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Problems on group modules coming from actions on C*-algebras

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1 Introduction

In this note, we report two results on modules over finite groups. Both of them arise in order to solve problems of actions of finite groups on C*-algebras. These problems as well as a brief introduction of C*-algebras can be found in Section 4.

In the next section, we introduce the notion of permutation presentations of modules over finite groups. Then we completely determined finite groups over which every modules have permutation presentations (Theorem 2.4). In Section 3, we introduce the notion of completely cohomologically trivial (CCT) modules over finite groups. Then we give various characterization of CCT modules and a relation with known notions (Theorem 3.9).

We consult the books of Brown [B] and Serre [Se] for a definition and results of the Tate cohomologies $\hat{H}^n(-,-)$.

2 Permutation presentations of modules

Let G be a finite group. By a G-module, we mean an abelian group with a left action of G. A G-module can be naturally considered as a $\mathbb{Z}G$-module where $\mathbb{Z}G$ is the group ring of G over the integer ring $\mathbb{Z}$. A G-set is a set with a left action of G.

Definition 2.1. For a G-set X, the free abelian group $\mathbb{Z}[X]$ whose basis is given by $\{[x]\}_{x\in X}$ is a G-module in a natural way. A G-module F which is isomorphic to this kind of G-modules is called a *permutation module*.

A permutation G-module is a direct sum of a G-module in the form $\mathbb{Z}[G/G']$ for a subgroup G' of G.

Definition 2.2. A *permutation presentation* of a G-module M is a G-equivariant exact sequence

$$0 \longrightarrow F \longrightarrow F \longrightarrow M \longrightarrow 0$$

where F is a permutation G-module.

Example 2.3. Let $G = \langle \sigma \mid \sigma^2 = 1 \rangle \cong \mathbb{Z}/2\mathbb{Z}$. Let $M = \mathbb{Z}/3\mathbb{Z}$ be a G-module where an action of G on M is defined by $\sigma(m) = -m$. Then M has a permutation presentation

$$0 \longrightarrow \mathbb{Z}[X] \xrightarrow{\varphi} \mathbb{Z}[X] \xrightarrow{\psi} M \longrightarrow 0$$

where $X = \{1, 2\}$ with the G-action defined by $\sigma(1) = 2$ and $\sigma(2) = 1$, and φ and ψ are defined by

$$\begin{aligned} \varphi([1]) &= 2[1] - [2], & \psi([1]) &= 1, \\ \varphi([2]) &= -[1] + 2[2], & \psi([2]) &= 2. \end{aligned}$$

One can see that the map φ can be expressed by a 2×2-matrix

$$D = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \in M_2(\mathbb{Z}).$$

In this way, a permutation presentation can be given by a G-set X and a "G-equivariant" $X \times X$ matrix with integer entries.

We are interested in problems to determine modules having permutation presentations, or determine finite groups over which all modules have permutation presentations. We solve the first problem partially (Proposition 2.6 and Lemma 2.9), and the second problem completely as follows.

Theorem 2.4 ([K1, Theorem 1.4]). *For a finite group G, every G-module has a permutation presentation if and only if every Sylow subgroup of G is cyclic.*

There exists an explicit description of finite groups all of whose Sylow subgroups are cyclic. Such a group is isomorphic to a semi-direct product $G = (\mathbb{Z}/m\mathbb{Z}) \rtimes (\mathbb{Z}/n\mathbb{Z})$ such that m and n are relatively prime integers and that $\mathbb{Z}/m\mathbb{Z}$ is the commutator group of G (for detail, see [Ro, 10.1.10] for example).

Arnold [A] considered a similar problem for finitely generated modules, and got a similar answer to Theorem 2.4 using a result by Endo and Miyata in [EM]. Many arguments on finitely generated modules in [A] and [EM] can be applied without paying large fees. However, in one step in [EM] a property of finitely generated modules, which is no longer true for general modules, was used. We need a new idea to complete this step in our case.

We first see the reduction of our problem to a problem on coflasque modules over cyclic p-groups (Theorem 2.12) following the ideas of [EM] and [A], and then mention a little bit about the proof of this problem. We sketch proofs of some results. For the detail, see [K1].

There exists a cohomological obstruction for a G-module to have a permutation presentation. Note that for $n \geq 1$ the Tate cohomologies $\hat{H}^n(-,-)$ coincide with the group cohomologies $H^n(-,-)$. Recall that the exponent $\exp(G)$ of a finite group G

is the smallest positive integer n satisfying $g^n = 1$ for all $g \in G$. It is easy to see that $\exp(G)$ divides the order $|G|$ of G, and we have $\exp(G) = |G|$ if and only if every Sylow subgroup of G is cyclic.

Lemma 2.5. *For a permutation G-module F, we have $\hat{H}^1(G, F) = 0$ and $\hat{H}^2(G, F)$ is annihilated by* $\exp(G)$.

Proof. This follows from Shapiro's lemma. □

Proposition 2.6. *If a G-module M has a permutation presentation, then $\hat{H}^1(G, M)$ is annihilated by* $\exp(G)$.

Proof. Apply Lemma 2.5 to the long exact sequence of the Tate cohomologies induced by a permutation presentation. □

The converse of this proposition does not hold (see [K1, Example 2.7]).

Recall that the augmentation ideal I_G is the kernel of the surjection $\pi\colon \mathbb{Z}G \to \mathbb{Z}$ defined by $\pi(g) = 1$ for all $g \in G$. Since I_G is an ideal of the group ring $\mathbb{Z}G$, it is a G-module.

Proposition 2.7. *If I_G has a permutation presentation, then every Sylow subgroup of G is cyclic.*

Proof. We can easily compute $\hat{H}^1(G, I_G) \cong \mathbb{Z}/|G|\mathbb{Z}$. From this computation and Proposition 2.6, we have $\exp(G) = |G|$ if I_G has a permutation presentation. Thus every Sylow subgroup of G is cyclic. □

This proposition proves the "only if" part of Theorem 2.4. We are going to see how to prove its "if" part. We need the following notions.

Definition 2.8. A G-module M is said to be *permutation projective* if it is isomorphic to a direct summand of some permutation module, and *coflasque* if M is free as an abelian group and $\hat{H}^1(G', M) = 0$ for all subgroups G' of G.

By Lemma 2.5, one can see that a permutation projective G-module is coflasque. The converse is not true in general, and actually the difference of these two notions measures the numbers of G-modules without having permutation presentations (see Lemma 2.9 and Proposition 2.10).

For a G-module M, let N_M be the kernel of the surjection $\pi\colon \mathbb{Z}[M] \to M$ defined by $\pi([m]) = m$:

$$0 \longrightarrow N_M \longrightarrow \mathbb{Z}[M] \xrightarrow{\pi} M \longrightarrow 0$$

Then one can show the following.

Lemma 2.9. *The G-module N_M is always coflasque, and it is permutation projective if and only if M has a permutation presentation.*

Proof. The first assertion follows from the long exact sequence of the Tate cohomologies, and the second one follows similarly to "Schanuel's lemma" ([B, Lemma VIII.4.2]). □

Using this lemma, we get the next proposition.

Proposition 2.10. *Every G-module has a permutation presentation if and only if every coflasque G-module is permutation projective.*

Proof. We only need the "if" part which easily follows from Lemma 2.9. □

In order to check that a finite group G satisfies the property that every coflasque module is permutation projective, it suffices to see that every Sylow subgroup G' of G has this property, by the next lemma.

Lemma 2.11. *A G-module M is coflasque (resp. permutation projective) if and only if M is a coflasque (resp. permutation projective) G'-module for every Sylow subgroup G' of G.*

Hence the "if" part of Theorem 2.4 follows from the next theorem.

Theorem 2.12. *Let G be a cyclic p-group. Then every coflasque G-module is permutation projective.*

Endo and Miyata proved this theorem for finitely generated modules using the fact that a *finitely generated* module over the Dedekind domain $\mathbb{Z}[\zeta_q]$, where q is a power of a prime number, is projective if and only if it is torsion-free. Since this fact is valid only for finitely generated modules, we cannot use their argument. We prove Theorem 2.12 by force using an induction and explicit computations of Tate cohomologies. We sketch our proof in [K1].

Let G be the cyclic group of order p^K where p is a prime number and K is a positive integer. Let $\sigma \in G$ be the generator, and define $\sigma_k = \sigma^{p^k}$ for $k = 0, 1, \ldots, K$. For each $k = 0, 1, \ldots, K$, let G_k be the subgroup of G generated by σ_k. Then $G_k \cong \mathbb{Z}/p^{K-k}\mathbb{Z}$ and $G/G_k \cong \mathbb{Z}/p^k\mathbb{Z}$ holds for $k = 0, 1, \ldots, K$, and

$$G = G_0 \supset G_1 \supset \cdots \supset G_{K-1} \supset G_K = \{1\}$$

exhausts all subgroups of G. We identify G/G_k-modules with G-modules on which σ_k act trivially. We see that a permutation G-module is a direct sum of free G/G_k-modules for $k = 0, 1, \ldots, K$. This description of permutation G-modules can be possible because we can list all subgroups of G and all of them are normal. For a general finite group G, there is no such nice description of permutation G-modules.

Definition 2.13. Let $k = 0, 1, \ldots, K$. We write $M \in \mathcal{C}_k$ if M is a coflasque G-module such that $M^{\sigma_k} := \{m \in M \mid \sigma_k m = m\}$ is a projective G/G_k-module.

Note that $M \in \mathcal{C}_0$ if and only if M is a coflasque G-module, and $M \in \mathcal{C}_K$ if and only if M is a projective G-module. The following technical proposition enables us to prove Theorem 2.12 by an inductive argument.

Proposition 2.14. *Let $k \in \{0, 1, \ldots, K-1\}$. For $M \in \mathcal{C}_k$, there exists a G-equivariant short exact sequence*

$$0 \longrightarrow F \longrightarrow M \oplus P \longrightarrow M' \longrightarrow 0$$

where F is a free G/G_k-module, P is a projective G/G_k-module, and $M' \in \mathcal{C}_{k+1}$.

Proof. Since M^{σ_k} is a projective G/G_k-module, we can find a projective G/G_k-module P such that $M^{\sigma_k} \oplus P$ is a free G/G_k-module. We need to find a free G/G_k-module $F \subset M^{\sigma_k} \oplus P$ such that $M' := (M \oplus P)/F$ is in $\mathcal{C}_{k+1}$. It is not difficult to get conditions on F so that M' is coflasque. To show that the G/G_{k+1}-module $M'' := (M')^{\sigma_{k+1}}$ is projective, we use the well-known theorem of Rim (see [B, Theorem VI.8.10]) which says, in this situation, that M'' is projective if and only if $\hat{H}^n(G/G_{k+1}, M'') = 0$ for all $n \in \mathbb{Z}$. These cohomology groups can be computed as

$$\hat{H}^n(G/G_{k+1}, M'') = \begin{cases} \{m \in M'' \mid sm = 0\}/(1-\sigma)M'' & \text{if } n \text{ is odd,} \\ \{m \in M'' \mid (1-\sigma)m = 0\}/sM'' & \text{if } n \text{ is even.} \end{cases}$$

where $s = \sum_{j=0}^{p^{k+1}-1} \sigma^j \in \mathbb{Z}G$ (see [B, Example III.1.2]). From these arguments, we can write down the conditions on a free G/G_k-module $F \subset M^{\sigma_k} \oplus P$ so that $M' := (M \oplus P)/F$ is in $\mathcal{C}_{k+1}$. A straightforward argument using bases shows that there exists such F. We are done. $\square$

From Proposition 2.14, we can prove Theorem 2.12.

Proof of Theorem 2.12. Take a coflasque G-module M_0. We have $M_0 \in \mathcal{C}_0$. By applying Proposition 2.14 inductively, we get G-modules $\{F_k, P_k\}_{k=0}^{K-1}$ and $\{M_k\}_{k=1}^{K}$ where F_k is a free G/G_k-module, P_k is a projective G/G_k-module and $M_k \in \mathcal{C}_k$ such that there exist G-equivariant short exact sequences

$$0 \longrightarrow F_k \longrightarrow M_k \oplus P_k \longrightarrow M_{k+1} \longrightarrow 0.$$

Since $M_K \in \mathcal{C}_K$ is a projective G-module, there exists a projective G-module P_K such that $F_K = M_K \oplus P_K$ is a free G-module. Then we can show that

$$M_k \oplus \bigoplus_{l=k}^{K} P_l \cong \bigoplus_{l=k}^{K} F_l$$

holds for $k = 0, 1, \ldots, K$ by the induction on k from above using the fact that an extension of a permutation module by a permutation module splits ([K1, Proposition 3.5]). Hence M_0 is a direct summand of the permutation G-module $\bigoplus_{l=0}^{K} F_l$. This completes the proof. $\square$

By Theorem 2.12, we get the "if" part of Theorem 2.4. Thus we complete the proof of Theorem 2.4.

3 Completely cohomologically trivial modules

Let G be a finite group. The following notion and the result were due to Nakayama and Rim (see [B, Section VI.8]).

Definition 3.1. A G-module M is said to be *cohomologically trivial* (abbreviated as CT) if $\hat{H}^n(G', M) = 0$ for all $n \in \mathbb{Z}$ and all subgroups $G' \subset G$.

Proposition 3.2 ([B, Theorem VI.8.12]). *A G-module M is CT if and only if there exists a G-equivariant short exact sequence $0 \to P' \to P \to M \to 0$ where P and P' are projective G-modules.*

In the study of the Rohlin actions on a certain class of C^*-algebras, Izumi introduced the following notion.

Definition 3.3 ([I, Definition 3.8]). A G-module M is said to be *completely cohomologically trivial* (abbreviated as CCT) if the G-module $kM := \{km \mid m \in M\} \subset M$ is CT for every positive integer k.

A CCT G-module is CT, but the converse is not true.

Example 3.4. Let $G = \langle \sigma \mid \sigma^2 = 1 \rangle \cong \mathbb{Z}/2\mathbb{Z}$ and $M = \mathbb{Z}/8\mathbb{Z}$ a G-module where an action of G is defined by $\sigma(m) = 3m$. Then G is a CT G-module, but it is not CCT because $\hat{H}^0(G, 2M) \cong \mathbb{Z}/2\mathbb{Z}$.

To solve one problem on Rohlin actions, we need a concrete description of CCT G-modules (see the end of Section 4). The following gives a concrete example of CCT G-modules.

Definition 3.5. For an abelian group V, let us define a G-action on an abelian group

$$V[G] := \Big\{ \sum_{h \in G} v_h[h] : v_h \in V \Big\} \cong \bigoplus_{h \in G} V$$

by $g(\sum_{h\in G} v_h[h]) = \sum_{h \in G} v_h[gh]$ for $g \in G$. A G-module is said to be *induced* if it is isomorphic to such a G-module $V[G]$ for some abelian group V. A G-module is said to be *relatively projective* if it is a direct summand of an induced G-module.

By Shapiro's lemma, we can see that an induced G-module is CCT. Since a direct summand of a CCT G-module is CCT, a relatively projective G-module is CCT. Thus we get the following implications for G-modules:

$$\text{induced} \Rightarrow \text{relatively projective} \Rightarrow \text{CCT} \Rightarrow \text{CT}$$

It is not difficult to find relatively projective G-modules which are not induced. However, the difference between relatively projective G-modules and CCT G-modules is more subtle, and we need detailed analysis to find CCT G-modules which are not relatively projective. The following gives various characterization of relatively projective G-modules. Note that for G-modules M, N, the G-action on the abelian group $\mathrm{Hom}(M, N)$ of all homomorphisms from M to N is defined by $(g\varphi)(m) = g(\varphi(g^{-1}m))$ for $g \in G$, $\varphi \in \mathrm{Hom}(M, N)$ and $m \in M$.

Theorem 3.6 ([K3]). *For a G-module M, the following conditions are equivalent:*

(i) *M is relatively projective.*

(ii) *For every G-module N,* $\mathrm{Hom}(M, N)$ *is relatively projective.*

(iii) *For every G-module N,* $\mathrm{Hom}(M, N)$ *is CCT.*

(iv) *For every G-module N,* $\mathrm{Hom}(M, N)$ *is CT.*

(ii)' *For every G-module N,* $\mathrm{Hom}(N, M)$ *is relatively projective.*

(iii)' *For every G-module N,* $\mathrm{Hom}(N, M)$ *is CCT.*

(iv)' *For every G-module N,* $\mathrm{Hom}(N, M)$ *is CT.*

(ii)" $\mathrm{Hom}(M, M)$ *is relatively projective.*

(iii)" $\mathrm{Hom}(M, M)$ *is CCT.*

(iv)" $\mathrm{Hom}(M, M)$ *is CT.*

(v) $\hat{H}^0(G, \mathrm{Hom}(M, M)) = 0$.

(vi) *For every G-module F, all surjective G-equivariant homomorphism $F \to M$ splitting as abelian groups has a G-equivariant splitting.*

Note that the condition (vi) is close to the original definition of relatively projective G-modules, or more precisely $(\mathbb{Z}G, \mathbb{Z})$-relatively projective modules (see [H]).

Using this theorem, we can find an example of a CCT G-module which is not relatively projective.

Example 3.7. Let $G = \langle \sigma \mid \sigma^2 = 1 \rangle \cong \mathbb{Z}/2\mathbb{Z}$ and $M = \mathbb{Z}[1/3] \oplus \mathbb{Z}$ a G-module where an action of G is defined by $\sigma((x, y)) = (x + y, -y)$. Then M is a torsion-free CT G-module, and hence a CCT G-module. We can check that $\hat{H}^0(G, \mathrm{Hom}(M, M)) \cong \mathbb{Z}/2\mathbb{Z}$. Hence M is not relatively projective by Theorem 3.6.

The main theorem of this section is several characterization of CCT G-modules. To get a similar result as Proposition 3.2 for CCT G-modules, we need the following notion.

Definition 3.8. A short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of abelian groups is said to be *pure* if for all positive integer k the sequence $0 \to kM_1 \to kM_2 \to kM_3 \to 0$ is exact.

Recall that a subgroup N of an abelian group M is said to be *pure* if for all positive integer k we have $N \cap kM = kN$. We can see that N is a pure subgroup of M if and only if the short exact sequence $0 \to N \to M \to M/N \to 0$ is pure. We can also see that if two of the three G-modules in a G-equivariant pure short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ are CCT then the rest is CCT.

Theorem 3.9 ([I, K3]). *For a G-module M, the following are equivalent:*

(i) *M is CCT.*

(ii) *There exists a G-equivariant pure short exact sequence $0 \to P' \to P \to M \to 0$ such that P and P' are relatively projective G-modules.*

(iii) *For every finitely generated G-module N, $\mathrm{Hom}(N, M)$ is CCT.*

(iv) *For every finitely generated G-module N, $\mathrm{Hom}(N, M)$ is CT.*

(v) *For every finitely generated G-module N, $\hat{H}^0(G, \mathrm{Hom}(N, M)) = 0$.*

(vi) *For every finitely generated G-module N, every G-equivariant homomorphism $\varphi\colon N \to M$ factors through a finitely generated induced module.*

(vii) *M is isomorphic to an inductive limit of finitely generated induced modules.*

Note that the equivalence (i) $\iff$ (ii) is an analogue of Proposition 3.2, and the implication (i) $\Rightarrow$ (vii) gives a concrete description of CCT G-modules. This implication was proved and used in C*-algebra theory by Izumi (see Theorem 4.20).

4 C*-algebras, K-theory and group actions

In this section, we give definitions of C*-algebras and their K-groups, and discuss how we get problems on modules over finite groups from problems on group actions on C*-algebras. For precise definitions, properties and examples of C*-algebras and K-theory, see [RLL] or [W] for example. All linear spaces are over the complex numbers field $\mathbb{C}$.

Definition 4.1. For a linear space $\mathcal{A}$, a map $\mathcal{A} \ni a \mapsto a^* \in \mathcal{A}$ is called an *involution* if it satisfies

$$(a+b)^* = a^* + b^*, \quad (\lambda a)^* = \overline{\lambda} a^*, \quad (a^*)^* = a$$

for $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$.

Definition 4.2. A *C*-algebra* is a Banach space with respect to a norm $\|\cdot\|$ and simultaneously an algebra with an involution $\mathcal{A} \ni a \mapsto a^* \in \mathcal{A}$ such that

$$\|ab\| \leq \|a\| \cdot \|b\|, \quad (ab)^* = b^* a^*, \quad \|a^* a\| = \|a\|^2$$

for $a, b \in \mathcal{A}$.

The condition $\|a^*a\| = \|a\|^2$ is called the *C*-condition*, and it implies $\|a^*\| = \|a\|$ for $a \in \mathcal{A}$. Note that a C*-algebra is not necessarily commutative nor unital.

Example 4.3. For a positive integer n, the algebra $M_n(\mathbb{C})$ of all $n \times n$ matrices is a C*-algebra with respect to the involution defined by the transpose conjugation and the operator norm (see the next example).

Example 4.4. Let $\mathcal{H}$ be a Hilbert space. A linear map $T\colon \mathcal{H} \to \mathcal{H}$ is called a *bounded operator* if its *operator norm*

$$\|T\| := \sup\{\|T(\xi)\| : \xi \in \mathcal{H}, \|\xi\| = 1\}$$

is finite. The set of all bounded operators on $\mathcal{H}$ is denoted by $B(\mathcal{H})$ which becomes a C*-algebra where for $T \in B(\mathcal{H})$, $T^* \in B(\mathcal{H})$ is defined by the unique bounded operator satisfying $(T(\xi), \eta) = (\xi, T^*(\eta))$ for $\xi, \eta \in \mathcal{H}$. When $\mathcal{H} = \mathbb{C}^n$, $B(\mathbb{C}^n)$ is nothing but $M_n(\mathbb{C})$.

A norm closed, $*$-invariant subalgebra of $B(\mathcal{H})$ becomes a C*-algebra. By the Gelfand-Naimark Theorem, every C*-algebra is isomorphic to this type of C*-algebras.

Example 4.5. For a compact space X, the algebra $C(X)$ of all complex continuous functions on X is a C*-algebra with respect to the sup norm defined by $\|f\| := \sup_{x\in X} |f(x)|$ and the involution defined by $f^*(x) := \overline{f(x)}$ for $f \in C(X)$ and $x \in X$.

Example 4.6. Let X be a locally compact space. A continuous function $f \in C(X)$ is said to *vanish at the infinity* if for each $\varepsilon > 0$ the subset $\{x \in X : |f(x)| \geq \varepsilon\}$ is compact. We denote by $C_0(X)$ the set of all continuous functions on X vanishing at the infinity. Then $C_0(X)$ becomes a C*-algebra by the same operations in Example 4.5.

We remark that $C_0(X) = C(X)$ for a compact space X. We also remark that $C_0(X) \cong \{f \in C(\widetilde{X}) : f(\infty) = 0\}$ where $\widetilde{X} = X \cup \{\infty\}$ is the one-point compactification of a locally compact space X.

Theorem 4.7 (Gelfand). *By $X \mapsto C_0(X)$, locally compact spaces correspond bijectively to commutative C*-algebras.*

Based on this theorem, we sometimes say that a non-commutative C*-algebra is the algebra of "continuous functions" vanishing at the infinity on a "non-commutative space". Examples of "Non-commutative spaces" include orbit spaces of group actions on spaces (such as $\mathbb{R}/\mathbb{Q}$), foliations, manifolds with singularity. From this point of view, one may extend some theories on locally compact spaces to non-commutative spaces or C*-algebras. One nice such example is *K-theory*.

Definition 4.8. For each C*-algebra $\mathcal{A}$, one can define abelian groups $K_0(\mathcal{A})$ and $K_1(\mathcal{A})$ called *K-groups* of $\mathcal{A}$. We denote by $K_*(\mathcal{A})$ the pair $(K_0(\mathcal{A}), K_1(\mathcal{A}))$ of abelian groups.

For a commutative C*-algebra $C_0(X)$, the K-groups $K_i(C_0(X))$ coincides with the ordinary K-groups $K^i(X)$ of X for $i = 0, 1$ defined by using line bundles, suspensions and so on.

For $i = 0, 1$, the map $\mathcal{A} \mapsto K_i(\mathcal{A})$ is a covariant functor from the category of C*-algebras to the category of abelian groups. Hence there exists a group homomorphism from the automorphism group $\mathrm{Aut}(\mathcal{A})$ of a C*-algebra $\mathcal{A}$ to the automorphism group $\mathrm{Aut}(K_*(\mathcal{A}))$ of its K-groups where

$$\mathrm{Aut}(K_*(\mathcal{A})) := \mathrm{Aut}(K_0(\mathcal{A})) \times \mathrm{Aut}(K_1(\mathcal{A})).$$

Let us take an action of a group G on a C*-algebra $\mathcal{A}$, which is a homomorphism from G to $\mathrm{Aut}(\mathcal{A})$. Then we get an action of G on the K-groups $K_0(\mathcal{A})$ and $K_1(\mathcal{A})$ of $\mathcal{A}$ by composing the natural map $\mathrm{Aut}(\mathcal{A}) \to \mathrm{Aut}(K_*(\mathcal{A}))$. Thus $K_0(\mathcal{A})$ and $K_1(\mathcal{A})$ become G-modules.

Problem 4.9. For a group G and a C*-algebra $\mathcal{A}$, do every actions $G \curvearrowright K_*(\mathcal{A})$ come from actions $G \curvearrowright \mathcal{A}$? In other words, do every homomorphisms $G \to \mathrm{Aut}(K_*(\mathcal{A}))$ lift to $G \to \mathrm{Aut}(\mathcal{A})$?

$$\begin{tikzcd} & G \arrow[dl, dashed] \arrow[d] \\ \mathrm{Aut}(\mathcal{A}) \arrow[r] & \mathrm{Aut}(K_*(\mathcal{A})) \end{tikzcd}$$

This problem has a negative answer in general.

Example 4.10. For a positive integer n, we have $K_0(M_n(\mathbb{C})) \cong \mathbb{Z}$ and $K_1(M_n(\mathbb{C})) = 0$. All actions $G \curvearrowright M_n(\mathbb{C})$ induce the trivial action on $K_*(M_n(\mathbb{C})) \cong (\mathbb{Z}, 0)$ although there exists a non-trivial action of $G = \mathbb{Z}/2\mathbb{Z}$ on $K_0(M_n(\mathbb{C})) \cong \mathbb{Z}$.

We concentrate the following class of C*-algebras.

Definition 4.11. A *Kirchberg algebra* is a simple separable nuclear purely infinite C*-algebra satisfying the UCT.

For the definition see [Rø]. In many literatures including [Rø], Kirchberg algebras are not assumed to satisfy the UCT. It is an open problem that this condition is automatically satisfied from other conditions.

Example 4.12. Let n be an integer grater than 1. Then the C*-algebra generated by operators $S_1, S_2, \ldots, S_n$ on a Hilbert space satisfying

$$S_1^*S_1 = S_2^*S_2 = \cdots = S_n^*S_n = \sum_{i=1}^{n} S_iS_i^* = 1$$

does not depend on the choices of $S_1, S_2, \ldots, S_n$. We denote this C*-algebra by $\mathcal{O}_n$, and call a *Cuntz algebra*. A Cuntz algebra $\mathcal{O}_n$ is a Kirchberg algebra, and we have $K_*(\mathcal{O}_n) = (\mathbb{Z}/(n-1)\mathbb{Z}, 0)$.

The properties on Kirchberg algebras we will use in this note can be summarized in the following theorem.

Theorem 4.13 (Kirchberg, Phillips, Rørdam). *The correspondence $\mathcal{A} \mapsto K_*(\mathcal{A}) = (K_0(\mathcal{A}), K_1(\mathcal{A}))$ is a bijection from the class of all non-unital Kirchberg algebras to the class of all pairs of countable abelian groups. Moreover for each non-unital Kirchberg algebra $\mathcal{A}$, the natural homomorphism $\mathrm{Aut}(\mathcal{A}) \to \mathrm{Aut}(K_*(\mathcal{A}))$ is surjective.*

A similar statement holds for unital Kirchberg algebras by considering the position of the element $[1_{\mathcal{A}}] \in K_0(\mathcal{A})$ defined by the unit $1_{\mathcal{A}} \in \mathcal{A}$. We are not serious about this point although in order to make statements precise we need to treat the non-unital case and the unital case separately, and to worry about the position of $[1_{\mathcal{A}}]$ in the unital case.

Theorem 4.13 implies that Problem 4.9 has an affirmative answer for $G = \mathbb{Z}$ and a Kirchberg algebra $\mathcal{A}$. So far, no counterexample to Problem 4.9 has been known for Kirchberg algebras $\mathcal{A}$. From now on we only consider the case that G is a finite group and $\mathcal{A}$ is a Kirchberg algebra. The first result on this direction is due to Benson, Kumjian and Phillips who solved Problem 4.9 affirmatively for $G = \mathbb{Z}/2\mathbb{Z}$ and Kirchberg algebras satisfying a certain condition ([BKP]). Several years after, Spielberg extended their result to the case G is a cyclic group with a prime order and $\mathcal{A}$ is an arbitrary Kirchberg algebra ([Sp]). The following theorem extends the two results mentioned above.

Theorem 4.14 ([K2, Theorem 3.5]). *Let G be a finite group all of whose Sylow subgroups are cyclic and $\mathcal{A}$ a Kirchberg algebra. Then every action $G \curvearrowright K_*(\mathcal{A})$ lifts to an action $G \curvearrowright \mathcal{A}$.*

We remark that Izumi also got a result on Problem 4.9 (see Theorem 4.20). Theorem 4.14 is easily deduced from Theorem 2.4 and Theorem 4.16 below. We explain how to prove Theorem 4.16. In [K2], a construction of a Kirchberg algebra $\mathcal{O}_{A,B}$ from two matrices $A, B \in M_N(\mathbb{Z})$ satisfying certain conditions was introduced. Here N is either a positive integer or the countable infinite cardinal ∞. We consider an element of $M_N(\mathbb{Z})$ as an endomorphism of the free abelian group $\mathbb{Z}^N$. Take an action of a finite group G on the set $\{1, 2, \ldots, N\}$. This induces an action $G \curvearrowright \mathbb{Z}^N$. When two endomorphisms $A, B \in M_N(\mathbb{Z})$ of $\mathbb{Z}^N$ commute with this G-action, we can define an action $G \curvearrowright \mathcal{O}_{A,B}$. This action makes $K_i(\mathcal{O}_{A,B})$ G-modules for $i = 0, 1$. On the other hand, since $A, B \in M_N(\mathbb{Z})$ are G-equivariant, the action $G \curvearrowright \mathbb{Z}^N$ induces actions of G on

$$\mathrm{coker}(I - A), \quad \ker(I - A), \quad \mathrm{coker}(I - B), \quad \text{and} \quad \ker(I - B).$$

In this setting, we get the following.

Proposition 4.15. *There exist G-equivariant isomorphisms*

$$K_0(\mathcal{O}_{A,B}) \cong \mathrm{coker}(I - A) \oplus \ker(I - B)$$
$$K_1(\mathcal{O}_{A,B}) \cong \mathrm{coker}(I - B) \oplus \ker(I - A).$$

From this computation, we can prove the following theorem.

Theorem 4.16 ([K2, Theorem 3.3]). *Let G be a finite group, and $\mathcal{A}$ a Kirchberg algebra. An action $G \curvearrowright K_*(\mathcal{A})$ lifts to an action $G \curvearrowright \mathcal{A}$ if the induced two G-modules $K_0(\mathcal{A})$ and $K_1(\mathcal{A})$ have permutation presentations.*

Proof. Let $D_0 \in M_{N_0}(\mathbb{Z})$ and $D_1 \in M_{N_1}(\mathbb{Z})$ be the matrices appeared in permutation presentations of $K_0(\mathcal{A})$ and $K_1(\mathcal{A})$, respectively, that is, $\ker D_0 = \ker D_1 = 0$, $\operatorname{coker} D_0 \cong K_0(\mathcal{A})$ and $\operatorname{coker} D_1 \cong K_1(\mathcal{A})$. We may assume that $N_0 = N_1$, and two actions $G \curvearrowright \{1, 2, \ldots, N_0\}$ and $G \curvearrowright \{1, 2, \ldots, N_1\}$ coincide. We can also arrange D_0, D_1 so that two matrices $A := I - D_0$ and $B := I - D_1$ satisfy the conditions that we can define a Kirchberg algebra $\mathcal{O}_{A,B}$. Then by Proposition 4.15 and Theorem 4.13, we get $\mathcal{O}_{A,B} \cong \mathcal{A}$ and the action $G \curvearrowright \mathcal{O}_{A,B}$ is the desired one. $\square$

Problem 4.9 asks the existence of a lifting $G \curvearrowright \mathcal{A}$ of a given action $G \curvearrowright K_*(\mathcal{A})$. We can ask the uniqueness of a lifting. In general cases, the uniqueness of a lifting fails in various reasons. However, for special actions called Rohlin actions, Izumi showed the uniqueness of lifting.

Definition 4.17 ([I, Definition 2.8]). Let G be a finite group, and $\mathcal{A}$ a unital C*-algebra. An action $\alpha\colon G \curvearrowright \mathcal{A}$ is said to be a *Rohlin action* if for all $\varepsilon > 0$ and all finite subset $\mathcal{F} \subset \mathcal{A}$, there exists a partition of unity $\{e_g\}_{g \in G}$ of $\mathcal{A}$ such that $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ and $\|e_g x - x e_g\| < \varepsilon$ for all $g, h \in G$ and $x \in \mathcal{F}$.

Here a partition of unity $\{e_g\}_{g \in G}$ of $\mathcal{A}$ means that e_g is a projection (i.e. satisfies $e_g^2 = e_g^* = e_g$) for every $g \in G$, and $\sum_{g \in G} e_g = 1_{\mathcal{A}}$.

Theorem 4.18 ([I, Theorem 4.2]). *If two Rohlin actions of a finite group G on a unital Kirchberg algebra $\mathcal{A}$ induce conjugate actions on K-groups $K_*(\mathcal{A})$, then they are conjugate.*

Recall that two actions $\alpha, \beta\colon G \curvearrowright X$ are said to be *conjugate* if there exists $\theta \in \operatorname{Aut}(X)$ such that $\alpha_g = \theta \circ \beta_g \circ \theta^{-1}$ holds for all $g \in G$.

This uniqueness theorem produces the following new existence problem.

Problem 4.19. Let G be a finite group and $\mathcal{A}$ a unital Kirchberg algebra. Which action $G \curvearrowright K_*(\mathcal{A})$ lifts to a Rohlin action $G \curvearrowright \mathcal{A}$?

Izumi solved this problem by introducing the concept of CCT G-modules.

Theorem 4.20 ([I, Corollary 5.4]). *Let G be a finite group and $\mathcal{A}$ a unital Kirchberg algebra. An action $G \curvearrowright K_*(\mathcal{A})$ lifts to a Rohlin action $G \curvearrowright \mathcal{A}$ if and only if G-modules $K_i(\mathcal{A})$ are CCT for $i = 0, 1$.*

Note that the lifting is unique up to conjugacy by Theorem 4.18. The "only if" part was proved by combining in a very beautiful way the standard arguments of Rohlin actions and cohomological arguments on G-modules $K_i(\mathcal{A})$ involving the (mod n) K-groups $K_i(\mathcal{A}, \mathbb{Z}/n\mathbb{Z})$ ([I, Theorem 3.3]). To get the "if" part, Izumi showed the following two propositions using model actions for Rohlin actions and the conjugacy argument which is used to prove the uniqueness theorem (Theorem 4.18), respectively.

Proposition 4.21. *An action $G \curvearrowright K_*(\mathcal{A})$ lifts to a Rohlin action $G \curvearrowright \mathcal{A}$ if G-modules $K_i(\mathcal{A})$ are induced for $i = 0, 1$.*

Proposition 4.22. *An action $G \curvearrowright K_*(\mathcal{A})$ lifts to a Rohlin action $G \curvearrowright \mathcal{A}$ if G-modules $K_i(\mathcal{A})$ are inductive limits of G-modules $\{K_i(\mathcal{A}_n)\}_{n=1}^{\infty}$ such that $G \curvearrowright K_*(\mathcal{A}_n)$ lifts to a Rohlin action $G \curvearrowright \mathcal{A}_n$.*

Now the "if" part of Theorem 4.20 follows from the two propositions above and the implication (i) $\Rightarrow$ (vii) of Theorem 3.9.

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