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Nonlinear functionals of Gaussian and Poisson noises

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1 Introduction

We shall discuss Random complex systems which are evolutional. For this purpose we follow the following steps:

Reduction → Synthesis → Analysis

I. Reduction. This step means that we construct a system of independent random variables such that they are functions of the given random system and have the same information as the given system. For a stochastic process and a random field, the *innovation* appears as the most natural concept to realize the reduction.

In significant cases, innovation appears as the time derivative of a Lévy process. According to the Lévy-Itô decomposition of a Lévy process, we know an important innovation consists of (Gaussian) white noise and compound Poisson noise. It is therefore understood that white noise and Poisson noise are typical atomic components of innovation. White noise is realized by taking the time derivative of a Brownian mnotion $B(t)$, so it is denoted by $\dot{B}(t)$, and a realization of Poisson noise, denoted by $\dot{P}(t)$, is the time derivative of a Poisson process $P(t)$. Each of these noises actually forms a system of *idealized elemental random variables.*

The next step is synthesis. Our aim is to form a functional of the noises so as to express a mathematical model of the given random phenomenon. Naturally, the functional is nonlinear in the variables and may depends of the time t. Once such a functional is presented, we can go to the analysis of the functional. We shall be able to deal with differential and integral calculus including Laplacians, harmonic analysis, in particular Fourier analysis, and so forth.It should be noted that everything is infinite dimensional, although this is not noticed at each time. That is the *white noise analysis.*

We do not go into details, but the advantages or characteristics of our analysis should now be made clear.

They are

1. First $\dot{B}(t)$ and $\dot{P}(t)$ have been given identity for every t. It is easy to understand that they are generalized stochastic process by parameterized with test function ξ like $\dot{B}(\xi)$ and $\dot{P}(\xi)$. But we give correct mathematical meaning to both $\dot{B}(t)$ and $\dot{P}(t)$, respectively. Indeed, they are generalized linear functionals of Brownian motion and Poisson process, respectively. In fact, they are taken to be variables of generalized functionals, where t runs through an interval $T \subset R$.

Naturally, we are led to introduce classes of *generalized functionals* of white noise and Poisson noise, respectively. We are indeed gien a quite wider classes of random functionals, so that various kinds of collaborations and applications not only within mathematics, but in quantum dynamics, information science and molecular biology.

2. Effective use of the infinite dimensional rotation group and infinite symmetric group. These two groups appear as invariance of white noise and Poisson noise, respectively. Our analysis has, therefore, an aspect of harmonic analysis, which is, of course, infinite dimensional. In particular, unitary representations of the groups give us a powerful tool of the analy-

sis. Moreover, it is expected some duality can be discovered between two noises. Motivation will be seen in I. Ojima's works, in particular in 10] and the references listed there. We are grateful to him for his suggestion.

3. Study of random fields (and further quantum fields) is a natural development of the analysis where functiuonals have expression in terms of idealized elemental random variables. this direction may be said an innovation approach. Some results can be seen in [7].

Remark The white noise analysis based on Gaussian variables has extensively developed with the significant properties mentioned in 1,2 and 3 above. One can expect similar results on the analysis of Poisson noise functionals. Similar analysis can be applied, although here arise difficulties. They can be overcome in some ways or other. What are really interesting story exist in the dissimilarity between two noises. We are interested in those dissimilarity and will show some of them.

2 Gaussian system of random variables

There are many significant properties which characterize a Gaussian distribution. There are interesting cases where linear operations are involved. Others to be mentioned are limit theorems involving the central limit theorem and domain of attraction for Gaussian distribution. Unfortunately there is no space to write those interesting characteristics, and we have to come to a system involving many Gaussian random variables.

We prepare a definition.

Definition A system $\mathbf{X} = \{X_\alpha, \alpha \in A\}$ of real valued random variables is called a *Gaussian system* if any finite linear combination of the X_α's is Gaussian.

Before we come to a general theory on a system involving infinitely many random variables, we prepare the main lemma due to P. Lévy for a system of two random variables.

Lemma (P. Lévy) For a pair (X, Y), suppose that there exists U and

V such that

$$Y = aX + U,$$

and

$$X = bY + V,$$

where U is independent of X and V is independent of Y, respectively, and where a and b are constant, then there are only three possibilities:

i) there exists an affine relation between X and Y,

ii) X and Y are independent,

iii) (X, Y) is a Gaussian system.

Remark For a system of Poisson random variables similar statement does not exist.

Example. Let $P(t), t \geq 0$, be a Poisson process. Take $P(s)$ and $P(t)$ with $s < t$. Then,

$$P(t) = P(s) + (P(t) - P(s))$$

is fine, since $P(s)$ and $P(t) - P(s)$ are independent. On the other hand, if we have

$$P(s) = bP(t) + V,$$

then, by the Raikov Theorem (1938), both $bP(t)$ and V should be Poisson random variables. Hence, $b = 1$, so that the above equality is not acceptable.

There one can see a dissimilarity between Gaussian and Poissson distributions.

Let $\mathbf{X} = \{X_\alpha, \alpha \in A\}$ be a Gaussian system and let A be an infinite set, countable or may not. If A is uncountable, we have to assume that the system is *separable.*

Definition

1. A random variable Y which is outside of $\mathbf{X}$ is said to be linearly dependent on $\mathbf{X}$, if Y is expressed in the form

$$Y = U + V,$$

where U is a memeber in $\mathbf{X}$, and V is independent of $\mathbf{X}$.

In the above expression, U and V are unique up to non-random constants.

2. A system $\mathbf{X}$ is said to be *completely linearly correlated*, if for any subclass $\mathbf{X}' \subset \mathbf{X}$, any Y outside of $\mathbf{X}'$ is linearly dependent on $\mathbf{X}'$.

Theorem 2.1 *A completely linearly correlated system* $\mathbf{X}$ *is Gaussian, except isolated members.*

More precisely, we can state the result as the **decomposition theorem** of a completely linearly correlated system.

Theorem 2.2 *For a separable completely linearly correlated system* $\mathbf{X} = \{X_\alpha, \alpha \in A\}$, *the set* A *admits a partition:*

$$A = A_0 + \sum_{n \geq 1} A_n,$$

where

1) $\mathbf{X}_0 = \{X_\alpha, \alpha \in A_0\}$ *is a Gaussian system,*

2) $\mathbf{X}_n = \{X_\alpha, \alpha \in A_n\}, n \geq 0$, *are mutually independent systems,*

3) for any $n > 0$, $\mathbf{X}_n$ *involves only one element except constant.*

3 Whitening of Gaussian processs

A particular Gaussian system is a Gaussian process $X(t), t \in T$, depending on the time parameter t, the set T being an interval of R. We are interested in the *causality* of functions or operations related to the process. For this purpose it is necessary to have the step of reduction of the complex random system given by the process $X(t)$. There we have to introduce generalized stochastic processes, which is idealised elemental stochastic process; it is still Gaussian.

We then come to the general theory of representation of Gaussian processes in terms of Gaussian noises (independent systems). Let $X(t), t \geq a$, be a Gaussian process satisfying

1) it is separable, and

2) it has no remote past.

Then we have a theorem (T.H. and H. Cramér).

Theorem 3.1 *Under the assumptions 1) and 2) with additional assumption that $E(X(t)) = 0$, there exist at most countably many Gaussian noises $\dot{Z}_n(t), n \geq 0$, and $L^2(dm_n)$ kernels $F_n(t,u)$ such that*

$$X(t) = \sum_n \int_a^t F_n(t,u)\dot{Z}_n(u)du,$$

for which

$$\mathbf{B}_t(X) = \vee_n \mathbf{B}_t(\dot{Z}_n)$$

holds for every t. The sum in the above formula means the lattice sum. The measures dm_n are arranged in an decreasing order.

The symbol like $\mathbf{B}_t(X)$ denotes the sigma field with respect to which all the $X(s), s \leq t$, are measurable. The measure dm_n is unique up to equivalence. Moreover $F_n(t,u)^2 dm_n(u)$ is unique.

This theorem shows how our idea is realized: getting the system $\{\dot{Z}_n\}$ is the step of **reduction**, and forming the representation in terms of the stochastic integrals corresponds to the step of **synthesis**.

The method of investigating a Gaussian process mentioned above leads us to a possible method of the study of more general Gaussian system.

We discuss a Gaussian system for which we ignore the causality (or time propagation). Hence, we may assume that a separable Gaussian system $\mathbf{X} = \{X_\alpha, \alpha \in A\}$, is given where the parameter set A is arbitrary. Then, we can prove a generalization of the theorem mentioned above.

Theorem 3.2 *The whitening is possible for a separable completely linearly correlated system $\mathbf{X} = \{X_\alpha, \alpha \in A\}$.*

Proof. To make the matter simple, we assume that $E(X_\alpha) = 0$. By using the theorem, the decomposition of the parameter set A is given

$$A = A_0 + \sum_{n \geq 1} A_n.$$

For the sets $\mathbf{X}_n = \{X_\alpha, \alpha \in A_n\}, n > 0$, the whitening is already made. So we must prove the theorem only for a Gaussian system $\mathbf{X}_0 = \{X_\alpha, \alpha \in A_0\}$.

Suppose there exists a subset $A'_0 \subset A_0$ for which whitening of $\mathbf{X}' = \{X_\alpha,\ \alpha \in A'_0\}$ is done (existence of such A'_0 is obvious). If A'_0 is not equal to A_0, we can find X_β which is linearly correlated to $\mathbf{X}'$. So, we can extend the system $\mathbf{X}'$. Hence there exists a linearly ordered (by inclusion) subsystems of $\mathbf{X}_0$. By Zorn's Lemma there exists a maximal subsystem, which must be the entire $\mathbf{X}_0$. Thus, the theorem is proved.

Remark. If A_0 is linearly ordered, then, with a slight generalization, the canonical representation theory can be applied.

We may now say that for any separable linearly correlated system the step of reduction can alaways be done.

4 Nonlinear functions of Gaussian systems. Quadratic forms I

Concerning the step of reduction, the most powerful technique seems to be the innovation approach, or whitening more general random complex system is expected. By this method, suppose we are given a Gaussian noise. To fix the idea, we shall assume a white noise is given. Let us denote it by $\dot{B}(t), t \in R$. This system is fitting to be taken as a variable system of functionals to be discussed.

Once a variable system is given, it is natural to discuss polynomials in those varia bles in the system. In the present case, we must discuss polynomials in $\dot{B}(t)$'s. Linear functions are Gaussian, and if it depends on the time parameter t, it is a Gaussian process that has been discussed in the last section.

We are now ready to come to quadratic forms. The simplest example is just a square of $\dot{B}(t)$, i.e. $\dot{B}(t)^2$. Needless to say, it has no meening at all. Remind the particular equation of the Itô formula:

$$(dB(t))^2 = dt.$$

Hence $\dot{B}(t)^2 = \frac{1}{dt}$, which is not suitable in our case. However, we should manage $\dot{B}(t)^2$ as the most basic functional (simplest monomial) of white noise beyond the classical formula. We carefully see both sides to recognize that the left side is still random and its expectation is equal to the right side. The difference can be *magnified to close up* randomness that remains. There arises the idea of renormalization to jump beyond the classical stochastic calculus. To give a rigorous meaning to this trick we need "renormalization". Note that it is entirely different from the orthogonalization done by using the Hermite polynomials.

The S-transform

To establish a more general method to introduce polynomials in $\dot{B}(t)$'s and continuous sum (not integrals based on them) we employ the S-transform. This transform looks like an infinite dimensional analogue of the Laplace transfrom in its epression, but entirely different. See the definition

$$S\varphi(\dot{B})(\xi) = e^{-\frac{1}{2}\|\xi\|^2} E[e^{\langle \dot{B}, \xi \rangle}\varphi(\dot{B})]$$

which carries $\dot{B}$-functionals to a space of ordinary functionals of the ξ. By using this transform we can get *visualized* expression as (non-random) ordinary functionals, so that we can immediately appeal to the classical theory of functionals.

In addition, the transform is obtained by the *phase average*, which turns out to be the *time average* due to the ergodic theorem. The second advantage is good for statistical apprecation. Note that the S-transform is different from the classical transform on Hilbert space (cf. Segal-Bargmann transform). Its image forms a Reproducing Kernel Hilbert space. A member of this space is called a U-functional.

Example Application to the data processing of the astrophysical statistics of X-ray.

A detailed, in fact rigorous, mathematical interpretation has been given in [8], before that the idea appeared in [6].

We are now ready to discuss typical quadratic functionals of $\dot{B}(t)$'s. Needless to say, quadratic forms are significant in many places in mathematics. Here the variables are taken to be $\dot{B}(t)$'s.

1) Ordinary functionals φ in H_2. Its S-transform is expressed in the form

$$\int\int F(u,v)\xi(u)\xi(v)dudv,$$

where F is a symmetric L^2-kernel. With this representation we obtain characteristic functions, orthogonal series expansion, and others from the kernel properties of F, like in the case of the stochastic area.

2) Quadratic forms in terms of $\dot{B}(t)$'s expressed as either linear functionals of $:\dot{B}(t)^2:$'s or integrals of $\dot{B}(t)\dot{B}(s), t \neq s$. There appear really generalized white noise functionals. We may write functionals in question in the S-transform:

$$(S\varphi)(\xi) = \int\int F(u,v)\xi(u)\xi(v)dudv,$$

where F is a symmetric generalized function on R^2.

We now rush to discuss differential calculus. Before we come to actual computation, we need to understand some hidden structure of white noise theory. Hidden structure means that unlike visualized expression, we have to recognize latent trait of infinite dimensional calculus. The system $\{\dot{B}(t)\}$ looks like a system of continuously many independent random variables depending on t which runs through a continuum. This understanding, however, is not good. The real meaning will be understood by observing the calculus that we are going to develop, where separability is behind.

Coming to visualized expression, that is, using the S-transform, the differential operator is defined. Note that it is not quite the same as $\frac{d}{dx}$. For a white noise functional $\varphi(x)$, we have a U-functional $U(\xi)$:

$$U(\xi) = (S\varphi)(\xi).$$

The variation is denoted by $\delta U(\xi)$. We assume that for $\delta\xi \in E$ we have

$$U(\xi+\delta\xi) - U(\xi) = \int U'(\xi,t)\delta\xi(t)dt + o(\delta\xi).$$

The first term of the right side is denoted by $\delta U(\xi)$ and $U'(\xi,t)$ is a generalized function of t for any ξ. The function of t (may be said that it is parametrized by t) is the *functional derivative in the Fréchet sense*, that is Fréchet derivative.

If $U'(\xi, t)$ is a U-functional for every t, then we write

$$(S^{-1}U'(\cdot, t))(x) = \partial_t \varphi(x).$$

Existence and the domain of the operator ∂_t are defined in the usual manner. Note that the domain of ∂_t includes (S). We often write

$$\partial_t = \frac{\partial}{\partial \dot{B}(t)}.$$

Remark It should be noted that we did use the Fréchet derivative, but not Gâteaux derivative.

The operator ∂_t acts as an *annihilation* operator acting on the space of white noise functionals. Its adjoint operator ∂_t^* can be defined, and it is calle a *creation* operator. It is used to define stochastic integrals, where integrands are not necessarily non-anticipating.

Their commutation relations are

$$[\partial_t, \partial_s^*] = \delta(t-s).$$

In an analogous manner to the finite dimensional case, generator $r_{t,s}$ of rotation is defined by

$$r_{t,s} = \partial_t^* \partial_s - \partial_s^* \partial_t.$$

5 Quadratic forms II. A general stochastic integrals

1. Laplacians.

Laplacians as a quadratic form of differential operators can be characterized by rotations.

Generally speaking a Laplacian is understood to be a quadratic form of a differential form that satisfies some invariance. Our setup is as follows.

We are concerned with the continuously many dimensional case, so that differential operators are taken to be $\partial_t, t \in T$, where T is an interval, say $[0,1]$.

Once again, we remind a second order functional derivatives in the space of U-functionals; let $(S\varphi)(\xi) = U(\xi)$. The second order variation of U is expressed in the form

$$\delta^2 U(\xi) = \int\int_{[0,1]^2} U''(\xi, u, v)\delta\xi(u)\delta\xi(v)dudv,$$

where $U''(\xi, u, v)$ is a symmetric generalized function of (u, v) for every ξ. It is the second order *functional derivative* in the *Fréchet* sense. The correspondence between white noise functionals and U-functionals tells us that

$$\partial_t \partial_s \varphi = S^{-1}(U''(\xi, t, s)),$$

if the right hand side exists.

The second order partial differential operator D is now defined by

$$D = \int\int_{[0,1]} F(u,v)\partial_u \partial_v dudv.$$

The choice of the class of the kernel $F(t, u)$, which is symmetric, of course depends on the problems to be discussed.

We can specify the operator D by using commutation relations with rotations $r_{s,t} = \partial_s^* \partial_t - \partial_t^* \partial_s, s, t \in [0,1]$. We may consider an analogy of the finite dimensional case where the Laplacian is determined. Namely we can prove

Theorem 5.1 *The operator D commutes with all rotations if and only if $F(u,v) = 0$ for off diagonal and $F(t,t)$ being a constant (ordinary or generalized) function.*

Proof is given by actual computations of the commutators $[D, r_{s,t}], s, t \in [0,1]$.

The situation is now divided into two cases.

1) $F(u, v) = c\delta(u - v)$. Then, $D = c\int_0^1 \partial_t^2 dt$, which is the Volterra Laplacian Δ_V by taking $c = 1$.

2) $F(u,u) = 1$. Then, we have the Lévy Laplacian $\Delta_L = \int_0^1 \partial_t^2 (dt)^2$, although we need some interpretation to this integral.

In the case 2), there are many kinds of explanation on why dt appears in two holds. Here we may say one dt is used to take average over $[0,1]$. Another explanation is that one dt is used to cancel the singularity when Δ_L is applied to generalized white noise functionals. There may be some others.

The idea of characterization of quadratic differential operators by rotations has appeared in the paper [8].

2. Comparison between Gaussian and Poisson noises.

A particular quadratic form of a Gaussian system, indeed that of a Brownian motion has appeared in the study of a duality between white noise and Poisson noise.(See [13]). The quadratic form in question can play significant roles in the unitary representation of infinite symmetric group and in the description of hidden characteristic of Gaussian systems.

First we take an increasing sequence of vector spaces spanned by the renormalized squares of increments of a Brownina motion $B(t), t \in [0,1]$. Let D_n be the partition of $[0,1]$: $D_n = \{\Delta_k^n, 1 \le k \le 2^n\}, |\Delta_k^n| = 2^{-n}$. The system $\{B_k^n \equiv : (\Delta_k B)^2 :, 1 \le k \le 2^n\}$, spans a 2^n-dimensional vector space L_n which is a subspace of $L^2(\Omega)$. The L_n is 2^n dimensional.

Now let us change our viewpoint to discuss a unitary representation of the symmetric group $S(n)$, and follow the technique proposed in [12].

To specify the situation, consider $S(2^n)$. Let π^n be a permutation of a subset $\{1, 2, \cdots, 2^n\}$, where $\pi^n(k)$ is the image of k. Let U_π^n be the operator defined by

$$U_\pi^n : B_k^n \quad \to \quad B_{\pi^n(k)}^n.$$

This extends to a unitary operator acting on L_n. Thus, the triple $(S(2^n), U_\pi^n, L_n)$ is a unitary representation of the group $S(2^n)$.

The special irreducible unitary representation is given by taking the one–dimensional subspace of L_n spanned by $\{c \sum_1^{2^n} Y_k^n\}$, where $Y_k^n, 1 \le k \le 2^n$, are the orthonormal vectors given by $Y_k^n =: (\frac{\Delta_k^n B}{\Delta_k^n})^2 : \frac{\Delta_k^n}{\sqrt{2}}$.

We are now ready to define the projection

$$(S(2^{n+1}), L_{n+1}) \;\rightarrow\; (S(2^n), L_n).$$

The mapping $S(2^{n+1}) \;\rightarrow\; S(2^n)$ is easily done by the usual method. While the projection of a vector needs interpretation. Let $\mathbf{B}_n$ be the smallest sigma–field with respect to which all the $Y_k^n, 1 \geq k \geq 2^n$ are measurable. Then we establish a theorem.

Theorem 5.2 *We have a relationship between the conditional expectations:*

$$E(\sum_1^{2^{n+1}} Y_k^{n+1}/\mathbf{B}_n) = \frac{1}{\sqrt{2}}\sum_1^{2^n} Y_k^n.$$

Proof comes from the computation of the conditional expectations.

We therefore establish a consistent family of irreducible representations by using the conditional expectations. Hence, the existence of the projective limit

$$\mathrm{proj}\cdot\lim(S(2^{n+1}), L_{n+1}) = (S(\infty), L_\infty).$$

One may ask how to understand the space $(S(\infty),\, L_\infty)$ or the limit of the sum $\sum Y_k^n$ of orthonormal vectors. Formally writing,

$$\sum \frac{:(\Delta_k B)^2:}{(\Delta_k)^2}\Delta_k.$$

We have, however, to have the average (arithmetic mean) to take an analogy of Cesaro limit. This can be done by dividing by 2^n, i.e. multiplying $|\Delta|$. Finally we come to

$$\int_0^1 :\dot{B}(t)^2:(dt)^2.$$

Noting that $\dot{B}(t) = \partial_t^* 1$, the above expression denotes the adjoint of the Lévy Laplacian Δ_L.

For further development of the theory we refer to the paper [13]. We shall however note one important fact. Namely, a generalization of the stochastic integral.

The following fact is well known.

The integral $\int_0^1 f(t) : \dot{B}(t)^2 : dt$ is well defined for $f \in L^2([0,1])$, and is a member of the space $H_2^{(-2)}$ involving generalized white noise functionals of degree 2. Its dual space $H_2^{(2)}$ consists of all quadratic test functionals of the $\dot{B}(t)$'s expressed in the form

$$\int_0^1 \int_0^1 F(t,s) : \dot{B}(t)\dot{B}(s) : dtds,$$

where F is a member of symmetric Sobolev space of order 3/2. We therefore have a Gel'fand triple:

$$H_2^{(2)} \subset H_2 \subset H_2^{(-2)},$$

where H_2 is the space of double Wiener integral. The sapces $H_2^{(2)}$ and $H_2^{(-2)}$ form a dual (cf. [3] Part 1. Chap. II, III.) pair.

We now propose another dual pair, one of which is

$$H_2^{(-2,1)} = \left\{ \int_0^1 : \dot{B}(t) :^2 dt; \ f \in L^2([0,1]) \right\},$$

a subspace of $H_2^{(-2)}$.

Our aim is to define an integral

$$\int_0^1 g(t) : \dot{B}(t) :^2 (dt)^2$$

by using the projective limit theorem, and then to show that

$$\left\{ \int_0^1 : \dot{B}(t) :^2 dt \right\} \text{ and } \left\{ \int_0^1 : \dot{B}(t) :^2 dt \right\}$$

stands in a dual relation. cf. P. Lévy [3] Part I.Chap.II.

The details of the proof of this fact comes from Theorem(5.2) and Si Si [9].

6 Poisson noise functionals

Dissimilarities in the analysis of Gaussian and Poisson noise functionals can be found in may ways. Among others

1. *Infinite symmetric group* gives an invariance of Poisson noise measure. This property is compared with the fact that the infinite dimensional rotation group give an invariance of white noise measure and even gives a characterization of Gaussian measure. Unitary representation of the infinite symmetric group shows a particularly potent properties of Poisson noise. See e.g. [13].

2. *Duality* between Gaussian and Poisson noises is most interesting topic to be investigated systematically. Professor I. Ojima highly recommends and in fact we have started a joint work on this problem.

3. Orthogonal polynomials. As is well known, Hermite polynomials define a complete orthonormal base (Fourier-Hermite polynomials) of the Hilbert space (L^2) spanned by functionals of Brownian motion with finite variance. It may be expected that similar role can be played by the Charlier (Poisson-Charlier) polynomials in Poisson case. This may be all right, in a sense. While, as is mentioned in the *Remark* in §11.2 of [5], this is not quite so. To make a long story short, we claim the following assertion. For simplicity, we assume that the time parameter runs through $[0, 1]$.

Proposition 6.1 *1) Two random variables $\langle \xi_i, \dot{B} \rangle, i = 1, 2$, are independent if and only if*

$$\int \xi_1(t)\xi_2(t)dt = 0.$$

2) On the other hand, $\langle \xi_i, \dot{P} \rangle, i = 1, 2$, are independent if and only if

$$\xi_1(t)\xi_2(t) \equiv 0.$$

Proof. 1) is well known.

2) Suppose that $\langle \xi_1, \dot{P} \rangle$ and $< \xi_2, \dot{P} >$ are independent. Let φ and $\varphi_i, i = 1, 2$, be characteristic function of the pair and each members, respectively. By assumption, we have

$$\varphi(z_1, z_2) = \varphi_1(z_1)\varphi_2(z_2), \ (z_1, z_2) \in R^2.$$

Using the formula of characteristic functional of $\dot{P}$, we can conclude that $\xi_1(t)\xi_2(t) \equiv 0$ must hold.

Because of 2), orthogonalization of Poisson noise functionals is somewhat different from the Gaussian case; namely we have to pay additional attention on the completeness of the Poisson-Chasrlier polynomials. (See [5].)

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