

# **Nuclearity in CFT**

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## QFT selection criterion.

A net of local observable von Neumann algebras on Minkowski spacetime:

$$\mathcal{O} \subset \mathbb{R}^4 \rightarrow \mathcal{A}(\mathcal{O}) \subset B(\mathcal{H})$$

$\Omega$  vacuum vector,  $U : g \in \mathcal{P}_+^\uparrow \rightarrow B(\mathcal{H})$  positive energy, unitary covariance representation of the Poincaré group.

Which nets are physical?

Quantum mechanics: states localised in a bounded region with an energy bound are finitely many (Laplacian in a box with Dirichlet boundary condition has finitely many eigenvalues  $\leq \text{const.}$ )

→ *Haag-Swieca*:  $E\mathcal{A}(\mathcal{O})_1\Omega$  compact subset of  $\mathcal{H}$  where  $E$  spectral projection of bounded energy (time-translation generator  $P$ ),  $\mathcal{O}$  bounded region.

*Split property.*  $\mathcal{O} \subset\subset \tilde{\mathcal{O}}$  bounded double cones

$$\mathcal{A}(\mathcal{O}) \vee \mathcal{A}(\tilde{\mathcal{O}})' \simeq \mathcal{A}(\mathcal{O}) \vee \mathcal{A}(\tilde{\mathcal{O}})'$$

natural isomorphism.

$\leftrightarrow$  statistical independence:  $\varphi_1, \varphi_2$  normal states on  $\mathcal{A}(\mathcal{O})$  and  $\mathcal{A}(\mathcal{O})' \Rightarrow \varphi_1 \otimes \varphi_2$  is normal on  $\mathcal{A}(\mathcal{O}) \vee \mathcal{A}(\tilde{\mathcal{O}})'$ .

split property  $\Rightarrow$  local charges

(integrated form of Noether thm.) (Doplicher, L.)

*Buchholz-Wichmann nuclearity* (quantitative Haag-Swieca compactness):

$$\Phi_{\mathcal{O}}^{\text{BW}}(\beta) : x \in \mathcal{A}(\mathcal{O}) \rightarrow e^{-\beta P} x \Omega \in \mathcal{H}$$

is nuclear,  $\mathcal{O}$  interval of  $\mathbb{R}$ ,  $\beta > 0$ . Moreover  $\|\Phi_I^{\text{BW}}(\beta)\|_1 \leq e^{cr^m/\beta^n}$  as  $\beta \rightarrow 0^+$ .

Recall:  $A : X \rightarrow Y$  is nuclear if  $\exists$  sequences  $f_k \in X^*$  and  $y_k \in Y$  s.t.  $\sum_k \|f_k\| \|y_k\| < \infty$  and

$$Ax = \sum_k f_k(x) y_k .$$

$$\|A\|_1 \equiv \inf \sum_k \|f_k\| \|y_k\| .$$

BW nuclearity  $\Rightarrow$  split property

Buchholz-Junglas:

nuclearity  $\Rightarrow$  KMS states for translations

“local KMS states” by the split property

weak limit point  $\longrightarrow$  global KMS states.

**Möbius covariant nets of vN algebras.** A (local) *Möbius covariant net*  $\mathcal{A}$  on  $S^1$  is a map

$$I \in \mathcal{I} \rightarrow \mathcal{A}(I) \subset B(\mathcal{H})$$

$\mathcal{I} \equiv$  family of proper intervals of  $S^1$ , that satisfies:

**A. Isotony.**  $I_1 \subset I_2 \implies \mathcal{A}(I_1) \subset \mathcal{A}(I_2)$

**B. Locality.**  $I_1 \cap I_2 = \emptyset \implies [\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\}$

**C. Möbius covariance.**  $\exists$  unitary rep.  $U$  of the Möbius group  $\text{Mob}$  on  $\mathcal{H}$  such that

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI), \quad g \in \text{Mob}, I \in \mathcal{I}.$$

**D. Positivity of the energy.** Generator  $L_0$  of rotation subgroup of  $U$  (conformal Hamiltonian) is positive.

**E. Existence of the vacuum.**  $\exists!$   $U$ -invariant vector  $\Omega \in \mathcal{H}$  (vacuum vector), and  $\Omega$  is cyclic

for the von Neumann algebra  $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I)$  and unique  $U$ -invariant.

## First consequences

*Irreducibility:*  $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I) = B(H)$ .

*Reeh-Schlieder theorem:*  $\Omega$  is cyclic and separating for each  $\mathcal{A}(I)$ .

*Bisognano-Wichmann property:* Tomita-Takesaki modular operator  $\Delta_I$  and conjugation  $J_I$  of  $(\mathcal{A}(I), \Omega)$ , are

$$\begin{aligned} U(\Lambda_I(2\pi t)) &= \Delta_I^{it}, \quad t \in \mathbb{R}, && \text{dilations} \\ U(r_I) &= J_I && \text{reflection} \end{aligned}$$

(Guido-L., Frölich-Gabbiani)

*Haag duality:*  $\mathcal{A}(I)' = \mathcal{A}(I')$

*Factoriality:*  $\mathcal{A}(I)$  is III<sub>1</sub>-factor (or  $\mathcal{A}(I) = \mathbb{C}$ ).

*Additivity:*  $I \subset \cup_i I_i \implies \mathcal{A}(I) \subset \vee_i \mathcal{A}(I_i)$  (Fredenhagen, Jorss).

### Further selection properties.

- *Split property.*  $\mathcal{A}$  is *split* if the von Neumann algebra

$$\mathcal{A}(I_1) \vee \mathcal{A}(I_2) \simeq \mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$$

(natural isomorphism) if  $\bar{I}_1 \cap \bar{I}_2 = \emptyset$ .

- Split is a property of the net (not of  $U$ ).
- Split is crucial, e.g. for local charges, complete rationality, hypofiniteness, classification...

- *Trace class condition.*

$$\text{Tr}(e^{-tL_0}) < \infty, \forall t > 0$$

- Trace class condition is standard in CFT

- Trace class condition  $\implies$  split

- Trace class condition can be refined to *log-ellipticity*

$$\log \operatorname{Tr}(e^{-tL_0}) \sim \frac{1}{t^\alpha}(a_0 + a_1 t + \dots) \quad \text{as } t \rightarrow 0^+$$

(Kawahigashi, L.)

- Trace class is a property of  $U$  (not of the net).

• *Buchholz-Wichmann nuclearity:*

$$\Phi_I^{\text{BW}}(\beta) : x \in \mathcal{A}(I) \rightarrow e^{-\beta P} x \Omega \in \mathcal{H}$$

is nuclear,  $I$  interval of  $\mathbb{R}$ ,  $\beta > 0$ .  $P$  translation generator (Hamiltonian).

- BW-nuclearity is a physical property (Haag-Swieca).



- BW-nuclearity is a property of the full Möbius covariant net.

- Can be refined with  $\|\Phi_I^{\text{BW}}(\beta)\|_1 \leq e^{cr^m/\beta^n}$  as  $\beta \rightarrow 0^+$  and  $\rightarrow$  KMS states for translations (Buchholz-Junglas).

*Problem:* Derive BW-nuclearity from the trace class condition.

• *Modular nuclearity*

$M$  von Neumann algebra,  $\Omega$  cyclic separating unit vector. Set

$$L^\infty(M) = M, \quad L^2(M) = \mathcal{H}, \quad L^1(M) = M_* .$$

Then we have the embeddings

$$\begin{array}{ccc}
 L^\infty(M) & \xrightarrow{x \rightarrow (x\Omega, J \cdot \Omega)} & L^1(M) \\
 & \searrow \Phi_{\infty,1}^M & \nearrow \\
 & \Phi_{\infty,2}^M & \Phi_{2,1}^M \\
 & \searrow \Phi_{\infty,2}^M & \nearrow \Phi_{2,1}^M \\
 x \rightarrow \Delta^{1/4} x \Omega & & \xi \rightarrow (\xi, J \cdot \Omega) \\
 & \searrow & \nearrow \\
 & L^2(M) &
 \end{array}$$

Now let  $N \subset M$  be an inclusion of vN algebras with cyclic and separating unit vector  $\Omega$ .

$L^{p,q}$ -nuclearity if  $\Phi_{p,q}^M|_N$  is a nuclear operator.

$L^{\infty,2}$ -nuclearity was called *modular nuclearity*, i.e.

$$\Phi_{\infty,2}^M|_N : x \in N \rightarrow \Delta_M^{1/4} x \Omega$$

is nuclear.

As  $\Phi_{\infty,1}^M = \Phi_{2,1}^M \Phi_{\infty,2}^M$ , we have

$$\|\Phi_{\infty,1}^M|_N\|_1 \leq \|\Phi_{2,1}^M\| \cdot \|\Phi_{\infty,2}^M|_N\|_1 \leq \|\Phi_{\infty,2}^M|_N\|_1 ,$$

Thus

Modular nuclearity  $\Rightarrow L^{\infty,1}$  – nuclearity.

indeed  $\Phi_{\infty,1}^M|_N = \Phi_{2,1}^N \cdot \Phi_{\infty,2}^M|_N$  and  $\|\Phi_{2,1}^N\| \leq 1$  so  $\|\Phi_{\infty,1}^M|_N\|_1 \leq \|\Phi_{\infty,2}^M|_N\|_1$ . (A certain converse holds) .

- If  $N$  or  $M$  is a factor and  $\Phi_{\infty,1}^M|_N$  is nuclear then  $N \subset M$  is a split inclusion ( $N \vee M' \simeq N \otimes M'$ ).

*Short proof.* By definition  $\Phi_{\infty,1}^M|_N$  nuclear means:  $\exists$  sequences of elements  $\varphi_k \in N^*$  and  $\psi_k \in M'_*$  ( $\simeq L^1(M)$ ) such that  $\sum_k \|\varphi_k\| \|\psi_k\| < \infty$  and

$$\omega(nm') = \sum_k \varphi_k(n) \psi_k(m'), \quad n \in N, m' \in M'.$$

where  $\omega \equiv (\cdot \Omega, \Omega)$ . As  $\Phi_{\infty,1}^M|_N$  is normal the  $\varphi_k$  can be chosen normal (take the normal part). Thus the state  $\omega$  on  $N \odot M'$  extends to  $N \otimes M'$  and this gives the split property.

Consider now the commutative diagram

$$\begin{array}{ccc} L^\infty(N) & \xrightarrow{\Phi_{\infty,1}^M|_N} & L^1(M) \\ \Phi_{\infty,2}^N \downarrow & & \uparrow \Phi_{2,1}^M \\ L^2(N) & \xrightarrow{T_{M,N} \equiv \Delta_M^{1/4} \Delta_N^{-1/4}} & L^2(M) \end{array}$$

$T_{M,N} \equiv \Phi_{2,2}^M|_N$ .  $L^2$ -nuclearity condition (or  $L^{2,2}$ -nuclearity) means that

$$\|T_{M,N}\|_1 < \infty$$

-  $L^2$ -nuclearity  $\Rightarrow$  modular nuclearity,

indeed  $\|\Phi_{\infty,2}^M|_N\|_1 \leq \|T_{M,N}\|_1$  because  
 $\Phi_{\infty,2}^M|_N = T_{M,N} \cdot \Phi_{\infty,2}^N$  and  $\|\Phi_{\infty,2}^N\| \leq 1$ .

## Standard real Hilbert subspaces

$\mathcal{H}$  complex Hilbert space and  $H \subset \mathcal{H}$  a real linear subspace.

Symplectic complement:

$$H' \equiv \{\xi \in \mathcal{H} : \Im(\xi, \eta) = 0 \quad \forall \eta \in H\}.$$

$$H' = (iH)^\perp \text{ (real orthogonal complement)}$$

$$H_1 \subset H_2 \Rightarrow H'_1 \supset H'_2 .$$

A *standard* subspace  $H$  of  $\mathcal{H}$  is a closed, real linear subspace of  $\mathcal{H}$  which is both cyclic ( $\overline{H + iH} = \mathcal{H}$ ) and separating ( $H \cap iH = \{0\}$ ).  $H$  is standard iff  $H'$  is standard.

$H$  standard subspace  $\rightarrow$  anti-linear operator  $S \equiv S_H : D(S) \subset \mathcal{H} \rightarrow \mathcal{H}$ , where  $D(S) \equiv H + iH$ ,

$$S : \xi + i\eta \mapsto \xi - i\eta , \quad \xi, \eta \in H .$$

$$S^2 = 1 \upharpoonright_{D(S)}.$$

Conversely,  $S$  densely defined, closed, anti-linear involution on  $\mathcal{H}$  gives

$$H = \{\xi : S\xi = \xi\} \quad \text{standard subspace}$$

$$H \leftrightarrow S \quad \text{bijection}$$

*Modular theory.* Set

$$S_H = J_H \Delta_H^{1/2}$$

polar decomposition of  $S = S_H$ . Then  $J$  is an anti-unitary involution  $\Delta \equiv S^*S > 0$

$$\Delta_H^{it} H = H, \quad J_H H = H'$$

*Standard subspace version of Borchers theorem.*  $H$  standard subspace,  $U$  a one-parameter group with positive generator

$$U(s)H \subset H \quad s \geq 0.$$

Then:

$$\begin{cases} \Delta_H^{it} U(s) \Delta_H^{-it} = U(e^{-2\pi t s}), \\ J_H U(s) J_H = U(-s), \end{cases} \quad t, s \in \mathbb{R}.$$

*Standard subspace version of Wiesbrock, Borchers, Araki-Zsido theorem*

Let  $H, K$  be standard subspaces. Assume half-sided modular inclusion:

$$\Delta_H^{it} K \subset K, \quad t \leq 0.$$

Then  $\{\Delta_K^{it}, \Delta_H^{is}\}$  generates a unitary representation of the “ $ax+b$ ” group with positive energy

$$\text{dilations} = \Delta_H^{-is/2\pi}$$

$$P = \frac{1}{2\pi} (\log \Delta_K - \log \Delta_H)$$

*von Neumann algebras and real Hilbert subspaces*

$M$  von Neumann algebra on  $\mathcal{H}$ ,  $\Omega \in \mathcal{H}$  a cyclic separating vector,

$$H_M \equiv \overline{M_{sa}\Omega}$$

is a standard subspace of  $H$

$$\Delta_M = \Delta_{H_M}, \quad J_M = J_{H_M}$$

In particular

$$H'_M = H_{M'}$$

## **Möbius covariant nets of real Hilbert subspaces**

A *local Möbius covariant net* of standard subspaces  $\mathcal{A}$  of real Hilbert subspaces on the intervals of  $S^1$  is a map

$$I \rightarrow H(I)$$

with

1. *Isotony* : If  $I_1, I_2$  are intervals and  $I_1 \subset I_2$ , then

$$H(I_1) \subset H(I_2) .$$



2. Möbius invariance: *There is a unitary representation  $U$  of  $\text{Mob}$  on  $\mathcal{H}$  such that*

$$U(g)H(I) = H(gI) , \quad g \in \text{Mob}, I \in \mathcal{I}.$$

3. Positivity of the energy :  $L_0 \geq 0$
4. Cyclicity : *the complex linear span of all spaces  $H(I)$  is dense in  $\mathcal{H}$ .*
5. Locality : *If  $I_1$  and  $I_2$  are disjoint intervals then*

$$H(I_1) \subset H(I_2)'$$

- Reeh-Schlieder theorem, Bisognano-Wichmann property, Haag duality, . . . hold.

- Net of factors  $\rightarrow$  Net of standard subspaces (not one-to-one)

- Net of standard subspaces  $\rightarrow$  Net of factors  
(second quantization)

### **Modular theory and representations of $SL(2, \mathbb{R})$** (Brunetti, Guido, L.)

$U$  is a unitary, positive energy representation  
 $U$  of Möbius on  $\mathcal{H}$  and  $J$  on  $\mathcal{H}$

$$JU(g)J = U(rgr) \quad g \in \text{Mob}$$

where  $r : z \mapsto \bar{z}$  reflection on  $S^1$  w.r.t. the  
upper semicircle  $I_1$ . Then define

$$J_I \equiv U(g)JU(g)^*$$

where  $g \in \text{Mob}$  maps  $I_1$  onto  $I$ .

$$\Delta_I^{it} \equiv U(\Lambda_I(-2\pi t)), \quad t \in \mathbb{R}$$

namely  $-\frac{1}{2\pi} \log \Delta_I$  generator of dilations of  $I$ ,

$$S_I \equiv J_I \Delta_I^{1/2}$$

is a densely defined, antilinear, closed involu-  
tion on  $\mathcal{H}$ .

$H(I)$  standard subspace associated with  $S_I \rightarrow$  Möbius covariant local net of real Hilbert spaces of  $\mathcal{H}$ .

A  $\pm$ hsm factorization of real subspaces is a triple  $K_0, K_1, K_2$ , where  $\{K_i, i \in \mathbb{Z}_3\}$  is a set of standard subspaces s.t.  $K_i \subset K'_{i+1}$  is a  $\pm$ hsm inclusion.

Factorization



Local Möbius covariant net of real Hilbert spaces



Positive energy representation of  $SL(2, \mathbb{R})$

(Guido, Wiesbrock, L.)

**$L^2$ -Nuclearity.** Let  $H \subset \tilde{H}$  be an inclusion of standard subspaces. Set

$$T_{\tilde{H}, H} \equiv \Delta_{\tilde{H}}^{1/4} \Delta_H^{-1/4}$$

then  $\|T_{\tilde{H},H}\| \leq 1$ . The inclusion is *nuclear* if  $T_{\tilde{H},H}$  is a nuclear (i.e. trace class) operator.

$U$  unitary, positive energy representation of  $\text{Mob}$ ,  $H(I)$  the associated net of standard subspaces.  $U$  satisfies  $L^2$  nuclearity if  $H(I) \subset H(\tilde{I})$  is nuclear if  $I \subset\subset \tilde{I}$ .

**$SL(2, \mathbb{R})$  identities.**

**Formula 0 (Schroer-Wiesbrock)**

$U$  positive energy unitary  $\text{Mob}$  rep.,  $\forall s \geq 0$ :

$$\Delta_1^{1/4} \Delta_2^{-is} \Delta_1^{-1/4} = e^{-2\pi s L_0}$$

$\Delta_1 = \Delta_{I_1}$ ,  $\Delta_2 = \Delta_{I_2}$ , with  $I_1, I_2$  upper and right semicircles.

*About the proof.* Use of double interpretation of  $\Delta_1, \Delta_2$ : modular (analyticity) and  $SL(2, \mathbb{R})$  (Lie algebra relations)

**Formula 1**  $U$  positive energy unitary representation:

$$T_{\tilde{I}, I} = e^{-sL_0} \Delta_2^{is/2\pi}$$

$s = \ell(\tilde{I}, I)$  is the inner distance (if  $I = (-1, 1)$  and  $\tilde{I} = (-e^s, e^s)$  on the real line, then  $\ell(\tilde{I}, I) = s$ ) thus

$$\|T_{\tilde{I}, I}\|_1 = \|e^{-sL_0}\|_1$$

*About the proof.*

$$\begin{aligned} e^{-2\pi s L_0} &= \Delta_1^{1/4} \Delta_2^{-is} \Delta_1^{-1/4} = \\ &\Delta_1^{1/4} \Delta_2^{-is} (\Delta_1^{-1/4} \Delta_2^{is}) \Delta_2^{-is} = T_{I_1, I_1, s} \Delta_2^{-is} \end{aligned}$$

**Formula 2**

$$T_{I, I_{a', a}} = e^{-a' P'_I} e^{-a P_I} e^{-ia P_I} e^{ia' P'_I} .$$

$I_{a', a} \equiv \tau'_{-a'} \tau_a I$  with  $a, a' > 0$ .

$$e^{-2sL_0} = e^{-\tanh(\frac{s}{2})P} e^{-\sinh(s)P'} e^{-\tanh(\frac{s}{2})P}$$

therefore

$$e^{-2sL_0} \leq e^{-2 \tanh(\frac{s}{2})P}$$

in particular  $e^{-i\pi L_0} = e^{iP} e^{iP'} e^{iP}$ .

*About the proof.* Consider  $\tilde{I} = (0, \infty)$ ,  $I = (t, \infty)$ , then

$$\begin{aligned} T_{\tilde{I}, I} &= \Delta_{\tilde{I}}^{1/4} \Delta_I^{-1/4} \\ &= \left( \Delta_{\tilde{I}}^{1/4} U(t) \Delta_{\tilde{I}}^{-1/4} \right) U(-t) \\ &= e^{-tP} e^{itP} \end{aligned}$$

where we have used the Borchers commutation relation  $\Delta_{\tilde{I}}^{is} e^{itP} \Delta_{\tilde{I}}^{-is} = e^{i(e^{-2\pi s})tP}$ . Any  $I \subset\subset \tilde{I}$  is obtain by iteration the above, get a formula and compare with formula 1.

### Formula 3

$$\|e^{-\tan(2\pi\lambda)d_I P} \Delta_I^{-\lambda}\| \leq 1, \quad 0 < \lambda < 1/4.$$

with  $d_I$  the usual lenght. Thus

$$e^{-2 \tan(2\pi\lambda)d_I P} \leq \Delta_I^{2\lambda}.$$

so we have

$$e^{-2d_I P} \leq \Delta_I^{1/4} \leq e^{\frac{2}{d_I} P'}$$

## Modular nuclearity and $L^2$ -nuclearity

$L^2$ -nuclearity implies modular nuclearity and  $\|\Delta_{\tilde{H}}^{1/4} E_H\|_1 \leq \|T_{\tilde{H}, H}\|_1$ .

## Comparison of nuclearity conditions

Let  $H$  be a Möbius covariant net of real Hilbert subspaces of a Hilbert space  $\mathcal{H}$ . Consider the following nuclearity conditions for  $H$ .

*Trace class condition:*  $\text{Tr}(e^{-sL_0}) < \infty$ ,  $s > 0$ ;

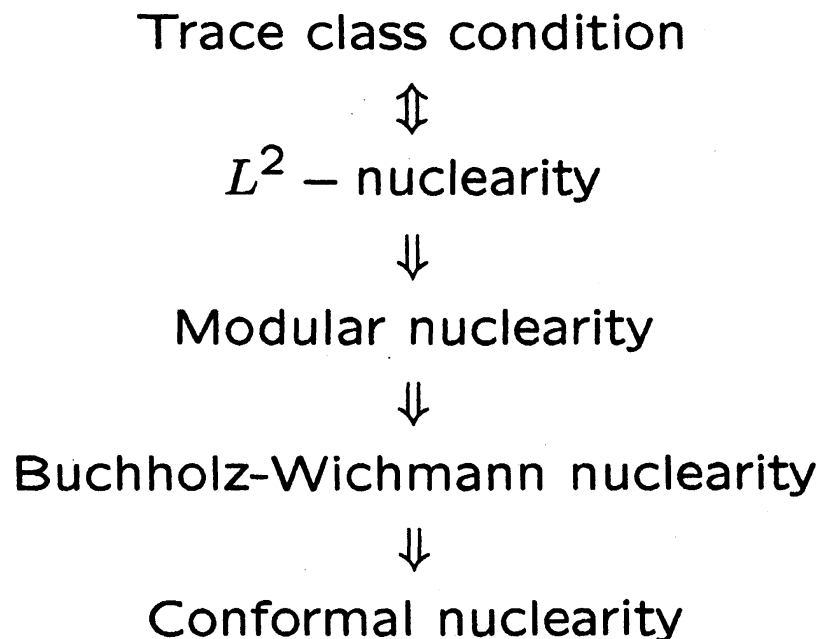
*$L^2$ -nuclearity:*  $\|T_{\tilde{I}, I}\|_1 < \infty$ ,  $\forall I \in \tilde{I}$ ;

*Modular nuclearity:*  $\Xi_{\tilde{I}, I} : \xi \in H(I) \rightarrow \Delta_{\tilde{I}}^{1/4} \xi \in \mathcal{H}$  is nuclear  $\forall I \subset \tilde{I}$ ;

*Buchholz-Wichmann nuclearity:*  $\Phi_I^{\text{BW}}(s) : \xi \in H(I) \rightarrow e^{-sP}\xi \in \mathcal{H}$  is nuclear,  $I$  interval of  $\mathbb{R}$ ,  $s > 0$  ( $P$  the generator of translations);

*Conformal nuclearity:*  $\Psi_I(s) : \xi \in H(I) \rightarrow e^{-sL_0}\xi \in \mathcal{H}$  is nuclear,  $I$  interval of  $S^1$ ,  $s > 0$ .

We shall show the following chain of implications:



Where all the conditions can be understood for a specific value of the parameter, that will be



determined, or for all values in the parameter range.

We have already discussed the implications “Trace class condition  $\Leftrightarrow L^2$ -nuclearity  $\Rightarrow$  Modular nuclearity” .

*Modular nuclearity  $\Rightarrow$  BW-nuclearity*

We have

$$\|\Phi_{I_0}^{\text{BW}}(d_I)\|_1 \leq \|\Xi_{I,I_0}\|_1$$

where  $d_I$  is the length of  $I$  on  $\mathbb{R}$ .

*BW-nuclearity  $\Rightarrow$  Conformal nuclearity*

By formula 2 there exists a bounded operator  $B$  with norm  $\|B\| \leq 1$  such that  $e^{-sL_0} = B e^{-\tanh(\frac{s}{2})H}$ , therefore

$$\Psi_I(s) = B \Phi_I^{\text{BW}}(\tanh(s/2))$$

$$\|\Psi_I(s)\|_1 \leq \|\Phi_I^{\text{BW}}(\tanh(s/2))\|_1.$$

## Consequences

- *Distal split property.* If  $\text{Tr}(e^{-sL_0}) < \infty$  for a fixed  $s > 0$ , then  $\mathcal{A}(I) \subset \mathcal{A}(\tilde{I})$  is split if  $I \subset \tilde{I}$  and  $\ell(\tilde{I}, I) > s$  e.g. free probability nets (D'Antoni, Radulescu, L.).

- *Constructing KMS states.*  $\mathcal{A}|_{\mathbb{R}}$  restriction of  $\mathcal{A}$  to  $\mathbb{R} \simeq S^1 \setminus \{-1\}$ ,  $\mathcal{A}_0$  the quasi-local C\*-algebra. i.e. the norm closure of  $\cup_I \mathcal{A}(I)$  as  $I$  varies in the bounded intervals of  $\mathbb{R}$ . Let  $\mathfrak{A} \subset \mathcal{A}_0$  the C\*-algebras of elements with norm continuous orbit, namely

$$\mathfrak{A} = \{X \in \mathcal{A}_0 : \lim_{t \rightarrow 0} \|\tau_t(X) - X\| = 0\}$$

$\tau$  translation automorphism group.

*Thm.* If the trace class condition holds for  $\mathcal{A}$  with the asymptotic bound

$$\text{Tr}(e^{-sL_0}) \leq e^{\text{const.} \frac{1}{s^\alpha}}, \quad s \rightarrow 0^+$$

for some  $\alpha > 0$ , then the BW-nuclearity holds with  $m = n = \alpha$ .

If the trace class condition holds with log-ellipticity (above asymptotics) then for every  $\beta > 0$  there exists a translation  $\beta$ -KMS state on  $\mathfrak{A}$ .

- *$L^2$ -Nuclearity and KMS states in higher dimensions.*

$\mathcal{O}$  a double cone in the Minkowski spacetime  $\mathbb{R}^{d+1}$ ,  $\mathcal{A}(\mathcal{O})$  the local von Neumann algebra associated with  $\mathcal{O}$  by the  $d+1$ -dimensional scalar, massless, free field.

With  $I$  an interval of the time-axis  $\{x = \langle x_0, \mathbf{x} \rangle : \mathbf{x} = 0\}$  we set

$$\mathcal{A}_0(I) \equiv \mathcal{A}(\mathcal{O}_I)$$

where  $\mathcal{O}_I$  is the double cone  $I'' \subset \mathbb{R}^{d+1}$ , the causal envelope of  $I$ . Then  $\mathcal{A}_0$  is a translation-dilation covariant net on  $\mathbb{R}$ .  $\mathcal{A}_0$  is local if  $d$  is

odd and twisted local if  $d$  is even. Moreover  $\mathcal{A}_0$  extends to a Möbius covariant net on  $S^1$  ( $d$  odd) as one has a natural factorization.

We have:

$$\mathcal{A}_0 = \bigotimes_{k=0}^{\infty} N_d(k) \mathcal{A}^{(k)}$$

where  $\mathcal{A}^{(k)}$  is the Möbius covariant net on  $S^1$  associated with the  $k^{\text{th}}$ -derivative of the  $U(1)$ -current algebra and  $N_d(k)$  is a multiplicity factor (see below).

This follows because the one-particle Mob representation  $U_0$  decomposes

$$U_0 = \bigoplus_{k=1}^{\infty} N_d(k) U^{(k)}$$

where  $U^{(k)}$  is the positive energy irreducible representation of  $PSL(2, \mathbb{R})$  with lowest weight  $k$ .

A spherical harmonics computations determines the multiplicity factor  $N_d(k)$ . As  $k \rightarrow \infty$ :

$$\begin{aligned} N_d(k+1) &= \dim(\mathcal{P}_k \ominus \mathcal{P}_{k-2}) \\ &= m_{d-1}(k-1) + m_{d-1}(k) \sim \frac{2}{(d-2)!} k^{d-2}, \end{aligned}$$

with  $\mathcal{P}_k \ominus \mathcal{P}_{k-2}$  the  $k$ -spherical harmonics and  $m_d(k) \sim \frac{1}{(d-1)!} k^{d-1}$ . Thus

$$\log \text{Tr}(e^{-sL_0}) \sim \frac{2}{s^d} \quad s \rightarrow 0^+,$$

where  $L_0$  is the conformal Hamiltonian of  $\mathcal{A}_0$ .

### Problems.

- $\text{Tr}(e^{-sL_0}) < \infty \Leftrightarrow$  split property?
- $e^{-sL_0}$  compact  $\Leftrightarrow$  split property?
- $\text{Tr}(e^{-sL_0}) < \infty \Rightarrow \text{Tr}(e^{-sL_{0,\rho}}) < \infty$  in every irreducible representation  $\rho$  of  $\mathcal{A}$ ?