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ON THE CROSSED BURNSIDE RINGS

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1. Bouc functors

1.1. Notation and Definition. *Let G and H be finite groups. An (H, G)-**biset**, or a **biset** shortly, is a set with a left $(H\times G^{op})$-action, i.e., a set U with a left H-action and a right G-action which commute.*

If K is another group, and V is a (K, H)-biset, then the product $V\times U$ by the right action of H given by $(v, u)h=(vh, h^{-1}u)$ for $v\in V$, $u\in U$, and $h\in H$. The class of (v, u) in $V\times_{H}U$ is denoted by $(v,_{H}u)$. The set $V\times_{H}U$ is a (K, G)-biset for the action given by

$$k(v,_{H}u)g=(kv,_{H}ug)$$

for $k\in K$, $g\in G$, $u\in U$, and $v\in V$.

Denote by $\mathcal{C}_{p}$ the following category:

- *The objects of $\mathcal{C}_{p}$ are the finite p-groups.*
- *If P and Q are finite p-groups, then $\mathrm{Hom}_{\mathcal{C}_{p}}(P, Q)=B(Q\times P^{op})$ is the Burnside group of finite (Q, P)-bisets. An element of this group is called a **virtual** (Q, P)-biset.*
- *The composition of morphisms is $\mathbb{Z}$-bilinear, and if P, Q, R are finite p-groups, if U is a finite (Q, P)-biset, and V is a finite (R, Q)-biset, then the composition of (the isomorphism classes of) V and U is the (isomorphism class) of $V\times_{Q}U$. The identity morphism Id_{P} of the p-group P is the class of the set P, with left and right action by multiplication.*

*Let $\mathcal{F}_{p}$ denote the category of additive functors from $\mathcal{C}_{p}$ to the category $\mathbb{Z}$-Mod of abelian groups. An object of $\mathcal{F}_{p}$ is called a **Bouc functor** (defined over p-groups, with values in $\mathbb{Z}$-Mod)(see* [Th06], [Bo06]*).*

1.2. Notation. The Bouc functor of *Burnside group* will denote by B. The Bouc functor of *rational representations* will denote by $R_{\mathbb{Q}}$. The G-poset of the family of all subgroups of a finite group G will denote by $\mathcal{S}(G)$. If X is a G-set, denote by $G\backslash X$ a family of G-orbits, and by $[G\backslash X]$ a set of representatives of $G\backslash X$.

2. The Dade group

2.1. Some known Dade groups: The structure of $D(P)$ is known for any 2-group P of normal 2-rank 1: when P is generalized quaternion, the result is due to Dade, and the other cases have been solved by Carlson and Thévenaz:

Theorem 2.2. (Dade [Da78a], Carlson-Thévenaz [CT00])
(1) $D(C_{2^{n}})\cong(\mathbb{Z}/2\mathbb{Z})^{n-1}$, *and* $D(C_{p^{n}})\cong(\mathbb{Z}/2\mathbb{Z})^{n}$, *if* $p\geq 3$.
(2) $D(D_{2^{n}})\cong \mathbb{Z}^{2n-3}$.
(3) $D(SD_{2^{n}})\cong \mathbb{Z}^{2n-4}\oplus \mathbb{Z}/2\mathbb{Z}$.
(4) $D(Q_{2^{n}})\cong \mathbb{Z}^{2n-5}\oplus \mathbb{Z}/4\mathbb{Z}\oplus \mathbb{Z}/2\mathbb{Z}$, *for* $n\geq 4$.
(5) $D(Q_{8})\cong \mathbb{Z}\oplus \mathbb{Z}/4\mathbb{Z}\oplus \mathbb{Z}/2\mathbb{Z}$, *if the ground field contains all cubic roots of unity, and* $D(Q_{8})\cong \mathbb{Z}\oplus \mathbb{Z}/4\mathbb{Z}$ *otherwise.*

The ingredient of the present note is

Theorem 2.3. (Bouc-Thévenaz [BT00] Theorem 10.4) *There is an exact sequence of functors*

$$0 \longrightarrow \mathbb{Q}D \xrightarrow{\alpha} \mathbb{Q}B \xrightarrow{\varepsilon} \mathbb{Q}R_{\mathbb{Q}} \longrightarrow 0$$

where $\varepsilon(P):\mathbb{Q}B(P)\rightarrow \mathbb{Q}R_{\mathbb{Q}}(P)$ is the morphism mapping a P-set to the corresponding permutation module over $\mathbb{Q}$.

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We could determine the difference $\operatorname{rank}B^c(P) - \operatorname{rank}R_{\mathbb{Q}}(\mathcal{D}(P))$ by using a result as follows:

Theorem 2.4. (Bouc-Thévenaz [BT00] Theorem A) *The torsion-free rank of the Dade group $D(P)$ is equal to the number of conjugacy classes of non-cyclic subgroups of P.*

3. The crossed Burnside ring and the rational representations of Drinfel'd double

3.1. Definition. Let M be one of the Bouc functors $\mathbb{Q}D$, $\mathbb{Q}B$ and $\mathbb{Q}R_{\mathbb{Q}}$. We use a construction of Dress for Mackey functors for obtaining a module from M. Let P be a p-group. Now we set

$$\begin{aligned} M(X) &= \left(\bigoplus_{x\in X} M(P_x)\right)^P \\ &= \left\{(m(x)) \in \bigoplus_{x\in X} M(P_x) \,\middle|\, {}^g(m(x)) = m(gx) \forall g \in P\right\}, \end{aligned}$$

where P_x is the stabilizer of x in P.

Corollary 3.2. *Let P be a p-group and X a P-set. Then there is an exact sequence of $\mathbb{Q}$-vector spaces*

$$0 \longrightarrow \mathbb{Q}D(X) \xrightarrow{\alpha} \mathbb{Q}B(X) \xrightarrow{\varepsilon} \mathbb{Q}R_{\mathbb{Q}}(X) \longrightarrow 0.$$

3.3. Notation. We denote by $B^c(P)$ the *crossed Burnside ring* of P, i.e. the Grothendieck ring of the category of finite crossed P-sets over P^c, for relations given by decomposition into disjoint union of crossed P-sets, the ring structure being induced by the product of crossed P-sets. Also we denote by $R_{\mathbb{Q}}(\mathcal{D}(P))$ the *rational representation ring* of the *Drinfel'd double* $\mathcal{D}(P) = (\mathbb{Q}P)^* \otimes \mathbb{Q}P$ for the group algebra $\mathbb{Q}P$.

Corollary 3.4. *Let P be a p-group. Then there is an exact sequence of $\mathbb{Q}$-vector spaces*

$$0 \longrightarrow \mathbb{Q}D(P^c) \xrightarrow{\alpha} \mathbb{Q}B^c(P) \xrightarrow{\varepsilon} \mathbb{Q}R_{\mathbb{Q}}(\mathcal{D}(P)) \longrightarrow 0.$$

In particular, we have

$$\operatorname{rank}B^c(P) = \operatorname{rank}R_{\mathbb{Q}}(\mathcal{D}(P)) + \dim_{\mathbb{Q}} \mathbb{Q}D(P^c).$$

Corollary 3.5. *Let P be a p-group. Then the following numbers are equal:*

(1) $\operatorname{rank}B^c(P)$.

(2) $\operatorname{rank}R_{\mathbb{Q}}(\mathcal{D}(P)) + \sum_{g\in[P\backslash P^c]} \dim_{\mathbb{Q}} \mathbb{Q}D(C_P(g))$.

(3) $\displaystyle\sum_{Q\in[P\backslash \mathcal{S}(P)]} \frac{|C_P(Q)|}{|N_P(Q)|} \cdot |Q| \left(\sum_{x\in Q/Q'} \frac{1}{|x|}\right)$.

(4) $\displaystyle\sum_{Q\in[P\backslash \mathcal{S}(P)]} |N_P(Q)\backslash C_P(Q))|$.

(5) $\displaystyle\sum_{g\in[P\backslash P^c]} |C_P(g)\backslash \mathcal{S}(C_P(g))|$.

Corollary 3.6. *Let P be a p-group. Then*

$$\operatorname{rank}B^c(P) = \operatorname{rank}R_{\mathbb{Q}}(\mathcal{D}(P)) + \sum_{g\in[P\backslash P^c]} |C_P(g)\backslash \mathcal{S}_{\mathrm{non}}(C_P(g))|,$$

where $\mathcal{S}_{\mathrm{non}}(C_P(g))$ is the $C_P(g)$-poset of non-cyclic subgroups of $C_P(g)$ with $C_P(g)$-action defined by conjugation.

Corollary 3.7. *Let P be a cyclic p-group. Then*

$$\operatorname{rank}B^c(P) = \operatorname{rank}R_{\mathbb{Q}}(\mathcal{D}(P)).$$

3.8. Some small 2-groups. We summarize basic facts on the structure of the centralizers of the representative of a conjugacy class of dihedral, semi-dihedral and generalized quaternion 2-groups (see, for instance, III.17 of [Er90]). In the rest of the paper, we always denote by z the central elements of order 2 of the group considered. Suppose that

$$D_{2^n} = \langle x, y | x^{2^{n-1}} = y^2 = 1,\ y^{-1}xy = x^{-1} \rangle$$

is a dihedral group of order 2^n $(n \geq 2)$. Then the centralizers of 1 and z are D_{2^n}. The centralizers of y and xy are Klein four groups. The centralizers of the representative of the other $2^{n-2} - 1$ conjugacy classes are cyclic subgroups $(n \geq 3)$. Suppose that

$$SD_{2^n} = \langle x, y | x^{2^{n-1}} = y^2 = 1,\ y^{-1}xy = x^{-1+2^{n-2}} \rangle$$

is a semi-dihedral group of order 2^n $(n \geq 4)$. Then the centralizers of 1 and z are SD_{2^n}. The centralizer of y is a Klein four group. The centralizers of the representative of the other 2^{n-2} conjugacy classes are cyclic subgroups. Suppose that

$$Q_{2^n} = \langle x, y | x^{2^{n-2}} = y^2,\ y^4 = 1,\ y^{-1}xy = x^{-1+2^{n-2}} \rangle$$

is a generalized quaternion group of order 2^n $(n \geq 3)$. Then the centralizers of 1 and z are Q_{2^n}. The centralizers of the representative of the other $2^{n-2} + 1$ conjugacy classes are cyclic subgroups.

Corollary 3.9. *Let P be a dihedral group D_{2^n} of order 2^n $(n \geq 2)$. Then*

$$\mathrm{rank} B^c(P) - \mathrm{rank} R_{\mathbb{Q}}(\mathcal{D}(P)) = 4n - 4.$$

Corollary 3.10. *Let P be a semi-dihedral group SD_{2^n} of order 2^n $(n \geq 4)$. Then*

$$\mathrm{rank} B^c(P) - \mathrm{rank} R_{\mathbb{Q}}(\mathcal{D}(P)) = 4n - 7.$$

Corollary 3.11. *Let P be a generalized quaternion group Q_{2^n} of order 2^n $(n \geq 3)$. Then*

$$\mathrm{rank} B^c(P) - \mathrm{rank} R_{\mathbb{Q}}(\mathcal{D}(P)) = 4n - 10.$$

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