数理解析研究所講究録
第 1563 巻 2007 年 89-108

Functional integral representations of a model in nonrelativistic QED with spin 1/2

九州大学大学院数理学研究院　　廣島 文生 * (Fumio Hiroshima)
Faculty of Mathematics, Kyushu University

1 Feynman-Kac type formula

This is a continuation of [Hir97, Hir05]. In this article we are concerned with a Feynman-Kac type formula in quantum field theory. In particular we investigate the so called Pauli-Fierz Hamiltonian $H_{\rm PF}$ with *spin* 1/2 in nonrelativistic QED. Since the model includes spin, we need the $3+1$ dimensional Levy process,

$$(\xi_t)_{t\geq 0}=(B_t,N_t)_{t\geq 0},$$

to construct the Feynman-Kac type formula, where B_t denotes the 3 dimensional Brownian motion and N_t a Poisson process taking dichotomic values. Then the Levy process ξ_t takes values in $\mathbb{R}^3\times\mathbb{Z}_2$, where $\mathbb{Z}_2$ denotes the additive group with degree two. The Pauli-Fierz Hamiltonian [Hir04, Spo04] describes a minimal interaction between nonrelativistic electrons and a quantized radiation field in the Coulomb gauge. Specifically we impose an ultraviolet cutoff on it. The field quanta of the quantized radiation field are massless bosons referred to as photons. While electrons are assumed to be in low energy and treated as quantum mechanical particles. Then the number of electrons are conserved, and for simplicity it is assumed to be one in this paper. The quantized radiation field is described by an infinite dimensional Gaussian random process

$$(\mathcal{A}_1(j_tf))_{f\in L^2(\mathbb{R}^3),t\in\mathbb{R}}.$$

*e-mail: hiroshima@ math.kyushu-u.ac.jp

Then combining the Levy process ξ_t and the Gaussian random process $\mathcal{A}_{1\mu}(j_t f)$, we construct the Feynman-Kac type formula. In the spinless case the functional integral representation of the heat semigroup is established by [Hir97] and in the translation invariant case by [Hir06]. Here we extend them to the Hamiltonian including spin 1/2.

2 Definition of the model

2.1 Fundamental facts

Let us begin with defining the Pauli-Fierz Hamiltonian as a self-adjoint operator on some Hilbert space. Let $h_{\text{ph}} := L^2(\mathbb{R}^3 \times \{-1,1\})$ denote the Hilbert space of one-particle states of photons, where $\mathbb{R}^3 \times \{-1,1\} \ni (k,j)$ is the momentum and the polarization of photon, respectively. The Boson Fock space, $\mathcal{F}_{\text{b}}$, over h_{ph} is defined by

$$\mathcal{F}_{\text{b}} := \bigoplus_{n=0}^{\infty} \left[\bigotimes_{\text{sym}}^{n} h_{\text{ph}} \right],$$

where $\otimes_{\text{sym}}^n$ denotes n-fold symmetric tensor product with $\otimes_{\text{sym}}^0 h_{\text{ph}} := \mathbb{C}$. $\mathcal{F}_{\text{b}}$ is the Hilbert space with the scalar product $(\Psi, \Phi)_{\mathcal{F}_{\text{b}}} := \sum_{n=0}^{\infty} (\Psi^{(n)}, \Phi^{(n)})_{\otimes^n h_{\text{ph}}}$. Let us define the free Hamiltonian H_{rad} on $\mathcal{F}_{\text{b}}$, which is given as the infinitesimal generator of a one-parameter unitary group. This unitary group is provided through second quantization and the second quantization through the functor Γ. The set of contraction operators from X to Y is denoted by $\mathcal{C}(X \to Y)$. We define the functor Γ,

$$\Gamma : \mathcal{C}(L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)) \to \mathcal{C}(\mathcal{F}_{\text{b}} \to \mathcal{F}_{\text{b}}),$$

by $\Gamma(T) := \oplus_{n=0}^{\infty} \underbrace{T \otimes \cdots \otimes T}_{n}$, where $\underbrace{T \otimes \cdots \otimes T}_{0}$ is the identity operator. Particulary for the self-adjoint operator h on h_{ph}, $\Gamma(e^{ith})$, $t \in \mathbb{R}$, is the strongly continuous one-parameter unitary group on $\mathcal{F}_{\text{b}}$. Then there exists the unique self-adjoint operator $d\Gamma(h)$ such that

$$\Gamma(e^{ith}) = e^{itd\Gamma(h)}, \quad t \in \mathbb{R}.$$

$d\Gamma(h)$ is called the second quantization of h. Now we define H_{rad}. Let ω_{b} be the multiplication operator on h_{ph} defined by $\omega_{\text{b}} : f \longmapsto \omega_{\text{b}}(k) f(k,j) = |k| f(k,j)$. Then

$H_{\rm rad} := d\Gamma(\omega_{\rm b})$ and the spectrum of $H_{\rm rad}$ is $[0,\infty)$ with the simple eigenvalue $\{0\}$. Of course the semigroup $e^{-tH_{\rm rad}}$ can be also expressed as $e^{-tH_{\rm rad}} = \Gamma(e^{-t\omega_{\rm b}})$. Next we define the annihilation operator and the creation operator on $\mathcal{F}_{\rm b}$. With each $f \in h_{\rm ph}$, one associates the creation operator $a^\dagger(f)$ defined by $(a^\dagger(f)\Psi)^{(n)} = \sqrt{n}S_n(f\otimes\Psi^{(n-1)})$ where S_n is the symmetrizer. The domain of $a^\dagger(f)$ is maximally defined. The annihilation operator $a(f)$ is defined to be the adjoint of $a^\dagger(\bar{f})$: $a(f) = (a^\dagger(\bar{f}))^*$. We symbolically write as $a^\sharp(f) = \sum_{j=\pm1}\int f(k,j)a^\sharp(k,j)dk$. The operators $a^\dagger(f)$ and $a(f)$ obey the canonical commutation relations;

$$[a(f), a^\dagger(g)] = (\bar{f}, g)1, \quad [a(f), a(g)] = 0, \quad [a^\dagger(f), a^\dagger(g)] = 0.$$

Let us define a quantized radiation field. Since the radiation field is quantized in the Coulomb gauge, polarization vectors are introduced. Let $e(k,+1)$ and $e(k,-1)$ be polarization vectors, i.e., $e(k,-1), e(k,+1), k/|k|$, $k \neq 0$, form the right-handed system in $\mathbb{R}^3$ with $e(k,-1)\times e(k,+1) = k/|k|$, $e(k,j)\cdot e(k,j') = \delta_{jj'}$ and $e(k,j)\cdot k/|k| = 0$. Thus the quantized radiation field with ultraviolet cutoff $\hat{\varphi}$ is defined by

$$A_{\hat{\varphi},\mu}(x) := \frac{1}{\sqrt{2}}\sum_{j=\pm1}\int e_\mu(k,j)\Big(\frac{\hat{\varphi}(k)}{\sqrt{\omega_{\rm b}(k)}}a^\dagger(k,j)e^{-ik\cdot x} + \frac{\hat{\varphi}(-k)}{\sqrt{\omega_{\rm b}(k)}}a(k,j)e^{+ik\cdot x}\Big)dk.$$

Here $\hat{\varphi}$ denotes the Fourier transform of φ, and $\hat{\varphi}/\sqrt{\omega_{\rm b}} \in L^2(\mathbb{R}^3)$ is assumed. By $k\cdot e(k,j) = 0$, the Coulomb gauge condition, $\sum_{\mu=1}^3[\partial_{x_\mu}, A_{\hat{\varphi}_\mu}(x)] = 0$, is obeyed. We assume that $\overline{\hat{\varphi}(k)} = \hat{\varphi}(-k) = \hat{\varphi}(k)$. Then $A_{\hat{\varphi},\mu}(x)$ is symmetric. The states of one electron coupled to the quantized radiation field are vectors of the composition of $L^2(\mathbb{R}^3_x;\mathbb{C}^2)$ and $\mathcal{F}_{\rm b}$:

$$\mathcal{H} := L^2(\mathbb{R}^3_x;\mathbb{C}^2)\otimes\mathcal{F}_{\rm b}.$$

To define the quantized radiation field on $\mathcal{H}$, we identify $\mathcal{H}$ with the set of $\mathbb{C}^2\otimes\mathcal{F}_{\rm b}$-valued L^2 function on $\mathbb{R}^3_x$. Then $A_{\hat{\varphi}_\mu}$ is given by $(A_{\hat{\varphi},\mu}F)(x) = A_{\hat{\varphi},\mu}(x)F(x)$. Now we are in the position to define the Pauli-Fierz Hamiltonian, which is given by

$$H_{\rm PF} := \frac{1}{2}\left(\sum_{j=1}^3\sigma_j(-i\nabla_j\otimes 1 - eA_{\hat{\varphi},j})\right)^2 + V\otimes 1 + 1\otimes H_{\rm rad}, \tag{2.1}$$

where $e \in \mathbb{R}$ is a coupling constant, V denotes an external potential and σ_j, $j = 1, 2, 3$, are the usual 2×2 Pauli matrices given by

$$\sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Using the formula [1] $\sigma_\mu \sigma_\nu = \delta_{\mu\nu} + i \sum_{\gamma=1}^3 \epsilon^{\mu\nu\gamma} \sigma_\gamma$, we can rewite (2.1) as

$$H_{\mathrm{PF}} = \frac{1}{2}(-i\nabla - eA_{\hat{\varphi}})^2 + V + H_{\mathrm{rad}} - \frac{e}{2} \sum_{j=1}^{3} \sigma_j B_{\hat{\varphi} j},$$

where we omit $\otimes$ for notational convenience and $B_{\hat{\varphi}}(x) = \mathrm{rot}_x A_{\hat{\varphi}}(x)$. Explicitly

$$B_{\hat{\varphi}\mu}(x) = \frac{-i}{\sqrt{2}} \sum_{j=\pm 1} \int (k \times e(k, j))_\mu \left(\frac{\hat{\varphi}(k)}{\sqrt{\omega_{\mathrm{b}}(k)}} a^\dagger(k, j) e^{-ik\cdot x} - \frac{\hat{\varphi}(-k)}{\sqrt{\omega_{\mathrm{b}}(k)}} a(k, j) e^{ik\cdot x} \right) dk.$$

The fundamental assumption to guarantee the self-adjointness of H_{PF} is as follows.

Assumption 2.1 *(1) $\sqrt{\omega_{\mathrm{b}}}\hat{\varphi}, \hat{\varphi}/\omega_{\mathrm{b}} \in L^2(\mathbb{R}^3)$ and $\overline{\hat{\varphi}(k)} = \hat{\varphi}(-k) = \hat{\varphi}(k)$. (2) V is relatively bounded with respect to $(-1/2)\Delta$ with a relative bound strictly smaller than one.*

Under Assumption 2.1, it is established in [Hir00b, Hir02] that H_{PF} is self-adjoint on $D(-\Delta) \cap D(H_{\mathrm{rad}})$ and bounded from below. Moreover it is essentially self-adjoint on any core of $-(1/2)\Delta + V + H_{\mathrm{rad}}$.

2.2 Symmetry and polarization

In this subsection we discuss the symmetry of H_{PF}. See [Hir06] for detail. When the form factor $\hat{\varphi}$ and the external potential V are translation invariant, i.e., $\hat{\varphi}(Rk) = \hat{\varphi}(k)$ and $V(Rx) = V(x)$ for arbitrary $R \in \mathrm{O}(3)$, then H_{PF} has the symmetry:

$$SU(2) \otimes O_{\mathrm{particle}}(3) \otimes O_{\mathrm{field}}(3) \otimes \text{helicity},$$

where $SU(2)$ and $O_{\mathrm{particle}}(3)$ come from spin and the angular momentum of the particle, respectively, $O_{\mathrm{field}}(3)$ and helicity from the angular momentum and the helicity of photons, respectively. Let $R \in \mathrm{SO}(3)$ and $\hat{k} = k/|k|$. Two orthogonal bases

[1] $\epsilon^{\mu\nu\gamma}$ is the antisymmetric tensor with $\epsilon^{123} = 1$.

$e(Rk,1), e(Rk,-1), \hat{Rk}$ and $Re(k,1), Re(k,-1), R\hat{k}$ in $\mathbb{R}^3$ at k satisfy

$$\begin{bmatrix} e(Rk,1) \\ e(Rk,-1) \\ \hat{Rk} \end{bmatrix} = \begin{bmatrix} \cos\theta 1_3 & -\sin\theta 1_3 & 0 \\ \sin\theta 1_3 & \cos\theta 1_3 & 0 \\ 0 & 0 & 1_3 \end{bmatrix} \begin{bmatrix} Re(k,1) \\ Re(k,-1) \\ R\hat{k} \end{bmatrix}, \tag{2.2}$$

where 1_3 denotes the 3×3 unit matrix and $\theta := \theta(R,k) := \arccos(Re(k,1) \cdot e(Rk,1))$ Let $R = R(\phi, n) \in \mathrm{SO}(3)$ be the rotation around $n \in \mathrm{S}^2 := \{k \in \mathbb{R}^3 | |k| = 1\}$ by angle $\phi \in \mathbb{R}$ and $\det R = 1$. Also, let $\ell_k := k \times (-i\nabla_k) = (\ell_{k1}, \ell_{k2}, \ell_{k3})$ be the triplet of angular momentum operators in $L^2(\mathbb{R}^3_k)$. Then (2.2) is rewritten as

$$e^{i\theta(R,k)X} e^{i\phi n \cdot \ell_k} \begin{bmatrix} e(k,1) \\ e(k,-1) \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} e(k,1) \\ e(k,-1) \end{bmatrix}, \tag{2.3}$$

where $X = -i \begin{bmatrix} 0 & -1_3 \\ 1_3 & 0 \end{bmatrix}$. To discuss the symmetry of H_{PF}, we introduce *coherent* polarization vectors in some direction. We have Assumption (P) as follows.

(P) *There exists* $(n, w) \in \mathrm{S}^2 \times \mathbb{Z}$ *such that polarization vectors* $e(\cdot,1)$ *and* $e(\cdot,-1)$ *satisfy for arbitrary* $R = R(n,\phi) \in \mathrm{SO}(3)$ *and* $\hat{k} \neq n$,

$$\begin{bmatrix} e(Rk,1) \\ e(Rk,-1) \end{bmatrix} = \begin{bmatrix} \cos(\phi w) 1_3 & -\sin(\phi w) 1_3 \\ \sin(\phi w) 1_3 & \cos(\phi w) 1_3 \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} e(k,1) \\ e(k,-1) \end{bmatrix} \tag{2.4}$$

or for each $\mu = 1,2,3$,

$$\begin{bmatrix} e_\mu(Rk,1) \\ e_\mu(Rk,-1) \end{bmatrix} = \begin{bmatrix} \cos(\phi w) & -\sin(\phi w) \\ \sin(\phi w) & \cos(\phi w) \end{bmatrix} \begin{bmatrix} (Re(k,1))_\mu \\ (Re(k,-1))_\mu \end{bmatrix}. \tag{2.5}$$

By assuming (P), we have by (2.5),

$$\exp\left\{ i\phi \left(w\tilde{X} + n \cdot \ell_k \right) \right\} \begin{bmatrix} e_\mu(k,1) \\ e_\mu(k,-1) \end{bmatrix} = \begin{bmatrix} (Re(k,1))_\mu \\ (Re(k,-1))_\mu \end{bmatrix}, \tag{2.6}$$

where $\tilde{X} = -i \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$. Here is an example for polarization vectors satisfying Assumption (P).

Example 2.2 *Let* $n \in \mathrm{S}^3$, *and* $e(k,-1) := \hat{k} \times n / \sin\theta$ *and* $e(k,+1) := (k/|k|) \times e(k,1)$, *where* $\theta = \arccos(\hat{k} \cdot n)$. *Then, since* $R = R(n,\phi)$ *satisfies that* $Rn = n$ *and* $Ru \times Rv = R(u \times v)$, $e(k,j)$ *obeys (2.4) with* $(n, 0) \in \mathrm{S}^2 \times \mathbb{Z}$.

Assume (P) with some $(n, w) \in S^2 \times \mathbb{Z}$. We define $S_f := d\Gamma(w\mathbf{X})$ and $L_f := d\Gamma(\ell_k)$. Here $\mathbf{X} := -i \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} : L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \longrightarrow L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$. S_f is called the helicity of the field and L_f the angular momentum of the field. Define $J_f := n \cdot L_f + S_f$. Then we have for translation invariant f,

$$e^{i\phi J_f} a^\sharp(fe^{-ik\cdot x} \begin{bmatrix} e_\mu(1) \\ e_\mu(-1) \end{bmatrix}) e^{-i\phi J_f} = a^\sharp(fe^{i\phi(\tilde{X}+n\ell_k)} e^{-ik\cdot x} \begin{bmatrix} e_\mu(1) \\ e_\mu(-1) \end{bmatrix})$$
$$= a^\sharp(fe^{-iRk\cdot x} \begin{bmatrix} (Re(1))_\mu \\ (Re(-1))_\mu \end{bmatrix}) = \sum_{\nu=1}^{3} R_{\mu\nu} a^\sharp(fe^{-ik\cdot R^{-1}x} \begin{bmatrix} e_\nu(1) \\ e_\nu(-1) \end{bmatrix}), \tag{2.7}$$

where $R = R(\phi, n)$. Let $J_p := n \cdot \ell_x + \frac{1}{2} n \cdot \sigma$ be the angular momentum plus spin for the particle, and define

$$J_{\mathrm{total}} := J_p \otimes 1 + 1 \otimes J_f.$$

Lemma 2.3 *Assume (P) and that $\hat{\varphi}$ and V are translation invariant. Then for arbitrary $\phi \in \mathbb{R}$,*

$$e^{i\phi J_{\mathrm{total}}} H_{\mathrm{PF}} e^{-i\phi J_{\mathrm{total}}} = H_{\mathrm{PF}}.$$

Proof: By $e^{i\phi J_f} = e^{i\phi S_f} e^{i\phi n\cdot L_f}$, (2.6) and (2.7), we see that ($R = R(n, \phi)$)

$$\begin{aligned}
&e^{i\phi J_f} H_{\mathrm{rad}} e^{-i\phi J_f} = H_{\mathrm{rad}}, && e^{i\phi J_f} P_{f\mu} e^{-i\phi J_f} = (RP_f)_\mu, \\
&e^{i\phi J_f} A_{\hat{\varphi}_\mu}(x) e^{-i\phi J_f} = (RA_{\hat{\varphi}}(R^{-1}x))_\mu, && e^{i\phi n\cdot \ell_x} x_\mu e^{-i\phi n\cdot \ell_x} = (Rx)_\mu, \\
&e^{i\phi n\cdot \ell_x}(-i\nabla_x)_\mu e^{-i\phi n\cdot \ell_x} = (R(-i\nabla_x))_\mu, && e^{i\phi n\cdot(1/2)\sigma} \sigma_\mu e^{-i\phi n\cdot(1/2)\sigma} = (R\sigma)_\mu.
\end{aligned}$$

Then we complete the proof. **qed**

Note that $\sigma(n \cdot (\ell_x + (1/2)\sigma)) = \mathbb{Z}_{1/2}$, $\sigma(n \cdot L_f) = \mathbb{Z}$ and $\sigma(S_f) = \mathbb{Z}$. Then $\sigma(J_{\mathrm{total}}) = \mathbb{Z}_{1/2}$ and we have the theorem below.

Theorem 2.4 *We assume the same assumptions as in Lemma 2.3. Then $\mathcal{H}$ and H_{PF} are decomposed as $\mathcal{H} = \bigoplus_{z\in\mathbb{Z}_{1/2}} \mathcal{H}(z)$ and $H_{\mathrm{PF}} = \bigoplus_{z\in\mathbb{Z}_{1/2}} H_{\mathrm{PF}}(z)$. Here $\mathcal{H}(z)$ is the subspace spanned by eigenvectors of J_{total} with eigenvalue $z \in \mathbb{Z}_{1/2}$ and $H_{\mathrm{PF}}(z) = H_{\mathrm{PF}}\lceil_{\mathcal{H}(z)}$.*

Proof: This follows from Lemma 2.3 and the fact that $\sigma(J_{\mathrm{total}}) = \mathbb{Z}_{1/2}$. **qed**

Next we consider incoherent polarization vectors. However we can show that the Pauli-Fierz Hamiltonians with different polarization vectors are isomorphic with each others. We will see it below. Let $e(1), e(-1)$ and $\eta(1), \eta(-1)$ be polarization vectors. The Pauli-Fierz Hamiltonian with polarization vector $e(1), e(-1)$ (resp. $\eta(1), \eta(-1)$) is denoted by H_{PF_e} (resp. H_{PF_η}).

Lemma 2.5 *H_{PF_e} and H_{PF_η} are isomorphic.*

Proof: We learned it from [Sas06]. Since both polarization vectors form orthogonal base on the plan perpendicular to the vector k, there exists θ_k such that

$$\begin{bmatrix} e(k,1) \\ e(k,-1) \end{bmatrix} = \begin{bmatrix} \cos\theta_k 1_3 & -\sin\theta_k 1_3 \\ \sin\theta_k 1_3 & \cos\theta_k 1_3 \end{bmatrix} \begin{bmatrix} \eta(k,1) \\ \eta(k,-1) \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} e_\mu(k,1) \\ e_\mu(k,-1) \end{bmatrix} = R_k \begin{bmatrix} \eta_\mu(k,1) \\ \eta_\mu(k,-1) \end{bmatrix},$$

where $R_k = \begin{bmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{bmatrix}$. Define $R : h_{\mathrm{ph}} \to h_{\mathrm{ph}}$ by $R \begin{bmatrix} f \\ g \end{bmatrix}(k) = R_k \begin{bmatrix} f(k) \\ g(k) \end{bmatrix}$ and $U : \mathcal{F}_{\mathrm{b}} \to \mathcal{F}_{\mathrm{b}}$ by the second quantization of R, i.e., $U := \Gamma(R)$. Then U is the unitary on $\mathcal{F}_{\mathrm{b}}$. Note that $R \begin{bmatrix} \eta_\mu(1) f \\ \eta_\mu(-1) f \end{bmatrix} = \begin{bmatrix} f e_\mu(1) \\ f e_\mu(-1) \end{bmatrix}$ which implies that $U H_{\mathrm{PF}_\eta} U^{-1} = H_{\mathrm{PF}_e}$. Hence the lemma follows. **qed**

Combining Lemma 2.5 and Theorem 2.4, we have the corollary below.

Corollary 2.6 *Suppose that $\hat{\varphi}$ and V are translation invariant. Then H_{PF} with arbitrary polarization vectors is isomorphic to $\bigoplus_{z \in \mathbf{Z}_{1/2}} H_{\mathrm{PF}}(z)$, where $H_{\mathrm{PF}}(z)$ is defined in Theorem 2.4.*

2.3 $\mathcal{Q}$-representations and dichotomic variables

To construct the functional integral representation, we have to take $\mathcal{Q}$-representation of H_{PF} instead of the Fock representation. To introduce $\mathcal{Q}$-representation, we define a bilinear form and construct the Gaussian random process with mean zero and covariance given by this bilinear form. Let us define the field operator $A_\mu(\hat{f})$ by

$$A_\mu(\hat{f}) := \frac{1}{\sqrt{2}} \sum_{j=\pm 1} \int e_\mu(k,j)(\hat{f}(k) a^\dagger(k,j) + \hat{f}(-k) a(k,j)) dk$$

and 3×3 matrix $D(k)$, $k \neq 0$, by $D(k) := (\delta_{\mu\nu} - k_\mu k_\nu / |k|^2)_{1 \le \mu,\nu \le 3}$. Note that $\sum_{j=\pm 1} e_\mu(k,j) e_\nu(k,j) = D_{\mu\nu}(k)$. Then the bilinear form $q_0 : \oplus^3 L^2(\mathbf{R}^3) \times \oplus^3 L^2(\mathbf{R}^3) \to \mathbf{C}$ is given by

$$q_0(f,g) := \sum_{\mu,\nu=1}^{3} (A_\mu(f_\mu)\Omega_{\mathrm{b}}, A_\nu(g_\nu)\Omega_{\mathrm{b}})_{\mathcal{F}_{\mathrm{b}}} = \frac{1}{2} \int_{\mathbf{R}^3} \overline{\hat{f}(k)} \cdot D(k) \hat{g}(k) dk.$$

Just as the Euclidean free field is exhibited as a kind of path integrals over the free Minkowski field in constructive quantum field theory [Sim74, Theorem III.6], we introduce an additional bilinear form q_1 to define an additional Gaussian random process. The bilinear form $q_1 : \oplus^3 L^2(\mathbb{R}^{3+1}) \times \oplus^3 L^2(\mathbb{R}^{3+1}) \to \mathbb{C}$ is given by

$$q_1(F,G) := \frac{1}{2}\int_{\mathbb{R}^{3+1}} \overline{\hat{F}(k,k_0)} \cdot D(k)\hat{G}(k,k_0)dkdk_0.$$

From now on β stands for 0 or 1. Let $\mathcal{S}_{\mathrm{r}\beta} := \oplus^3 \mathcal{S}_{\mathrm{r}}(\mathbb{R}^{3+\beta})$, where $\mathcal{S}_{\mathrm{r}}(\mathbb{R}^{3+\beta})$ denotes the set of real-valued Schwartz test functions. Define

$$C_\beta(f) := \exp(-q_\beta(f,f)), \quad f \in \mathcal{S}_{\mathrm{r}\beta}.$$

It is immediate to check that (1) $\sum_{i,j=1}^n \bar{z}_i z_j C_\beta(f_i - f_j) \geq 0$ for $z_i \in \mathbb{C}$, $i = 1, ..., n$, (2) $C_\beta(g)$ is strongly continuous in g, (3) $C_\beta(0) = 1$. Let $\langle \phi, f\rangle_\beta$ denote the pairing between $\mathcal{Q}_\beta := \mathcal{S}_{\mathrm{r}\beta}'$ and $\mathcal{S}_{\mathrm{r}\beta}$. By the Bochner-Minlos theorem there exists a probability space $(\mathcal{Q}_\beta, \mathcal{B}_{\mathcal{Q}_\beta}, \mu_\beta)$ such that $\mathcal{B}_{\mathcal{Q}_\beta}$ is the smallest σ-field generated by $\{\langle \phi, f\rangle_\beta, f \in \mathcal{S}_{\mathrm{r}\beta}\}$ and $\langle \phi, f\rangle_\beta$ is the Gaussian random variable with mean zero and the covariance given by

$$\int_{\mathcal{Q}_\beta} e^{i\langle \phi, f\rangle_\beta} d\mu_\beta(\phi) = e^{-q_\beta(f,f)}, \quad f \in \mathcal{S}_{\mathrm{r}\beta}. \tag{2.8}$$

For a general $f = f_{\mathrm{Re}} + i f_{\mathrm{Im}} \in \oplus^3 \mathcal{S}(\mathbb{R}^{3+\beta})$, we set $\langle \phi, f\rangle_\beta := \langle \phi, f_{\mathrm{Re}}\rangle_\beta + i\langle \phi, f_{\mathrm{Im}}\rangle_\beta$. Since $\mathcal{S}(\mathbb{R}^{3+\beta})$ is dense in $L^2(\mathbb{R}^{3+\beta})$ and

$$\int_{\mathcal{Q}_\beta} |\langle \phi, f\rangle_\beta|^2 d\mu_\beta(\phi) \leq \|f\|^2_{\oplus^3 L^2(\mathbb{R}^{3+\beta})}$$

by (2.8), we can define $\langle \phi, f\rangle_\beta$ for $f \in \oplus^3 L^2(\mathbb{R}^{3+\beta})$ by a limiting argument.

So we define the multiplication operator $\mathcal{A}_\beta(f)$ labeled by $f \in \oplus^3 L^2(\mathbb{R}^{3+\beta})$ in $L^2(\mathcal{Q}_\beta)$ by $(\mathcal{A}_\beta(f)F)(\phi) := \langle \phi, f\rangle_\beta F(\phi)$ for $\phi \in \mathcal{Q}_\beta$. We denote the identity functions in $L^2(\mathcal{Q}_\beta)$ by $1_{\mathcal{Q}_\beta}$ and the function $\mathcal{A}_\beta(f)1_{\mathcal{Q}_\beta}$ by $\mathcal{A}_\beta(f)$ unless confusion may arise. It is known that $L^2(\mathcal{Q}_\beta)$ is divided in the infinite direct sum as

$$L^2(\mathcal{Q}_\beta) = \bigoplus_{n=0}^\infty L^2_n(\mathcal{Q}_\beta),$$

where $L^2_n(\mathcal{Q}_\beta) = \overline{\mathrm{L.H.}\{:\mathcal{A}_\beta(f_1)\cdots\mathcal{A}_\beta(f_n): | f_j \in \oplus^3 L^2(\mathbb{R}^{3+\beta}), j = 1,2,...,n\}}$, $n \geq 1$, with $L^2_0(\mathcal{Q}_\beta) = \{\alpha 1_{\mathcal{Q}_\beta} | \alpha \in \mathbb{C}\}$ and $:X:$ denotes the Wick product. Next let us define the second quantization $\Gamma_{\beta\beta'}$ on $\mathcal{Q}$-representation, which is also the functor

$$\Gamma_{\beta\beta'} : \mathcal{C}(L^2(\mathbb{R}^{3+\beta}) \to L^2(\mathbb{R}^{3+\beta'})) \to \mathcal{C}(L^2(\mathcal{Q}_\beta) :\to L^2(\mathcal{Q}_{\beta'}))$$

defined by

$$\Gamma_{\beta\beta'}(T)1_{\mathcal{Q}_\beta} := 1_{\mathcal{Q}_{\beta'}}, \quad \Gamma_{\beta\beta'}(T){:}\mathcal{A}_\beta(f_1)\cdots\mathcal{A}_\beta(f_n){:} = {:}\mathcal{A}_{\beta'}(Tf_1)\cdots\mathcal{A}_{\beta'}(Tf_n){:}.$$

Simply we write as Γ_β for $\Gamma_{\beta\beta}$. For each self-adjoint operator h in $L^2(\mathbb{R}^{3+\beta})$, $\Gamma_\beta(e^{ith})$ is the one parameter unitary group. Then $\Gamma_\beta(e^{ith}) = e^{itd\Gamma_\beta(h)}$, $t \in \mathbb{R}$, for the unique self-adjoint operator $d\Gamma_\beta(h)$ in $L^2(\mathcal{Q}_\beta)$. Thus we can see that $\mathcal{F}_{\mathrm{b}}$, $A_\mu(\hat{f})$ and $d\Gamma(h)$ are isomorphic to $L^2(\mathcal{Q}_0)$, $\mathcal{A}_0(\oplus_{\nu=1}^3 \delta_{\mu\nu} f)$ and $d\Gamma_0(\hat{h})$, respectively, where $\hat{h} = FhF^{-1}$ and F denotes the Fourier transform on $L^2(\mathbb{R}^3)$. This isomorphism maps H_{PF} to the self-adjoint operator on $L^2(\mathbb{R}^3;\mathbb{C}^2)\otimes L^2(\mathcal{Q}_0)$. We will see it bellow. Let $\lambda := (\hat{\varphi}/\sqrt{\omega_{\mathrm{b}}})^\vee$ and $\mathcal{A}_{0\mu}(\lambda(\cdot - x)) := \mathcal{A}_0(\oplus_{\nu=1}^3 \delta_{\mu\nu}\lambda(\cdot - x))$. Then we have $\mathcal{H} \cong L^2(\mathbb{R}^3;\mathbb{C}^2) \otimes L^2(\mathcal{Q}_0)$ and

$$\begin{aligned} H_{\mathrm{PF}} &\cong \frac{1}{2}(-i\nabla - e\mathcal{A}_0)^2 + V + d\Gamma_0(\omega_{\mathrm{b}}(-i\nabla)) - \frac{e}{2}\sum_{j=1}^3 \sigma_j \mathcal{B}_{0j} \\ &= \frac{1}{2}(-i\nabla - e\mathcal{A}_0)^2 + V + d\Gamma_0(\omega_{\mathrm{b}}(-i\nabla)) - \frac{e}{2}\begin{bmatrix} \mathcal{B}_{03} & \mathcal{B}_{01} - i\mathcal{B}_{02} \\ \mathcal{B}_{01} + i\mathcal{B}_{02} & -\mathcal{B}_{03} \end{bmatrix}. \end{aligned} \tag{2.9}$$

In this representation $A_{\hat{\varphi}_\mu}$ and $B_{\hat{\varphi}_\nu}$ are transformed to the multiplication operator $\mathcal{A}_{0\mu}$ and $\mathcal{B}_{0\nu}$, respectively. From now on we write the right-hand side of (2.9) (resp. $d\Gamma_0(\omega(-i\nabla))$ as H_{PF} (resp. H_{rad}) without confusion may arises. Preserving the discrete structure of spin components as discrete random variables, we introduce dichotomic variable σ with values in the additive group $\mathbb{Z}_2 = \{-1, 1\}$. Then the Hamiltonian under consideration is the self-adjoint operator on the Hilbert space

$$\tilde{\mathcal{H}} := L^2(\mathbb{R}^3 \times \mathbb{Z}_2) \otimes L^2(\mathcal{Q}_0)$$

defined by

$$(H_{\mathrm{PF}}F)(\sigma) = \left\{\frac{1}{2}(-i\nabla - e\mathcal{A}_0)^2 + V + H_{\mathrm{rad}} - \frac{e}{2}\sigma\mathcal{B}_{03}\right\} F(\sigma) - e^{\log[\frac{e}{2}(\mathcal{B}_{01} + i(-\sigma)\mathcal{B}_{02})]} F(-\sigma). \tag{2.10}$$

In the last term we take $\log z = \log|z| + i \arg z$, $0 \le \arg z < 2\pi$. The right-hand side of (2.10) is our main object, i.e., we want to construct the functional integral representation of semigroup generated by this.

3 Functional integral representation of $e^{-tH_{\mathrm{PF}}}$

3.1 Levy processes

Let us begin with defining notation on the wiener measure and the Brownian motion. Let $(B_t)_{t\geq 0} = (B_{i,t})_{t\geq 0, 1\leq i\leq 3}$ be the three dimensional Brownian motion on $(W, \mathcal{B}_W, P^x)$ with the natural filtration $\mathcal{F}_t = \sigma(B_s, s \leq t)$, $t \geq 0$, where $W = C([0,\infty); \mathbb{R}^3)$ and P^x denotes the wiener measure such that $P^x(B_0 = x) = 1$. I.e., $B_{i,t}(w) = w_i(t)$ for $w = (w_1, w_2, w_3) \in W$.

In order to construct a Feynman-Kac type formula of $e^{-tH_{\mathrm{PF}}}$, in addition to the Brownian motion, we need a Poisson point process. Here we explain minimum properties of Poisson point processes and counting measures we need. Let $(S, \mathcal{S}, P)$ be a probability space with a right-continuous increasing family of sub σ-fields $(\mathcal{S}_t)_{t\geq 0}$. Let $\mathbb{E}_P$ denote the expectation with respect to P. We fix a measurable space $(\mathcal{M}, B_{\mathcal{M}})$ and a stationary $(\mathcal{S}_t)$-Poisson point process p on $\mathcal{M}$ defined on $(S, \mathcal{S}, P)$ with intensity $\Lambda(t, U) := \mathbb{E}_P[N_p(t, U)] = tn(U)$ for some measure n on $\mathcal{M}$ with $n(\mathcal{M}) = 1$, where N_p denotes the counting measure on $((0,\infty) \times \mathcal{M}, \mathcal{B}_{(0,\infty)} \times B_{\mathcal{M}})$ defined by

$$N_p(t, U) := \#\{s \in D(p) | s \in (0, t], p(s) \in U\}, \quad t > 0, \quad U \in B_{\mathcal{M}},$$

where $\mathcal{B}_{(0,\infty)}$ is the Borel σ-field on $(0, \infty)$. Hence $\mathbb{E}_P[N_p(t, U) = N] = e^{-\Lambda(t)}\Lambda(t)^N/N!$. We set $N_t := N_{p(\omega)}(t, \mathcal{M})$ and $dN_t := \int_{\mathcal{M}} N_p(dtdm)$. Since $\#\{s \in D(p) | 0 < s \leq t\}$ is finite, for each $\tau \in S$, there exists $N = N(\tau) \in \mathbb{N}$, $0 < s_1 = s_1(\tau), ..., s_N = s_N(\tau) \leq t$ such that

$$\int_0^{t+} f(s, N_s) dN_s = \sum_{\substack{s' \in D(p) \\ 0 < s' \leq t}} f(s', N_{s'}) = \sum_{j=1}^N f(s_j, N_{s_j}) = \sum_{j=1}^N f(s_j, j). \tag{3.1}$$

Finally we note that the expectation of (3.1) is reduced to the Lebesgue integral:

$$\mathbb{E}_P[\int_0^{t+} f(s, N_s) dN_s] = \mathbb{E}_P[\int_0^t f(s, N_s) ds] = \int_0^t \sum_{n=0}^\infty f(s, n) \frac{s^n}{n!} e^{-s} ds.$$

Set $(\Omega, \mathcal{B}_\Omega, P_\Omega^x) := (W \times S, \mathcal{B}_W \times \mathcal{S}, P_W^x \otimes P)$ and $\omega := w \times \tau \in W \times S = \Omega$. For $\omega = w \times \tau$, we put $B_t(\omega) := B_t(w)$ and $p(s, \omega) := p(s, \tau)$. Let $\Omega_t = \mathcal{F}_t \times \mathcal{S}_t$, $t \geq 0$.

Let $\mathbb{Z}_2$ be the additive group. We denote the sum in $\mathbb{Z}_2$ by $\oplus_{\mathbb{Z}_2}$, i.e., $1 \oplus_{\mathbb{Z}_2} 1 = -1$, $-1 \oplus_{\mathbb{Z}_2} 1 = 1$, $-1 \oplus_{\mathbb{Z}_2} -1 = 1$. Then the $\mathbb{Z}_2$-valued random process, $\sigma_t : \mathbb{Z}_2 \times \Omega \to \mathbb{Z}_2$, is defined by

$$\sigma_t := \sigma \oplus_{\mathbb{Z}_2} N_t = \sigma(-1)^{N_t}, \quad \sigma \in \mathbb{Z}_2.$$

So we constructed the $(3+1)$-dimensional Levy process

$$\xi_t = (B_t, N_t)$$

on $(\Omega, \mathcal{B}_\Omega, P_\Omega^x)$. We set for simplicity

$$\mathbb{E}^{x,\sigma}[f(\xi_\cdot)] := \int_\Omega f(x + B_\cdot, \sigma \oplus_{\mathbb{Z}_2} N_\cdot) dP_\Omega^0 = \int_\Omega f(x + B_\cdot, \sigma(-1)^{N_\cdot}) dP_\Omega^0$$

and $\sum_\sigma \int dx\, f(x,\sigma) := \sum_{\sigma \in \mathbb{Z}_2} \int_{\mathbb{R}^3} dx f(x,\sigma)$.

3.2 Functional integral representations

In addition to Assumption 2.1, we need specify the class of external potentials V. We assume the assumption below:

Assumption 3.1 *V satisfies that* $V_M := \sup_{x \in \mathbb{R}^3} \mathbb{E}^x[e^{-\int_0^t V(B_s) ds}] < \infty$.

The Kato class potentials satisfy Assumption 3.1 and, especially, the Coulomb potential does. We study the self-adjoint operator $\tilde{H}_{\mathrm{PF0}}(\phi)$ defined for each $\phi \in \mathcal{Q}_0$. Assume that $\lambda \in C_0^\infty(\mathbb{R}^3)$ in a moment. Then $\mathcal{A}_{0\mu}(\lambda(\cdot - x), \phi) = \langle \phi, \oplus_{\nu=1}^3 \delta_{\mu\nu} \lambda(\cdot - x) \rangle_0 \in C_b^\infty(\mathbb{R}_x^3)$. Define the multiplication operators $\mathcal{A}_{0\mu}(\phi)$ and $\mathcal{B}_{0\mu}(\phi)$, $\mu = 1,2,3$, in $L^2(\mathbb{R}^3)$ by

$$\mathcal{A}_{0\mu}(\phi) = \int_{\mathbb{R}^3}^\oplus \mathcal{A}_{0\mu}(\lambda(\cdot - x), \phi) dx, \qquad \mathcal{B}_{0\mu}(\phi) = \int_{\mathbb{R}^3}^\oplus \mathcal{B}_{0\mu}(\lambda(\cdot - x), \phi) dx$$

and the Pauli operator on $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ by

$$(\tilde{H}_{\mathrm{PF0}}(\phi) f)(x,\sigma) := \left\{ \frac{1}{2}(-i\nabla - e\mathcal{A}_0(\phi))^2 + V + V_\phi(x,\sigma) \right\} f(x,\sigma) - e^{W_\phi(x,-\sigma)} f(x,-\sigma),$$

where we set

$$V_\phi(x,\sigma) := -\frac{e}{2}\sigma \mathcal{B}_{03}(\phi), \quad W_\phi(x,-\sigma) := \log[\frac{e}{2}(\mathcal{B}_{01}(\phi) - i\sigma\mathcal{B}_{02}(\phi))].$$

Lemma 3.2 *For each $\phi \in \mathcal{Q}_0$, $\tilde{H}_{\mathrm{PF0}}(\phi)$ is self-adjoint on $D(-\Delta)$ and it follows that $(e^{-t\tilde{H}_{\mathrm{PF0}}(\phi)}g)(x,\sigma) = \mathbb{E}^{x,\sigma}[e^{-\int_0^t V(B_s)ds}e^{\tilde{Z}_\phi(t)}g(\xi_t)]$, where*

$$\tilde{Z}_\phi(t) = -i\sum_{\mu=1}^{3}\int_0^t \mathcal{A}_{0\mu}(\lambda(\cdot - B_s),\phi)dB_{\mu,s} - \int_0^t V_\phi(B_s,\sigma_s)ds + \int_0^{t+} W_\phi(B_s,-\sigma_s)dN_s.$$

Proof: Since $\tilde{H}_{\mathrm{PF0}}(\phi)$ is a Pauli operator with the sufficiently smooth and compactly supported vector potential $\mathcal{A}_0(\phi)$, the lemma follows from [ALS83]. qed

Define $\tilde{H}_{\mathrm{PF0}} := \int_{\mathcal{Q}_0}^{\oplus} \tilde{H}_{\mathrm{PF0}}(\phi)d\mu_0$ and

$$\tilde{H}_{\mathrm{PF}} := \tilde{H}_{\mathrm{PF0}} \dot{+} H_{\mathrm{rad}}.$$

Here $\dot{+}$ denotes the quadratic form sum. The next lemma is the key lemma in this note.

Lemma 3.3 *Assume that $\lambda \in C_0^\infty(\mathbb{R}^3)$. Then $(F, e^{-tH_{\mathrm{PF}}}G) = (F, e^{-t\tilde{H}_{\mathrm{PF}}}G)$.*

Proof: Let $L^2_{\mathrm{fin}}(\mathcal{Q}_0)$ denote the finite particle subspace of $L^2(\mathcal{Q}_0)$. Define the dense subspace $\tilde{\mathcal{H}}_0 := C_0^\infty(\mathbb{R}^3 \times \mathbb{Z}_2)\hat{\otimes}L^2_{\mathrm{fin}}(\mathcal{Q}_0)$, where $\hat{\otimes}$ denotes the algebraic tensor product. It is seen that $\tilde{H}_{\mathrm{PF}} = H_{\mathrm{PF}}$ on $\tilde{\mathcal{H}}_0$, which implies that $\tilde{H}_{\mathrm{PF}} = H_{\mathrm{PF}}$ as a self-adjoint operator, since $\tilde{\mathcal{H}}_0$ is a core of H_{PF}. Hence the lemma follows. qed

By Lemma 3.3 it is enough to construct a functional integral representation of $(F, e^{-t\tilde{H}_{\mathrm{PF}}}G)$ instead of $(F, e^{-tH_{\mathrm{PF}}}G)$. By the Trotter-Kato product formula for the quadratic form sum [KM78], we have $(F, e^{-t\tilde{H}_{\mathrm{PF}}}G) = \lim_{n\to\infty}(F, (e^{-(t/n)\tilde{H}_{\mathrm{PF0}}}e^{-(t/n)H_{\mathrm{rad}}})^n G)$. To compute its right-hand side, we factorize $e^{-tH_{\mathrm{rad}}}$ as usual. Let $j_t : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^{3+1})$, $t \geq 0$, be defined by

$$\widehat{j_t f}(k,k_0) := \frac{e^{-itk_0}}{\sqrt{\pi}}\sqrt{\frac{\omega_{\mathrm{b}}(k)}{\omega_{\mathrm{b}}(k)^2 + |k_0|^2}}\hat{f}(k), \quad (k,k_0) \in \mathbb{R}^3 \times \mathbb{R}.$$

Thus j_t is a reality-preserving operator and $j_t^* j_s = e^{-|t-s|\omega_{\mathrm{b}}(-i\nabla)}$, $s,t \in \mathbb{R}$, follows. Define $J_t : L^2(\mathcal{Q}_0) \to L^2(\mathcal{Q}_1)$ by $J_t := \Gamma_{01}(j_t)$. Hence $J_t^* J_s = e^{-|t-s|H_{\mathrm{rad}}}$ follows on $L^2(\mathcal{Q}_0)$. We denote the L^p-norm on $(\mathcal{Q}_\beta, \mu_\beta)$ by $\|\cdot\|_p$. As is explained previously, $\Gamma_\beta(T)$ for $\|T\| \leq 1$ is a contraction operator on $L^2(\mathcal{Q}_\beta)$. It has also a particularly strong property, so-called hypercontractivity. From this the lemma below is proved in [HL07].

Lemma 3.4 *Let $\Phi \in L^1(\mathcal{Q}_1)$ and $F, G \in L^2(\mathcal{Q}_1)$. Then, for $a \neq b$, $(J_a F)\Phi(J_b G) \in L^1(\mathcal{Q}_1)$ and*

$$\int_{\mathcal{Q}_1} |(J_a F)\Phi(J_b G)| d\mu_1 \leq \|\Phi\|_1 \|F\|_2 \|G\|_2. \tag{3.2}$$

Let $E_{[a,b]}$ be the projection to the range of J_t, $t \in [a,b]$.

Lemma 3.5 *Assume that $\lambda \in C_0^\infty(\mathbb{R}^3)$. Let $0 \leq \ell < s \leq t$, $F \in \mathcal{E}_{[0,\ell]}$ and $G \in \mathcal{E}_{[s,t]}$. Then*

$$(F, J_s e^{-t\tilde{H}_{\rm PF0}} J_s^* G) = \sum_\sigma \int dx\, \mathbb{E}^{x,\sigma}[e^{-\int_0^t V(B_{s'})ds'} \int_{\mathcal{Q}_1} \overline{F(\xi_0)} e^{\tilde{X}_s(0,t)} E_s G(\xi_t) d\mu_1]. \tag{3.3}$$

Here $\tilde{X}_s(0,t)$ is defined by

$$\begin{aligned} \tilde{X}_s(0,t) &= -ie\sum_{\mu=1}^3 \int_0^t \mathcal{A}_{1\mu}(j_s\lambda(\cdot - B_{s'}))dB_{\mu,s'} - \int_0^t (-\frac{e}{2})\sigma_{s'}\mathcal{B}_{03}(j_s\lambda(\cdot - B_{s'}))ds' \\ &\quad + \int_0^{t+} \log[\frac{e}{2}(\mathcal{B}_{01}(j_s\lambda(\cdot - B_{s'}) - i\sigma_{s'}\mathcal{B}_0(j_s\lambda(\cdot - B_{s'})))]dN_{s'}. \end{aligned} \tag{3.4}$$

Now we define the $L^2(\mathbb{R}^{3+1})$-valued stochastic integral $\int_0^t j_s\lambda(\cdot - B_s)dB_{\mu,s}$ by a limiting procedure. Let $\chi_n(s)$ be the step function on the interval $[0,t]$ given by

$$\chi_n(s) := \sum_{j=1}^n \frac{t(j-1)}{n} \chi_{(t(j-1)/n, tj/n]}(s) \tag{3.5}$$

Define the sequence of the $L^2(\mathbb{R}^{3+1})$-valued random variable $\xi_n^\mu : \Omega \to L^2(\mathbb{R}^{3+1})$ by $\xi_n^\mu := \int_0^t j_{\chi_n(s)}\lambda(\cdot - B_s)dB_{\mu,s}$. Since this sequence is Cauchy, we define

$$\int_0^t j_s\lambda(\cdot - B_s)dB_{\mu,s} := s - \lim_{n\to\infty} \xi_n^\mu, \quad \mu = 1,2,3,$$

and set

$$\int_0^t \mathcal{A}_{0\mu}(j_s\lambda(\cdot - B_s))dB_{\mu,s} := \mathcal{A}_{0\mu}(\int_0^t j_s\lambda(\cdot - B_s)dB_{\mu,s}).$$

The next theorem is the main results of our investigation.

Theorem 3.6 *It follows that*

$$(F, e^{-tH_{\rm PF}} G) = \sum_\sigma \int dx\, \mathbb{E}^{x,\sigma}[e^{-\int_0^t V(B_s)ds} \int_{\mathcal{Q}_1} d\mu_1 \overline{J_0 F(\xi_0)} e^{X(0,t)} J_t G(\xi_t)]. \tag{3.6}$$

Here the exponent $X(0,t)$ is given by

$$\begin{aligned} X(0,t) &= -ie\sum_{\mu=1}^{3}\int_0^t \mathcal{A}_{1\mu}(j_s\lambda(\cdot - B_s))dB_{\mu,s} - \int_0^t (-\frac{e}{2})\sigma_s \mathcal{B}_{13}(j_s\lambda(\cdot - B_s))ds \\ &\quad + \int_0^{t+} \log[\frac{e}{2}(\mathcal{B}_{11}(j_s\lambda(\cdot - B_s)) - i\sigma_s\mathcal{B}_{12}(j_s\lambda(\cdot - B_s)))]dN_s. \end{aligned}$$

Proof: We outline the proof. See [HL07] for detail. In a moment we assume that $\hat{\varphi}/\sqrt{\omega_b} \in C_0^\infty(\mathbb{R}^3)$. We can see that $\mathbb{E}^{x,\sigma}[e^{-\int_0^t V(B_l)dl}e^{\tilde{X}_s(0,t)}G(\xi_t)] \in \tilde{\mathcal{H}}$ for $G \in \tilde{\mathcal{H}}$. Then we define $S_{t,s} : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ by

$$(S_{t,s}G)(x,\sigma) := \mathbb{E}^{x,\sigma}[e^{-\int_0^t V(B_l)dl}e^{\tilde{X}_s(0,t)}G(\xi_t)].$$

Here $\tilde{X}_s(0,t)$ is defined in (3.4). By making use of Markov properties of both B_s and N_s, we can see that

$$(S_{t',s'}S_{t,s}G)(x,\sigma) = \mathbb{E}^{x,\sigma}[e^{-\int_0^{t+t'} V(B_l)dl}e^{\tilde{X}_{s'}(0,t')+\tilde{X}_s(t',t+t')}G(\xi_{t+t'})]. \tag{3.7}$$

Let $E_t = J_tJ_t^*$ and $\prod_{j=1}^n T_j := T_1T_2\cdots T_n$ up to the order. Then using the identity $H_{\mathrm{PF}} = \tilde{H}_{\mathrm{PF}}$, we have

$$\begin{aligned} (F, e^{-tH_{\mathrm{PF}}}G) &= (F, e^{-t(\tilde{H}_{\mathrm{PF}0} \dot{+} H_{\mathrm{rad}})}G) \\ &= \lim_{n\to\infty}(F, (e^{-(t/n)\tilde{H}_{\mathrm{PF}0}}e^{-(t/n)H_{\mathrm{rad}}})^n G) \\ &= \lim_{n\to\infty}(J_0F, \left(\prod_{j=0}^{n-1} J_{jt/n}e^{-(t/n)\tilde{H}_{\mathrm{PF}0}}J_{jt/n}^*\right) J_tG) \\ &= \lim_{n\to\infty}(J_0F, \left(\prod_{j=0}^{n-1} E_{jt/n}S_{t/n,j}E_{jt/n}\right) J_tG) \\ &= \lim_{n\to\infty}(J_0F, \left(\prod_{j=0}^{n-1} S_{t/n,tj/n}\right) J_tG) \\ &= \lim_{n\to\infty}\sum_\sigma\int dx\, \mathbb{E}^{x,\sigma}[e^{-\int_0^t V(B_{s'})ds'}\int_{Q_1} d\mu_1 \overline{J_0F(\xi_0)}e^{X_n(0,t)}J_tG(\xi_t)], \end{aligned}$$

where we used the formula $J_s^*J_t = e^{-|t-s|H_{\mathrm{rad}}}$ in the third line, Lemma 3.5 in the forth line, the Markov property of $E_{[\ldots]}$ in the fifth line, and (3.7) in the sixth line. Here we set

$$X_n(0,t) = X_{1,n}(t) + X_{2,n}(t) + X_{3,n}(t),$$

where

$$
\begin{aligned}
X_{1,n}(t) &= -ie\mathcal{A}_1(\oplus_{\mu=1}^3 \int_0^t j_{\chi_n(s)}\lambda(\cdot - B_s)dB_{\mu,s}), \\
X_{2,n}(t) &= -\int_0^t V_{\chi_n(s)}(B_s, \sigma_s)ds, \\
X_{3,n}(t) &= \int_0^t W_{\chi_n(s)}(B_s, -\sigma_s)dN_s,
\end{aligned}
$$

and

$$V_s(x,\sigma) := -\frac{e}{2}\sigma\mathcal{B}_{13}(j_s\lambda(\cdot - x)), \tag{3.8}$$

$$W_s(x,-\sigma) := \log[\frac{e}{2}(\mathcal{B}_{11}(j_s\lambda(\cdot - x)) - i\sigma\mathcal{B}_{12}(j_s\lambda(\cdot - x)))]. \tag{3.9}$$

We have

$$
\begin{aligned}
&\sum_\sigma \int dx\, \mathbb{E}^{x,\sigma}\int_{Q_1} d\mu_1 e^{-\int_0^t V(B_s)ds}|J_0F(\xi_0)||J_tG(\xi_t)||e^{X_n(t)} - e^{X(t)}| \\
&\le V_M\|G\|_{\tilde{\mathcal{H}}}\mathbb{E}^{x,\sigma}[\Big(\sum_\sigma\int dx\, \|F(x,\sigma)\|_2^2\|e^{X_n(t)} - e^{X(t)}\|_1^2\Big)^{1/2}]. \qquad (3.10)
\end{aligned}
$$

We show that the right-hand side above goes to zero as $n \to \infty$. For each $\omega \in \Omega$, there exists $N = N(\omega) \in \mathbb{N}$ such that

$$\|e^{X_n(0,t)}\|_1^2 \le \exp\left(\frac{e^2}{4}t^2\int_{\mathbb{R}^3}|\hat{\varphi}(k)|^2|k|dk\right)(\frac{e}{2})^{2N}N!\|\sqrt{|k|}\hat{\varphi}\|^{2N} := C(\omega).$$

Then $\mathbb{E}^{x,\sigma}[C(\cdot)^{1/2}] < \infty$ follows. Similarly $\|e^{X(t)}\|_1 < C'(\omega)$ and $\mathbb{E}^{x,\sigma}[C'(\cdot)^{1/2}] < \infty$ follows for some $C'(\omega)$. Note that C and C' are independent of $(x,\sigma) \in \mathbb{R}^3 \times \mathbb{Z}_2$ and n. Thus by (3.10) and the dominated convergence theorem, it is enough to show that for almost every $\omega \in \Omega$, $e^{X_n(t)} \to e^{X(t)}$ as $n \to \infty$ in $L^1(Q_1)$. We have

$$
\begin{aligned}
e^{X_n(0,t)} - e^{X(0,t)} &= \underbrace{e^{X_{1,n}(t)}e^{X_{2,n}(t)}e^{X_{3,n}(t)} - e^{X_1(t)}e^{X_{2,n}(t)}e^{X_{3,n}(t)}}_{=\mathrm{I}} \\
&\quad + \underbrace{e^{X_1(t)}e^{X_{2,n}(t)}e^{X_{3,n}(t)} - e^{X_1(t)}e^{X_2(t)}e^{X_{3,n}(t)}}_{=\mathrm{II}} \\
&\quad + \underbrace{e^{X_1(t)}e^{X_2(t)}e^{X_{3,n}(t)} - e^{X_1(t)}e^{X_2(t)}e^{X_3(t)}}_{=\mathrm{III}}. \qquad (3.11)
\end{aligned}
$$

We estimate I, II and III. We have

$$\|\mathrm{I}\|_1 \le \|e^{X_{1,n}(t)} - e^{X_1(t)}\|_2\|e^{X_{2,n}(t)}e^{X_{3,n}(t)}\|_2, \tag{3.12}$$

$$\|\mathrm{II}\|_1 \le \|e^{X_{2,n}(t)} - e^{X_2(t)}\|_2\|e^{X_{3,n}(t)}\|_2, \tag{3.13}$$

$$\|\mathrm{III}\|_1 \le \|e^{X_2(t)}\|_2\|e^{X_{3,n}(t)} - e^{X_3(t)}\|_2. \tag{3.14}$$

and that there exists $N = N(\omega)$ such that

$$\|e^{X_{2,n}(t)}e^{X_{3,n}(t)}\|_2^2 \le e^{4(e/2)^2t^2\|\sqrt{|k|}\hat\varphi\|^2}(e/2)^{4N}(2N)!\|\sqrt{|k|}\hat\varphi\|^{4N}, \tag{3.15}$$

$$\|e^{X_{3,n}(t)}\|_2^2 \le (e/2)^{2N}N!\|\sqrt{|k|}\hat\varphi\|^{2N}, \tag{3.16}$$

$$\|e^{X_{2,n}(t)}\|_2^2 \le e^{(e/2)^2t^2\|\sqrt{|k|}\hat\varphi\|^2}. \tag{3.17}$$

From (3.12)-(3.17) and the dominated convergence theorem, it is enough to show that $\|e^{X_{j,n}(t)} - e^{X_j(t)}\|_2 \to 0$ as $n \to \infty$ for $j = 1,2,3$ for almost every $\omega \in \Omega$.

First we estimate I. Let $\varrho_n = \oplus_{\mu=1}^3 \int_0^t \left\{j_{\chi_n(s)}\lambda(\cdot - B_s) - j_s\lambda(\cdot - B_s)\right\} dB_\mu$. Then we have $(e^{X_{1,n}(t)}, e^{X_1(t)})_2 = \exp\left(-\frac{e^2}{2}q_1(\varrho_n,\varrho_n)\right)$. Since

$$\mathbb{E}^{x,\sigma}[q_1(\varrho_n,\varrho_n)] \le \frac{3}{2}\mathbb{E}^{x,\sigma}[\int_0^t \left\{2\|\lambda\|^2 - 2\Re(\lambda(\cdot - B_s), e^{-|\chi_n(s)-s|\omega_b}\lambda(\cdot - B_s))\right\} ds] \to 0$$

as $n \to 0$. This implies that there exists a subsequence n' such that for almost every $\omega \in \Omega$, $\lim_{n\to\infty}(e^{X_{1,n'}(t)}, e^{X_1(t)})_2 = 1$ and then $\|e^{X_{1,n'}(t)} - e^{X_1(t)}\|_2 \to 0$. We reset n' as n. Then $\lim_{n\to\infty}\|\mathrm{I}\|_1 = 0$ follows from (3.12). Second we estimate II. A direct computation yields that

$$\begin{aligned}
&\|e^{X_{2,n}(t)}\|_2^2 \\
&= \exp\left((\frac{e}{2})^2\int_0^t ds\int_0^t ds'\sigma_s\sigma_{s'}\int dk\frac{|\hat\varphi(k)|^2}{\omega_b(k)}e^{-ik(B_s-B_{s'})}(|k_1|^2+|k_2|^2)e^{-|\chi_n(s)-\chi_n(s')|\omega_b(k)}\right) \\
&\to \exp\left((\frac{e}{2})^2\int_0^t ds\int_0^t ds'\sigma_s\sigma_{s'}\int dk\frac{|\hat\varphi(k)|^2}{\omega_b(k)}e^{-ik(B_s-B_{s'})}(|k_1|^2+|k_2|^2)e^{-|s-s'|\omega_b(k)}\right) \\
&= \|e^{X_2(t)}\|_2^2
\end{aligned}$$

and

$$\begin{aligned}
&(e^{X_{2,n}(t)}, e^{X_2(t)})_2 \\
&= \exp\left(\frac{1}{4}(\frac{e}{2})^2\int_0^t ds\int_0^t ds'\sigma_s\sigma_{s'}\int dk\frac{|\hat\varphi(k)|^2}{\omega_b(k)}e^{-ik\cdot(B_s-B_{s'})}(|k_1|^2+|k_2|^2)\right. \\
&\qquad \left.\times(e^{-|s-s'|\omega_b(k)} + e^{-|s-\chi_n(s')|\omega_b(k)} + e^{-|s'-\chi_n(s)|\omega_b(k)} + e^{-|\chi_n(s)-\chi_n(s')|\omega_b(k)})\right) \\
&\to \exp\left((\frac{e}{2})^2\int_0^t ds\int_0^t ds'\sigma_s\sigma_{s'}\int dk\frac{|\hat\varphi(k)|^2}{\omega_b(k)}e^{-ik\cdot(B_s-B_{s'})}(|k_1|^2+|k_2|^2)e^{-|s-s'|\omega_b(k)}\right) \\
&= \|e^{X_2(t)}\|_2^2
\end{aligned}$$

as $n \to \infty$. Then $\lim_{n\to\infty} \|\mathrm{II}\|_2 = 0$ follows from (3.13). Finally we estimate III. For the notational convenience, we set $\mathcal{B}_{1\mu}(j_l\lambda(\cdot - B_s)) := \mathcal{B}_{1\mu}(l, s)$. For each $\omega \in \Omega$ we have

$$\exp(X_{3,n}(t)) = \prod_{j=1}^{n} \prod_{\substack{s\in D(p)\\ t(j-1)/n\le s\le tj/n}} \frac{e}{2}(\mathcal{B}_{11}(t(j-1)/n, s) - i\sigma(-1)^{N_s}\mathcal{B}_{12}(t(j-1)/n, s)).$$

For sufficiently large n, the number of s_i's contained in the interval $(t(j-1)/n, tj/n]$ is at most one. Then assume that n is sufficiently large and we denote the interval containing s_j by $(n(s_j), n(s_j) + t/n]$, $j = 1, ..., N$. Hence

$$\exp(X_{3,n}(t))1_{Q_1} = \prod_{j=1}^{N} \frac{e}{2}(\mathcal{B}_{11}(n(s_j), s_j) - i\sigma(-1)^{N_{s_j}}\mathcal{B}_{12}(n(s_j), s_j))1_{Q_1}$$

$$\to \prod_{j=1}^{N} \frac{e}{2}(\mathcal{B}_{11}(s_j, s_j) - i\sigma(-1)^{N_{s_j}}\mathcal{B}_{12}(s_j, s_j))1_{Q_1}$$

$$= \exp\left(\int_0^{t+} \log[\frac{e}{2}(\mathcal{B}_{11}(j_s\lambda(\cdot - B_s)) - i\sigma_s\mathcal{B}_{12}(j_s\lambda(\cdot - B_s)))]dN_s\right) 1_{Q_1} = \exp(X_3(t))1_{Q_1}$$

strongly as $n \to \infty$, since $n(s_j) \to s_j$ as $n \to \infty$. Then $\lim_{n\to\infty} \|e^{X_{3,n}(t)} - e^{X_3(t)}\|_2 = 0$ and $\lim_{n\to\infty} \|\mathrm{III}\| = 0$ follows from (3.14). Combining these estimates we can conclude (3.6). Finally we show (3.6) for $\hat{\varphi}$ such that $\sqrt{\omega_b}\hat{\varphi}, \hat{\varphi}/\sqrt{\omega_b} \in L^2(\mathbb{R}^3)$ by a limiting argument. qed

4 Concluding remarks

4.1 Breaking of degenerate ground states

It is established that H_{PF} has degenerate ground states for sufficiently small coupling constants [HS01, Hir05]. Let us consider some toy model defined by H_{PF} with spin interaction replaced by the Fermion harmonic oscillator:

$$H(\epsilon) = \frac{1}{2}(-i\nabla - e\mathcal{A}_0)^2 + V + H_{\mathrm{rad}} + \underbrace{\frac{1}{2}\epsilon(\sigma_3 + i\sigma_2)(\sigma_3 - i\sigma_2) - \frac{1}{2}\epsilon}_{=-\epsilon\sigma_1}, \quad \epsilon \in \mathbb{R}.$$

When $\epsilon = 0$, the ground state of $H(0)$ is two fold degenerate for arbitrary values of coupling constants. Nevertheless we will show that the ground state of $H(\epsilon)$ for $\epsilon \neq 0$ is unique for arbitrary values of coupling constants.

Corollary 4.1 *Let* $\theta = e^{-i(\pi/2)N}$. *Then* $\theta^{-1}e^{H(\epsilon)}\theta$ *is positivity improving for* $\epsilon > 0$ *and, in particular, the ground state of* $H(\epsilon)$, $\epsilon \neq 0$ *is unique whenever it exists.*

Proof: Note that $H(\epsilon)$ and $H(-\epsilon)$ are isomorphic. Let $\epsilon > 0$. By a direct computation, we have[2]

$$(F, \theta^{-1}e^{-tH(\epsilon)}\theta G) = \int dx \mathbf{E}_{P_W^x} e^{-\int_0^t V(B_s)ds}$$
$$\times \sum_{\sigma \in \mathbb{Z}_2} [(F(x,\sigma), T_t G(B_t,\sigma)) \cosh \epsilon t + (F(x,\sigma), T_t G(B_t, -\sigma)) \sinh \epsilon t],$$

where $A = e\sum_{\mu=1}^3 \int_0^t \mathcal{A}_{1\mu}(j_s\lambda(\cdot - B_s))dB_{\mu,s}$ and $T_t := J_0^*\theta^{-1}e^{-iA}\theta J_t$. Then for $0 \leq F, G \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2 \times \mathcal{Q}_0)$ but $F \not\equiv 0$ and $G \not\equiv 0$, $(F, \theta^{-1}e^{-tH(\epsilon)}G) > 0$, since T_t is positivity improving which is proven in [Hir00a]. Then $e^{-tH(\epsilon)}$ is positivity improving. The uniqueness of the ground state follows from the infinite dimensional version of Perron Frobenius theorem. qed

4.2 Energy inequality

We can also derive some energy inequality from the functional integral representation which is an extension of the so-called the diamgnetic inequality. Although $A_{\hat\varphi}$ and $B_{\hat\varphi}$ are connected with $\mathrm{rot}A_{\hat\varphi} = B_{\hat\varphi}$, $A_{\hat\varphi}$ and $B_{\hat\varphi}$ are regarded as independent operators. The bottom of the spectrum of H_{PF} is denoted by

$$\inf \sigma(H_{\mathrm{PF}}) = E(A_{\hat\varphi}, B_{\hat\varphi 1}, B_{\hat\varphi 2}, B_{\hat\varphi 3}).$$

Then $E(0,0,0,0) \leq E(A_{\hat\varphi},0,0,0)$, is called diamagnetic inequality. We extend this inequality. We define $H_{\mathrm{PF}}^{\perp}$ by

$$H_{\mathrm{PF}}^{\perp} = H_{\mathrm{p}} + H_{\mathrm{rad}} - \frac{e}{2}\begin{bmatrix} \mathcal{B}_{03} & \sqrt{\mathcal{B}_{01}^2 + \mathcal{B}_{02}^2} \\ \sqrt{\mathcal{B}_{01}^2 + \mathcal{B}_{02}^2} & -\mathcal{B}_{03} \end{bmatrix}.$$

[2]Assume that $\epsilon \neq 0$. Then

$$(F, e^{-tH(\epsilon)}G) = \sum_\sigma \int dx\, \mathbf{E}^{x,\sigma}[\epsilon^{N_t} e^{-\int_0^t V(B_s)ds}(J_0F(\xi_0), e^{-iA}J_tG(\xi_t))].$$

Since the interaction term is infinitesimally small with respect to the decoupled Hamiltonian $H_p + H_{rad}$, $H_{PF}^{\perp}$ is self-adjoint on $D(-\Delta) \cap D(H_{rad})$ and bounded from below. By Theorem 3.6 the functional integral representation of $e^{-tH_{PF}^{\perp}}$ is as follows:

$$(F, e^{-tH_{PF}^{\perp}}G) = \sum_{\sigma} \int dx\, \mathbb{E}^{x,\sigma}[e^{-\int_0^t V(B_s)ds} \int_{Q_1} d\mu_1 \overline{J_0 F(\xi_0)} e^{X_{\perp}(t)} J_t G(\xi_t)],$$

where

$$X_{\perp}(t) = \int_0^t \frac{e}{2}\sigma_s \mathcal{B}_{13}(j_s\lambda(\cdot - B_s))ds + \int_0^{t+} \log\left[\frac{e}{2}\sqrt{\mathcal{B}_{11}(j_s\lambda(\cdot - B_s))^2 + \mathcal{B}_{12}(j_s\lambda(\cdot - B_s))^2}\right] dN_s.$$

Corollary 4.2 *It follows that* $|(F, e^{-tH_{PF}}G)| \leq (|F|, e^{-tH_{PF}^{\perp}}|G|)$ *and*

$$\max \left\{ \begin{array}{l} E(0, \sqrt{\mathcal{B}_{01}^2 + \mathcal{B}_{02}^2}, 0, \mathcal{B}_{03}) \\ E(0, \sqrt{\mathcal{B}_{03}^2 + \mathcal{B}_{01}^2}, 0, \mathcal{B}_{02}) \\ E(0, \sqrt{\mathcal{B}_{02}^2 + \mathcal{B}_{03}^2}, 0, \mathcal{B}_{01}) \end{array} \right\} \leq E(\mathcal{A}_0, \mathcal{B}_{01}, \mathcal{B}_{02}, \mathcal{B}_{03}). \tag{4.1}$$

Proof: Since $H_{PF}^{\perp}$ is unitarily equivalent to $H_{PF}^{\perp}$ with e replaced by $-e$, we may assume that $e > 0$ without loss of generality. By the functional integral representation of $e^{-tH_{PF}}$, we have

$$|(F, e^{-tH_{PF}}G)| \leq \sum_{\sigma} \int dx\, \mathbb{E}^{x,\sigma}[e^{-\int_0^t V(B_s)ds} \int_{Q_1} d\mu_1 (J_0|F(\xi_0)|)(J_t|G(\xi_t)|) e^{X_{\perp}(t)}],$$

where we used that $|J_t G| \leq J_t|G|$, since J_t is positivity preserving. Then the desired inequality follows. From this, $E(0, \sqrt{\mathcal{B}_{01}^2 + \mathcal{B}_{02}^2}, 0, \mathcal{B}_{03}) \leq E(A_{\varphi}, \mathcal{B}_{01}, \mathcal{B}_{02}, \mathcal{B}_{03})$ is obtained. (4.1) follows from the symmetry. Then the corollary is complete. qed

4.3 Translation invariance

Let $V = 0$. Then H_{PF} is translation invariant and decomposed as $H_{PF} = \int_{\mathbb{R}^3}^{\oplus} H_{PF}(P)dP$ with respect to the spectrum of the total momentum. In the spinless case the functional integral representation of $e^{-tH_{PF}(P)}$ is constructed for each $P \in \mathbb{R}^3$ and some energy inequality is shown in [Hir06]. When $H_{PF}(P)$ includes spin, we can also construct it. See [HL07] for detail.

Acknowledgments This work is financially supported by Grant-in-Aid for Science Research (C) 17540181 from JSPS.

References

[ALS83] G. F. De Angelis, G.J. Lasinio and M. Sirugue, Probabilistic solution of Pauli type equations, *J. Phys. A:Math. Gen.* **16** (1983), 2433–2444.

[Hir97] F. Hiroshima, Functional integral representations of quantum electrodynamics, *Rev. Math. Phys.* **9** (1997), 489–530.

[Hir00a] F. Hiroshima, Ground states of a model in nonrelativistic quantum electrodynamics II, *J. Math. Phys.* **41** (2000), 661–674.

[Hir00b] F. Hiroshima, Essential self-adjointness of translation-invariant quantum field models for arbitrary coupling constants, *Commun. Math. Phys.* **211** (2000), 585–613.

[Hir02] F. Hiroshima, Self-adjointness of the Pauli-Fierz Hamiltonian for arbitrary values of coupling constants, *Ann. Henri Poincaré* **3** (2002), 171–201.

[Hir04] F. Hiroshima, Analysis of ground states of atoms interacting with a quantized radiation field, *Topics in the theory of Schrödinger operators* ed. H. Araki and H. Ezawa, 145–272, World Scientific, 2004.

[Hir05] F. Hiroshima, Multiplicity of ground states in quantum field models: applications of asymptotic fields, *J. Funct. Anal.* **224** (2005), 431–470.

[Hir06] F. Hiroshima, Fiber Hamiltonians in nonrelativistic quantum electrodynamics, submitted.

[HL07] F. Hiroshima and J. Lőrinczi, Functional integral representation of nonrelativistic quantum electrodynamics with spin 1/2, preprint, 2007.

[HS01] F. Hiroshima and H. Spohn, Ground state degeneracy of the Pauli-Fierz model with spin, *Adv. Theor. Math. Phys.* **5** (2001), 1091–1104.

[KM78] T. Kato and K. Masuda, Trotter's product formula for nonlinear semigroups generated by the subdifferentiables of convex functionals, *J. Math. Soc. Japan* **30** (1978), 169-178.

[Sas06] I. Sasaki, Ground state of a model in relativistic quantum electrodynamics with a fixed total momentum, math-ph/0606029, preprint 2006.

[Sim74] B. Simon, *The $P(\phi)_2$ Euclidean Quantum Field Theory*, Princeton University Press 1974.

[Spo04] H. Spohn, Dynamics of charged particles and their radiation field, Cambridge University Press, 2004.