\|_{((H^{(s)})^k,(H^{(s)})^k)} \leq K$. We now define notations for some operators. In the notation $\mathcal{J}_{(z^{(l')},z^{(l)})}$ below, we are solely interested in the form of the operator and by its norm estimate rather than by its precise definition. Thus, in the definition of $\mathcal{J}_{(z^{(l')},z^{(l)})}$, the operators denoted by $\mathcal{K}$ may change from one term to another. We choose to make this abuse of notation for the sake of concision.

Definition 3.7. *Let $N \in \mathbb{N}$. Let $0 = z^{(0)} \leq z^{(1)} \leq \cdots \leq z^{(N)} \leq Z$. For $0 \leq l \leq l' \leq N$, we set*

$$\mathcal{G}_{(z^{(l')},\ldots,z^{(l)})} := \begin{cases} \mathrm{Id} & \text{if } l' = l, \\ \mathcal{G}_{(z^{(l')},z^{(l'-1)})} & \text{if } l' - 1 = l, \\ \mathcal{G}_{(z^{(l')},z^{(l'-1)})} \circ \cdots \circ \mathcal{G}_{(z^{(l+1)},z^{(l)})} & \text{otherwise,} \end{cases}$$

$$\mathcal{H}^{lr}_{(z^{(l')},\ldots,z^{(l)})} := \begin{cases} \mathrm{Id} & \text{if } l' = l, \\ \mathcal{H}^{lr}_{(z^{(l')},z^{(l'-1)})} = \mathcal{G}_{(z^{(l')},z^{(l'-1)})} & \text{if } l' - 1 = l, \\ \mathcal{H}^{l}_{(z^{(l')},z^{(l'-1)})} \circ \mathcal{H}_{(z^{(l'-1)},z^{(l'-2)})} \circ \cdots \circ \mathcal{H}_{(z^{(l+2)},z^{(l+1)})} \circ \mathcal{H}^{r}_{(z^{(l+1)},z^{(l)})} & \text{otherwise,} \end{cases}$$

and

$$\mathcal{H}^{r}_{(z^{(l')},\ldots,z^{(l)})} := \begin{cases} \mathrm{Id} & \text{if } l' = l, \\ \mathcal{H}^{r}_{(z^{(l')},z^{(l'-1)})} & \text{if } l' - 1 = l, \\ \mathcal{H}_{(z^{(l')},z^{(l'-1)})} \circ \cdots \circ \mathcal{H}_{(z^{(l+2)},z^{(l+1)})} \circ \mathcal{H}^{r}_{(z^{(l+1)},z^{(l)})} & \text{otherwise.} \end{cases}$$

The reader should note that $\mathcal{H}^{lr}_{(z^{(l')},z^{(l)})} = \mathcal{G}_{(z^{(l')},z^{(l)})}$ but $\mathcal{H}^{lr}_{(z^{(l')},\ldots,z^{(l)})} \neq \mathcal{G}_{(z^{(l')},\ldots,z^{(l)})}$, if $l' - l \geq 2$. For $0 \leq l \leq l' \leq N$, we *denote by* $\mathcal{J}_{(z^{(l')},z^{(l)})}$ *an operator of the form* ($\mathcal{J}_{(z^{(l)},z^{(l)})} = \mathrm{Id}$)

$$
\begin{aligned}
(6)\qquad \mathcal{J}_{(z^{(l')},z^{(l)})} =& \mathcal{H}^{lr}_{(z^{(l')},\dots,z^{(l)})} \\
&+ \Delta \sum_{l+1\le m_1\le l'-1} \mathcal{H}^{lr}_{(z^{(l')},\dots,z^{(m_1+1)})} \circ \mathcal{K} \circ \mathcal{H}^{r}_{(z^{(m_1-1)},\dots,z^{(l)})} \\
&\qquad\vdots \\
&+ \Delta^{r} \sum_{\substack{l+2r-1\le m_r\le l'-1\\ \vdots \\ l+3\le m_2\le m_3-2\\ l+1\le m_1\le m_2-2}} \mathcal{H}^{lr}_{(z^{(l')},\dots,z^{(m_r+1)})} \circ \mathcal{K} \circ \mathcal{H}^{r}_{(z^{(m_r-1)},\dots,z^{(m_{r-1}+1)})} \circ \cdots \\
&\qquad\qquad \cdots \circ \mathcal{H}^{r}_{(z^{(m_2-1)},\dots,z^{(m_1+1)})} \circ \mathcal{K} \circ \mathcal{H}^{r}_{(z^{(m_1-1)},\dots,z^{(l)})} \\
&\qquad\vdots \\
&+ \Delta^{r'} \sum_{\substack{l+2r'-1\le m_{r'}\le l'-1\\ \vdots \\ l+3\le m_2\le m_3-2\\ l+1\le m_1\le m_2-2}} \mathcal{H}^{lr}_{(z^{(l')},\dots,z^{(m_{r'}+1)})} \circ \mathcal{K} \circ \mathcal{H}^{r}_{(z^{(m_{r'}-1)},\dots,z^{(m_{r'-1}+1)})} \circ \cdots \\
&\qquad\qquad \cdots \circ \mathcal{H}^{r}_{(z^{(m_2-1)},\dots,z^{(m_1+1)})} \circ \mathcal{K} \circ \mathcal{H}^{r}_{(z^{(m_1-1)},\dots,z^{(l)})},
\end{aligned}
$$

with $r' = \mathrm{E}((l'-l)/2)$.

In the sequel, the "order" of a term will refer to the power of Δ appearing in it.

Remark 3.8. Observe that in the definition of $\mathcal{J}_{(z^{(l')},z^{(l)})}$, in the case where $l'-l$ is even, the last term is in fact $\Delta^{(l'-l)/2}\mathcal{K}\circ\cdots\circ\mathcal{K}$, with the generic operator $\mathcal{K}$ appearing $(l'-l)/2$ times (we do not write $\mathcal{K}^{(l'-l)/2}$ since the operator $\mathcal{K}$ may not be the same each time). In the case where $l'-l$ is odd, then, there remains one operator of the type $\mathcal{H}^{r}_{(z^{(m-1)},\dots,z^{(m'+1)})}$ or $\mathcal{H}^{lr}_{(z^{(l')},\dots,z^{(m+1)})}$ in each term of order $\mathrm{E}((l'-l)/2) = (l'-1-l)/2$. Basically, in each term in the sums above, the operator $\mathcal{K}$ replaces the occurrence of two consecutive operators of the type given in (1)–(4) and we cover allpossible cases in the sums. We write the first examples of the operators $\mathcal{J}_{(z^{(l')},z^{(l)})}$ to get used to their form:

$$
\begin{aligned}
\mathcal{J}_{(z^{(l+1)},z^{(l)})} &= \mathcal{H}^{lr}_{(z^{(l+1)},z^{(l)})}, \\
\mathcal{J}_{(z^{(l+2)},z^{(l)})} &= \mathcal{H}^{lr}_{(z^{(l+2)},\dots,z^{(l)})} + \Delta\mathcal{K}, \\
\mathcal{J}_{(z^{(l+3)},z^{(l)})} &= \mathcal{H}^{lr}_{(z^{(l+3)},\dots,z^{(l)})} + \Delta\left(\mathcal{K}\circ\mathcal{H}^{r}_{(z^{(l+1)},z^{(l)})} + \mathcal{H}^{lr}_{(z^{(l+3)},z^{(l+2)})}\circ\mathcal{K}\right), \\
\mathcal{J}_{(z^{(l+4)},z^{(l)})} &= \mathcal{H}^{lr}_{(z^{(l+4)},\dots,z^{(l)})} \\
&\quad + \Delta\left(\mathcal{K}\circ\mathcal{H}^{r}_{(z^{(l+2)},\dots,z^{(l)})} + \mathcal{H}^{lr}_{(z^{(l+4)},z^{(l+3)})}\circ\mathcal{K}\circ\mathcal{H}^{r}_{(z^{(l+1)},z^{(l)})} + \mathcal{H}^{lr}_{(z^{(l+4)},\dots,z^{(l+2)})}\circ\mathcal{K}\right) \\
&\quad + \Delta^2\mathcal{K}\circ\mathcal{K},
\end{aligned}
$$

etc.

As in Section 2, we shall use uniform subdivisions of $[0,Z]$ but the method and results presented here can be naturally adapted to any subdivision of $[0,Z]$. We give an estimation of the operator norm of $\mathcal{J}_{(z^{(l')},z^{(l)})}$ that relies on the sharp estimation of the Sobolev operator norm of the thin-slab propagator obtained in Theorem 1.20 in the case of a symmetric system. The result of Theorem 1.20 in fact applies to the operator $\mathcal{H}_{(z',z)}$, defined in (3), by Assumption 3.1.

Lemma 3.9. *There exist* $S\ge 0$ *and* $C\ge 0$ *such that, for all subdivision* $\mathfrak{P}$ *of* $[0,Z]$, $\mathfrak{P} = \{z^{(0)}, z^{(1)}, \dots, z^{(N)}\}$, *with* $0 = z^{(0)} < z^{(1)} < \cdots < z^{(N)} = Z$ *and* $z^{(i+1)} - z^{(i)} = \Delta_{\mathfrak{P}}$, *we have*

$$
(7)\qquad \|\mathcal{J}_{(z^{(l')},z^{(l)})}\|_{((H^{(s)})^k,(H^{(s)})^k)} \le S^2 e^{CZ}, \quad 0\le l\le l'\le N,
$$

for $\Delta_{\mathfrak{P}}$ *sufficiently small.*

Proof. From Proposition 1.4 and Theorem 1.20, there exist $S\ge 0$ and $M\ge 0$ such that

$$
\begin{aligned}
&\|\mathcal{H}^{r}_{(z',z)}\|_{((H^{(s)})^k,(H^{(s)})^k)} \le S, \quad \|\mathcal{H}^{lr}_{(z',z)}\|_{((H^{(s)})^k,(H^{(s)})^k)} \le S, \\
&\|\mathcal{H}^{l}_{(z',z)}\|_{((H^{(s)})^k,(H^{(s)})^k)} \le S, \quad \text{and, } \|\mathcal{H}_{(z',z)}\|_{((H^{(s)})^k,(H^{(s)})^k)} \le 1 + M\Delta,
\end{aligned}
$$

uniformly w.r.t. z' and z, $0 \le z \le z' \le Z$, for $\Delta = z' - z$ sufficiently small.

We choose $S > 1$ and $\Delta_{\mathfrak{P}}$ sufficiently small such that $1 + M\Delta \le S$ and to apply the invoked properties. There is no loss of generality in assuming $l = 0$.

If we consider the generic term in the sum of order r in the definition of $\mathcal{J}_{(z^{(l')},z^{(0)})}$ we find

$$\| \mathcal{H}^{lr}_{(z^{(l')},\dots,z^{(m_r+1)})} \circ \mathcal{K} \circ \mathcal{H}^{r}_{(z^{(m_r-1)},\dots,z^{(m_{r-1}+1)})} \circ \cdots$$
$$\cdots \circ \mathcal{H}^{r}_{(z^{(m_2-1)},\dots,z^{(m_1+1)})} \circ \mathcal{K} \circ \mathcal{H}^{r}_{(z^{(m_1-1)},\dots,z^{(0)})} \|_{((H^{(s)})^{\star},(H^{(s)})^{\star})} \le S^{r+2} K^r (1 + M\Delta)^{l'-3r-2}.$$

The number of terms in the sum of order r is less than $(l'-1)(l'-3)\cdots(l'-2r+1)/r!$. In any case, l' being even or odd, we can estimate this number of terms from above by $2^r\binom{\mathrm{E}(l'/2)}{r}$. In fact, the number of terms in the sum is over estimated but this estimation will suffice to our purpose. We obtain

$$\left\| \Delta^r \sum_{\substack{2r-1 \le m_r \le l'-1 \\ \vdots \\ 1 \le m_1 \le m_2 - 2}} \begin{array}{l} \mathcal{H}^{lr}_{(z^{(l')},\dots,z^{(m_r+1)})} \circ \mathcal{K} \circ \mathcal{H}^{r}_{(z^{(m_r-1)},\dots,z^{(m_{r-1}+1)})} \circ \cdots \\ \quad \cdots \circ \mathcal{H}^{r}_{(z^{(m_2-1)},\dots,z^{(m_1+1)})} \circ \mathcal{K} \circ \mathcal{H}^{r}_{(z^{(m_1-1)},\dots,z^{(0)})} \end{array} \right\|_{((H^{(s)})^{\star},(H^{(s)})^{\star})}$$
$$\le (2\Delta)^r \binom{\mathrm{E}(l'/2)}{r} K^r S^{r+2} (1 + M\Delta)^{l'-3r-2}.$$

Observe that this estimation is sharp in the case $r = 0$ but becomes rather crude when r becomes large, from the over-estimation made above. In particular, for $r = \mathrm{E}(l'/2)$ this estimate is much larger than the estimates $\Delta^{\mathrm{E}(l'/2)} K^r$ (in the case l' is even) and $\frac{l'+1}{2} S\, \Delta^{\mathrm{E}(l'/2)} K^r$ (in the case l' is odd) that we can directly obtain.

Summing the estimates we obtain

$$\|\mathcal{J}_{(z^{(l')},z^{(0)})}\|_{((H^{(s)})^{\star},(H^{(s)})^{\star})} \le \sum_{r=0}^{\mathrm{E}(l'/2)} (2\Delta)^r \binom{\mathrm{E}(l'/2)}{r} K^r S^{r+2} (1 + M\Delta)^{l'-3r-2}$$
$$= S^2 (1 + M\Delta)^{-2} \sum_{r=0}^{\mathrm{E}(l'/2)} \binom{\mathrm{E}(l'/2)}{r} \left(\frac{2\Delta S K}{1 + M\Delta} \right)^r \left((1 + M\Delta)^2 \right)^{l'/2-r}.$$

In the case where l' is even we obtain

$$\|\mathcal{J}_{(z^{(l')},z^{(0)})}\|_{((H^{(s)})^{\star},(H^{(s)})^{\star})} \le S^2 (1 + M\Delta)^{-2} \left(\frac{2\Delta S K}{1 + M\Delta} + (1 + M\Delta)^2 \right)^{\mathrm{E}(l'/2)}.$$

Thus, there exists $C \ge 0$ such that

$$\|\mathcal{J}_{(z^{(l')},z^{(0)})}\|_{((H^{(s)})^{\star},(H^{(s)})^{\star})} \le S^2 (1 + C\Delta)^{\mathrm{E}(l'/2)} \le S^2 (1 + ZC/N)^N,$$

which is bounded, with $S^2 \exp(CZ)$ as an upper bound. The case where l' is odd yields a similar bound. ∎

With the results of Propositions 3.4 to 3.6 we now compute $\mathcal{H}^{lr}_{(z^{(l'+1)},z^{(l')})} \circ \mathcal{J}_{(z^{(l')},z^{(l)})}$, which will be needed below.

Lemma 3.10. *For $\Delta_{\mathfrak{P}}$ sufficiently small, i.e. N large, for $l' - l \ge 3$, we have*

$$\mathcal{H}^{lr}_{(z^{(l'+1)},z^{(l')})} \circ \mathcal{J}_{(z^{(l')},z^{(l)})} = \mathcal{J}_{(z^{(l'+1)},z^{(l)})} + \Delta_{\mathfrak{P}} \mathcal{M}_{z^{(l'+1)}} \circ \mathcal{K} \circ \widetilde{\mathcal{J}}_{(z^{(l'-3)},z^{(l)})},$$

where $\widetilde{\mathcal{J}}_{(z^{(l')},z^{(l)})}$ *is given by* ($\widetilde{\mathcal{J}}_{(z^{(l)},z^{(l)})} = \mathrm{Id}$)

$$
\begin{aligned}
\widetilde{\mathcal{J}}_{(z^{(l')},z^{(l)})} &= \mathcal{H}^r_{(z^{(l')},\dots,z^{(l)})} \\
&+ \Delta_{\mathfrak{P}} \sum_{l+1 \le m_1 \le l'-1} \mathcal{H}^r_{(z^{(l')},\dots,z^{(m_1+1)})} \circ \mathcal{K} \circ \mathcal{H}^r_{(z^{(m_1-1)},\dots,z^{(l)})} \\
&\quad \vdots \\
&+ \Delta^r_{\mathfrak{P}} \sum_{\substack{l+2r-1 \le m_r \le l'-1 \\ \vdots \\ l+3 \le m_2 \le m_3-2 \\ l+1 \le m_1 \le m_2-2}} \mathcal{H}^r_{(z^{(l')},\dots,z^{(m_r+1)})} \circ \mathcal{K} \circ \mathcal{H}^r_{(z^{(m_r-1)},\dots,z^{(m_{r-1}+1)})} \circ \cdots \\
&\qquad\qquad \cdots \circ \mathcal{H}^r_{(z^{(m_2-1)},\dots,z^{(m_1+1)})} \circ \mathcal{K} \circ \mathcal{H}^r_{(z^{(m_1-1)},\dots,z^{(l)})} \\
&\quad \vdots \\
&+ \Delta^{r'}_{\mathfrak{P}} \sum_{\substack{l+2r'-1 \le m_{r'} \le l'-1 \\ \vdots \\ l+3 \le m_2 \le m_3-2 \\ l+1 \le m_1 \le m_2-2}} \mathcal{H}^r_{(z^{(l')},\dots,z^{(m_{r'}+1)})} \circ \mathcal{K} \circ \mathcal{H}^r_{(z^{(m_{r'}-1)},\dots,z^{(m_{r'-1}+1)})} \circ \cdots \\
&\qquad\qquad \cdots \circ \mathcal{H}^r_{(z^{(m_2-1)},\dots,z^{(m_1+1)})} \circ \mathcal{K} \circ \mathcal{H}^r_{(z^{(m_1-1)},\dots,z^{(l)})}.
\end{aligned}
$$

with $r' = \mathrm{E}((l'-l)/2)$. *We have*

$$
\text{(8)} \qquad \|\widetilde{\mathcal{J}}_{(z^{(l')},z^{(l)})}\|_{((H^{(s)})^k,(H^{(s)})^k)} \le S\, e^{CZ}, \quad 0 \le l \le l' \le N.
$$

for the same constants S *and* C *as in (7).*

Note that the definition of $\widetilde{\mathcal{J}}_{(z^{(l')},z^{(l)})}$ is similar to that of $\mathcal{J}_{(z^{(l')},z^{(l)})}$ with the terms $\mathcal{H}^{lr}_{(z',\dots,z)}$ replaced by $\mathcal{H}^r_{(z',\dots,z)}$.

We now focus on the estimation of the operator norm of $\mathcal{G}_{(z^{(l)},\dots,z^{(0)})} = \mathcal{H}^{lr}_{(z^{(l)},z^{(l-1)})} \circ \cdots \circ \mathcal{H}^{lr}_{(z^{(1)},z^{(0)})}$, i.e. the question of the stability of the Ansatz $\mathcal{W}_{\mathfrak{P},z}$. In the method we shall use, operators of the form of $\widetilde{\mathcal{J}}_{(z^{(l')},z^{(l)})}$ appear, for which we can now bound the operator norm uniformly w.r.t. $N = |\mathfrak{P}|$. We have seen above, in Proposition 3.4, that

$$
\begin{aligned}
\mathcal{G}_{(z^{(2)},\dots,z^{(0)})} &= \mathcal{H}^{lr}_{(z^{(2)},z^{(1)})} \circ \mathcal{H}^{lr}_{(z^{(1)},z^{(0)})} = \mathcal{H}^{l}_{(z^{(2)},z^{(1)})} \circ \mathcal{H}^{r}_{(z^{(1)},z^{(0)})} + \Delta_{\mathfrak{P}}\mathcal{K} + \mathcal{M}_{z^{(2)}} \\
&= \mathcal{H}^{lr}_{(z^{(2)},\dots,z^{(0)})} + \Delta_{\mathfrak{P}}\mathcal{K} + \mathcal{M}_{z^{(2)}} = \mathcal{J}_{(z^{(2)},z^{(0)})} + \mathcal{M}_{z^{(2)}}.
\end{aligned}
$$

Composing with $\mathcal{H}^{lr}_{(z^{(3)},z^{(2)})}$ on the l.h.s. we obtain

$$
\begin{aligned}
\mathcal{G}_{(z^{(3)},\dots,z^{(0)})} &= \mathcal{H}^{lr}_{(z^{(3)},z^{(2)})} \circ \mathcal{H}^{lr}_{(z^{(2)},z^{(1)})} \circ \mathcal{H}^{lr}_{(z^{(1)},z^{(0)})} \\
&= \mathcal{H}^{lr}_{(z^{(3)},z^{(2)})} \circ \left(\mathcal{H}^{lr}_{(z^{(2)},\dots,z^{(0)})} + \Delta_{\mathfrak{P}}\mathcal{K} + \mathcal{M}_{z^{(2)}}\right) \\
&= \mathcal{H}^{lr}_{(z^{(3)},\dots,z^{(0)})} + \Delta_{\mathfrak{P}}\left(\mathcal{K} \circ \mathcal{H}^{r}_{(z^{(1)},z^{(0)})} + \mathcal{H}^{lr}_{(z^{(3)},z^{(2)})} \circ \mathcal{K}\right) + \mathcal{H}^{lr}_{(z^{(3)},z^{(2)})} \circ \mathcal{M}_{z^{(2)}} \\
&= \mathcal{J}_{(z^{(3)},z^{(0)})} + \mathcal{H}^{lr}_{(z^{(3)},z^{(2)})} \circ \mathcal{M}_{z^{(2)}} = \mathcal{J}_{(z^{(3)},z^{(0)})} + \mathcal{M}_{z^{(3)}} + \Delta_{\mathfrak{P}}\mathcal{K},
\end{aligned}
$$

by Proposition 3.5 and Proposition 3.6. We carry on with these explicit computations to derive the form of $\mathcal{G}_{(z^{(5)},\dots,z^{(0)})} = \mathcal{H}^{lr}_{(z^{(5)},z^{(4)})} \circ \cdots \circ \mathcal{H}^{lr}_{(z^{(1)},z^{(0)})}$. We have

$$
\begin{aligned}
\mathcal{G}_{(z^{(4)},\dots,z^{(0)})} &= \mathcal{H}^{lr}_{(z^{(4)},z^{(3)})} \circ \cdots \circ \mathcal{H}^{lr}_{(z^{(1)},z^{(0)})} = \mathcal{H}^{lr}_{(z^{(4)},z^{(3)})} \circ (\mathcal{J}_{(z^{(3)},z^{(0)})} + \mathcal{M}_{z^{(3)}} + \Delta_{\mathfrak{P}}\mathcal{K}) \\
&= \mathcal{J}_{(z^{(4)},z^{(0)})} + \Delta_{\mathfrak{P}}\mathcal{M}_{z^{(4)}} \circ \mathcal{K} \circ \widetilde{\mathcal{J}}_{(z^{(0)},z^{(0)})} + \mathcal{M}_{z^{(4)}} + \Delta_{\mathfrak{P}}\mathcal{K} + \Delta_{\mathfrak{P}}\mathcal{H}^{lr}_{(z^{(4)},z^{(3)})} \circ \mathcal{K} \\
&= \mathcal{J}_{(z^{(4)},z^{(0)})} + \mathcal{M}_{z^{(4)}} \circ \left(\mathrm{Id} + \Delta_{\mathfrak{P}}\mathcal{K} \circ \widetilde{\mathcal{J}}_{(z^{(0)},z^{(0)})}\right) + \Delta_{\mathfrak{P}}\mathcal{K} + \Delta_{\mathfrak{P}}\mathcal{H}^{lr}_{(z^{(4)},z^{(3)})} \circ \mathcal{K},
\end{aligned}
$$

by Proposition 3.5 and Proposition 3.6 and Lemma 3.10. Similarly, we obtain

$$
\begin{aligned}
\mathcal{G}_{(z^{(5)},\dots,z^{(0)})} &= \mathcal{H}^{lr}_{(z^{(5)},z^{(4)})} \circ \cdots \circ \mathcal{H}^{lr}_{(z^{(1)},z^{(0)})} \\
&= \mathcal{H}^{lr}_{(z^{(5)},z^{(4)})} \circ \left[\mathcal{J}_{(z^{(4)},z^{(0)})} + \mathcal{M}_{z^{(4)}} \circ \left(\mathrm{Id} + \Delta_{\mathfrak{P}}\mathcal{K} \circ \widetilde{\mathcal{J}}_{(z^{(0)},z^{(0)})}\right) + \Delta_{\mathfrak{P}}\mathcal{K} + \Delta_{\mathfrak{P}}\mathcal{H}^{lr}_{(z^{(4)},z^{(3)})} \circ \mathcal{K}\right] \\
&= \mathcal{J}_{(z^{(5)},z^{(0)})} + \Delta_{\mathfrak{P}}\mathcal{M}_{z^{(5)}} \circ \mathcal{K} \circ \widetilde{\mathcal{J}}_{(z^{(1)},z^{(0)})} + (\mathcal{M}_{z^{(5)}} + \Delta_{\mathfrak{P}}\mathcal{K}) \circ \left(\mathrm{Id} + \Delta_{\mathfrak{P}}\mathcal{K} \circ \widetilde{\mathcal{J}}_{(z^{(0)},z^{(0)})}\right) \\
&\qquad + \Delta_{\mathfrak{P}}(\mathcal{H}^{lr}_{(z^{(5)},z^{(4)})} \circ \mathcal{K} + \mathcal{H}^{lr}_{(z^{(5)},z^{(4)})} \circ \mathcal{H}^{lr}_{(z^{(4)},z^{(3)})} \circ \mathcal{K}) \\
&= \mathcal{J}_{(z^{(5)},z^{(0)})} + \Delta_{\mathfrak{P}}\mathcal{G}_{(z^{(5)},\dots,z^{(3)})} \circ \mathcal{K} + \mathcal{M}_{z^{(5)}} \circ \left(\mathrm{Id} + \Delta_{\mathfrak{P}}\mathcal{K} \circ \widetilde{\mathcal{J}}_{(z^{(0)},z^{(0)})} + \Delta_{\mathfrak{P}}\mathcal{K} \circ \widetilde{\mathcal{J}}_{(z^{(1)},z^{(0)})}\right) \\
&\qquad + \Delta_{\mathfrak{P}}\mathcal{G}_{(z^{(5)},z^{(4)})} \circ \mathcal{K} + \Delta_{\mathfrak{P}}\mathcal{K} \circ \left(\mathrm{Id} + \Delta_{\mathfrak{P}}\mathcal{K} \circ \widetilde{\mathcal{J}}_{(z^{(0)},z^{(0)})}\right).
\end{aligned}
$$

By induction we can now obtain

Lemma 3.11. *Let $l \geq 5$, we have*

$$(9)\quad \mathcal{G}_{(z^{(l)},\dots,z^{(0)})} = \mathcal{J}_{(z^{(l)},z^{(0)})} + \Delta_{\mathfrak{P}}\mathcal{G}_{(z^{(l)},\dots,z^{(3)})} \circ \mathcal{K} + \mathcal{M}_{z^{(l)}} \circ \left(\mathrm{Id} + \Delta_{\mathfrak{P}} \sum_{j=0}^{l-4} \mathcal{K} \circ \widetilde{\mathcal{J}}_{(z^{(j)},z^{(0)})}\right)$$
$$+ \Delta_{\mathfrak{P}} \sum_{j=4}^{l} \left(\mathcal{G}_{(z^{(l)},\dots,z^{(j)})} \circ \mathcal{K} \circ \left(\mathrm{Id} + \Delta_{\mathfrak{P}} \sum_{i=0}^{j-5} \mathcal{K} \circ \widetilde{\mathcal{J}}_{(z^{(i)},z^{(0)})}\right)\right),$$

with the convention $\sum_{i=0}^{-1} \mathcal{K} \circ \widetilde{\mathcal{J}}_{(z^{(i)},z^{(0)})} = 0$.

From Lemma 3.9 and Lemma 3.10 we observe that

$$\left\| \mathrm{Id} + \Delta_{\mathfrak{P}} \sum_{j=0}^{l} \mathcal{K} \circ \widetilde{\mathcal{J}}_{(z^{(j)},z^{(0)})} \right\|_{((H^{(s)})^{k},(H^{(s)})^{k})} \leq C, \qquad 0 \leq l \leq N,$$

with C uniform w.r.t. $\mathfrak{P}$ and l, since $\Delta_{\mathfrak{P}} = Z/N$. As $\|\mathcal{M}_z\|_{((H^{(s)})^k,(H^{(s)})^k)}$ is bounded uniformly w.r.t. z, we obtain the existence of $A \geq 0$ and $B \geq 0$ such that

$$\|\mathcal{G}_{(z^{(l)},z^{(0)})}\|_{((H^{(s)})^k,(H^{(s)})^k)} \leq A + \Delta_{\mathfrak{P}} B \sum_{j=3}^{l} \|\mathcal{G}_{(z^{(l)},z^{(j)})}\|_{((H^{(s)})^k,(H^{(s)})^k)},$$

from Lemma 3.11, which gives, with $V_{l',l} = \|\mathcal{G}_{(z^{(l')},z^{(l)})}\|_{((H^{(s)})^k,(H^{(s)})^k)}$,

$$V_{l,0} \leq A + \Delta_{\mathfrak{P}} B \sum_{j=3}^{l} V_{l,j} \leq A + \Delta_{\mathfrak{P}} B \sum_{j=1}^{l} V_{l,j}.$$

Above, we have chosen to use $z^{(0)} = 0$ as the starting value for z. However, similarly, we obtain

$$V_{l',l} \leq A + \Delta_{\mathfrak{P}} B \sum_{j=l+1}^{l'} V_{l',j}, \quad 0 \leq l \leq l' \leq N.$$

Define the finite sequence, $(W_l)_{0 \leq l \leq N}$ by

$$W_0 = 1, \quad W_{l+1} = A + \Delta_{\mathfrak{P}} B \sum_{j=0}^{l} W_j, \quad 0 \leq l \leq N-1.$$

Since $\mathcal{G}_{(z^{(l)},z^{(l)})} = \mathrm{Id}$, $0 \leq l \leq N$, we have $V_{l,l} = 1$ and a simple induction gives

$$V_{l',l} \leq W_{l'-l}, \quad 0 \leq l \leq l' \leq N.$$

We now observe that for all l, $l = 1, \dots, N$,

$$W_l = W_{l-1} + \Delta_{\mathfrak{P}} B W_{l-1} = (1 + \Delta_{\mathfrak{P}} B) W_{l-1} = (1 + \Delta_{\mathfrak{P}} B)^l W_0$$
$$= (1 + \Delta_{\mathfrak{P}} B)^l \leq (1 + \frac{BZ}{N})^N \leq e^{BZ}.$$

For the Ansatz $\mathcal{W}_{\mathfrak{P},z}$ in the *symmetrizable* case, we thus have the following counterpart to Proposition 2.2.

Theorem 3.12. *Let $s \in \mathbb{R}$. Under Assumptions 3.1 and 3.2, there exists $K' \geq 0$ such that for every subdivision $\mathfrak{P} = \{z^{(0)}, z^{(1)}, \dots, z^{(N)}\}$ of $[0, Z]$ with $0 = z^{(0)} < z^{(1)} < \dots < z^{(N)} = Z$ and $\mathcal{W}_{\mathfrak{P},z}$ as defined in Definition 2.1 we have*

$$\forall z \in [0, Z], \quad \|\mathcal{W}_{\mathfrak{P},z}\|_{((H^{(s)})^k,(H^{(s)})^k)} \leq K',$$

for $\Delta_{\mathfrak{P}}$ sufficiently small.

With the stability of $\mathcal{W}_{\mathfrak{P},z}$ established, we can proceed with the analysis of its convergence as in Section 2. There is no difference in the argumentation between the symmetric and the symmetrizable cases there. We thus obtain a theorem similar to Theorem 2.7, which gives a representation of the solution operator of the Cauchy problem (1)–(2) by an infinite product of matrix-phase FIOs.

Theorem 3.13. *Let Assumptions 3.1 and 3.2 hold and let xassume that* $a(z,.)$ *belongs to* $\mathscr{C}^{0,\alpha}([0,Z], \mathcal{M}_k S^1(\mathbf{R}^n \times \mathbf{R}^n))$, *i.e. Hölder continuous w.r.t. z, with values in* $\mathcal{M}_k S^1(\mathbf{R}^n \times \mathbf{R}^n)$, *in the sense that, for some* $0 < \alpha < 1$

$$a(z',x,\xi) - a(z,x,\xi) = (z'-z)^\alpha\, \tilde{a}(z',z,x,\xi), \quad 0 \le z \le z' \le Z,$$

or Lipschitz ($\alpha = 1$), *with* $\tilde{a}(z',z,x,\xi)$ *bounded w.r.t.* z' *and* z *with values in* $\mathcal{M}_k S^1(\mathbf{R}^n \times \mathbf{R}^n)$. *Let* $s \in \mathbf{R}$ *and* $0 \le r < 1$. *Then the approximation Ansatz* $\mathcal{W}_{\mathfrak{P},z}$ *converges to the solution operator* $U(z,0)$ *of the Cauchy problem (1)–(2) in* $L((H^{(s+1)}(\mathbf{R}^n))^k, (H^{(s+r)}(\mathbf{R}^n))^k)$ *uniformly w.r.t. z as* $\Delta_{\mathfrak{P}}$ *goes to 0 with a convergence rate of order* $\alpha(1-r)$:

$$\|\mathcal{W}_{\mathfrak{P},z} - U(z,0)\|_{((H^{(s+1)})^k,(H^{(s+r)})^k)} \le C\Delta_{\mathfrak{P}}^{\alpha(1-r)}, \quad z \in [0,Z].$$

Furthermore, the operator $\mathcal{W}_{\mathfrak{P},z}$ *strongly converges to the solution operator* $U(z,0)$ *uniformly w.r.t.* $z \in [0,Z]$ *in* $L((H^{(s+1)}(\mathbf{R}^n))^k, (H^{(s+1)}(\mathbf{R}^n))^k)$.

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