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Hitsuda–Skorohod Quantum Stochastic Integrals in Terms of Quantum Stochastic Gradients

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1 Introduction

Since Hudson–Parthasarathy [2], the quantum stochastic integrals of Itô type have been studied extensively by many authors, see the books by Meyer [10] and Parthasarathy [14], and their extensions to non-adapted quantum stochastic integrals by Belavkin [1], Lindsay [8], among others. In this note, using the functional analytic method (kernel theorem and duality), we introduce the *quantum stochastic gradients* and thereby define the *Hitsuda–Skorohod quantum stochastic integrals* for non-adapted integrands.

The (classical) Hitsuda–Skorohod integral is defined by means of the adjoint action of the stochastic gradient (also called the Malliavin gradient) and provides a method of generalizing the Itô integral for non-adapted integrands, see e.g., Kuo [7], Malliavin [9], Nualart [11]. Let us explain slightly more in detail in terms of white noise theory (see Section 2). Let

$$(E) \subset \Gamma(H) = \Gamma(L^2(\mathbb{R})) \subset (E)^*$$

be the Hida–Kubo–Takenaka space of white noise functions. The stochastic gradient ∇ is defined for a white noise function ϕ (in a suitable domain) by

$$\nabla\phi(t) = a_t\phi,$$

where a_t is the annihilation operator at a point $t \in \mathbb{R}$. Then ∇ becomes a linear map from a suitable domain of white noise functions into a space of L^2-functions with values in white noise functions, i.e., a space of stochastic processes. The adjoint map δ of ∇ maps an L^2-function Φ with values in white noise functions (i.e., a stochastic process) to a white noise function. We call $\delta(\Phi)$ the Hitsuda–Skorohod integral, see Sections 3.1 and 4.1.

In the quantum context there are three quantum stochastic integrals, namely, against the annihilation, creation and conservation processes. Accordingly, we need to introduce three stochastic gradients. Our idea is based on the kernel theorem that ensures the isomorphism

$$\mathcal{L}((E), (E)^*) \cong (E)^* \otimes (E)^*.$$

Here an element in $\mathcal{L}((E), (E)^*)$ is called a *white noise operator*. A time-indexed white noise operator $\Xi = \{\Xi(t)\,;\, t \in \mathbb{R}\}$ is our quantum stochastic process for which we define integrals.

Thanks to the canonical isomorphism, we may define quantum stochastic gradients for white noise operators via the tensor product spaces. Taking a suitable domain, we define

$$\nabla^+[\phi\otimes\psi](t) = (\nabla\phi(t))\otimes\psi,$$
$$\nabla^-[\phi\otimes\psi](t) = \phi\otimes(\nabla\psi(t)),$$
$$\nabla^0[\phi\otimes\psi](t) = (\nabla\phi(t))\otimes(\nabla\psi(t)).$$

Through the canonical isomorphism, each ∇^ϵ is regarded as a linear map from a certain space of white noise operators into an L^2-space with values in white noise operators (i.e., a space of quantum stochastic processes). These maps are called the *annihilation, creation* and *conservation gradients*, respectively. The precise definition will be given in Section 3.

The adjoint map of ∇^ϵ, denoted by δ^ϵ, maps an L^2-function Ξ with values in white noise operators (a quantum stochastic process) to a white noise operator. We call $\delta^+(\Xi)$, $\delta^-(\Xi)$ and $\delta^0(\Xi)$ the *creation, annihilation* and *conservation integrals*, respectively. The details will be found in Section 4. Our approach is expected to be advantageous to systematic study of regularity properties of quantum stochastic integrals and quantum martingales, for relevant study see Ji [3], Ji–Obata [5, 6].

2 Quantum White Noise Calculus

2.1 White Noise Distributions

Let $L^2_{\mathbb{R}}(\mathbb{R}, dt)$ be the Hilbert space of $\mathbb{R}$-valued square-integrable functions on the real line $\mathbb{R}$, which is often considered as the time axis. Let $\mathcal{S}(\mathbb{R})$ be the space of rapidly decreasing functions equipped with the canonical topology, and $\mathcal{S}'(\mathbb{R})$ its dual space, i.e., the space of so-called tempered distributions. The real Gelfand triple:

$$\mathcal{S}(\mathbb{R}) \subset L^2_{\mathbb{R}}(\mathbb{R}, dt) \subset \mathcal{S}'(\mathbb{R}) \tag{2.1}$$

is our starting point. Since the inner product of $L^2_{\mathbb{R}}(\mathbb{R}, dt)$ and the canonical bilinear form on $\mathcal{S}'(\mathbb{R})\times\mathcal{S}(\mathbb{R})$ are compatible, they are denoted by the same symbol $\langle\cdot,\cdot\rangle$. For simplicity, the complexification of (2.1) is denoted by

$$E \subset H = L^2(\mathbb{R}) \subset E^*.$$

(Throughout this paper $L^2(\dots)$ means the complex L^2-space.) The canonical $\mathbb{C}$-bilinear form on $E^*\times E$ is denoted again by $\langle\cdot,\cdot\rangle$ so the norm of H, denoted by $|\cdot|_0$, satisfies $|\xi|_0^2 = \langle\overline{\xi},\xi\rangle$ for $\xi\in H$.

It is well known that the topology of E is defined by means of the differential operator $A = 1 + t^2 - d^2/dt^2$ acting in H. For each $p \ge 0$, $E_p = \mathrm{Dom}(A^p)$ becomes a Hilbert space with norm $|\xi|_p = |A^p\xi|_0$ and E_{-p} denotes the completion of H with respect to the norm $|\xi|_{-p} = |A^{-p}\xi|_0$. Then we obtain a chain of Hilbert spaces:

$$\cdots\subset E_p\subset\cdots\subset H\subset\cdots\subset E_{-p}\subset\cdots.$$

Note that E_{-p} is identified with the strong dual space of E_p through the canonical $\mathbb{C}$-bilinear form. Finally, we have topological isomorphisms:

$$E \cong \operatorname*{proj\,lim}_{p\to\infty} E_p, \qquad E^* \cong \operatorname*{ind\,lim}_{p\to\infty} E_{-p}.$$

There exists an orthonormal basis $\{e_i\}_{i=0}^{\infty} \subset E$ of H such that $Ae_i = (2i+2)e_i$, $i = 0, 1, 2, \ldots$. Hence A^{-1} is of Hilbert–Schmidt type and E is a countably Hilbert nuclear space. The constant number

$$\rho = \|A^{-1}\|_{\mathrm{OP}} = \frac{1}{2}$$

plays an important role in norm estimates in white noise calculus.

The (Boson) Fock space over E_p is defined by

$$\Gamma(E_p) = \left\{ \phi = (f_n)_{n=0}^{\infty} \, ; \, f_n \in E_p^{\hat{\otimes} n}, \, \|\phi\|_p^2 = \sum_{n=0}^{\infty} n! \, |f_n|_p^2 < \infty \right\}.$$

Then we obtain a chain of Fock spaces:

$$\cdots \subset \Gamma(E_p) \subset \cdots \subset \Gamma(H) \subset \cdots \subset \Gamma(E_{-p}) \cdots$$

and, as limit spaces we define

$$(E) = \operatorname*{proj\,lim}_{p\to\infty} \Gamma(E_p), \qquad (E)^* = \operatorname*{ind\,lim}_{p\to\infty} \Gamma(E_{-p}).$$

It is known that (E) is a countably Hilbert nuclear space. Consequently, we obtain a complex Gelfand triple:

$$(E) \subset \Gamma(H) \subset (E)^*,$$

which is referred to as the *Hida–Kubo–Takenaka space*. The dual space $\Gamma(H)$ is identified with itself through the canonical $\mathbb{C}$-bilinear form.

By definition the topology of (E) is defined by the norms

$$\|\phi\|_p^2 = \sum_{n=0}^{\infty} n! \, |f_n|_p^2, \qquad \phi = (f_n),$$

where p runs over $\mathbb{R}$. On the other hand, for each $\Phi \in (E)^*$ there exists $p \geq 0$ such that $\Phi \in \Gamma(E_{-p})$ and

$$\|\Phi\|_{-p}^2 \equiv \sum_{n=0}^{\infty} n! \, |F_n|_{-p}^2 < \infty, \qquad \Phi = (F_n).$$

The canonical $\mathbb{C}$-bilinear form on $(E)^* \times (E)$ takes the form:

$$\langle\langle \Phi, \phi \rangle\rangle = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle, \qquad \Phi = (F_n) \in (E)^*, \quad \phi = (f_n) \in (E).$$

2.2 White Noise Operators

A continuous linear operator from (E) into $(E)^*$ is called a *white noise operator*. The space of white noise operators is denoted by $\mathcal{L}((E),(E)^*)$ and is equipped with the bounded convergence topology. The white noise operators cover a wide class of Fock space operators, for example, $\mathcal{L}((E),(E))$, $\mathcal{L}((E)^*,(E))$ and $\mathcal{L}(\Gamma(H),\Gamma(H))$ are subspaces of $\mathcal{L}((E),(E)^*)$. Note that $\mathcal{L}(\Gamma(H),\Gamma(H))$ is the space of bounded operators on $\Gamma(H)$.

Since (E) is a nuclear space, by the kernel theorem we have the canonical isomorphism:

$$\mathcal{K} : \mathcal{L}((E),(E)^*) \xrightarrow{\cong} (E)^* \otimes (E)^*, \tag{2.2}$$

which is defined by

$$\langle\!\langle \Xi\phi, \psi \rangle\!\rangle = \langle\!\langle \mathcal{K}\Xi, \psi \otimes \phi \rangle\!\rangle, \qquad \phi, \psi \in (E).$$

In (2.2) the symbol $\otimes$ means the π-tensor product of topological vector spaces. As restrictions of $\mathcal{K}$ we obtain similar isomorphisms such as

$$\mathcal{L}((E)^*, (E)) \cong (E) \otimes (E), \qquad \mathcal{L}((E), (E)) \cong (E) \otimes (E)^*,$$
$$\mathcal{L}((E)^*, (E)^*) \cong (E)^* \otimes (E), \qquad \mathcal{L}((E)^*, \Gamma(H)) \cong \Gamma(H) \otimes (E), \quad \text{etc.}$$

Particular attention should be paid to $\mathcal{L}(\Gamma(H), \Gamma(H))$. Note that the Hilbert space tensor product $\Gamma(H) \otimes \Gamma(H)$ is isomorphic to the space of Hilbert–Schmidt operators $\mathcal{L}_2(\Gamma(H), \Gamma(H))$.

For each $x \in E^*$ we define

$$a(x) : \phi = (f_n)_{n=0}^{\infty} \mapsto ((n+1)x \otimes_1 f_{n+1})_{n=0}^{\infty},$$

where $x \otimes_1 f_n$ stands for the contraction. It is well known that $a(x) \in \mathcal{L}((E), (E))$. We call $a(x)$ the *annihilation operator* associated with x. The adjoint operator $a^*(x) \in \mathcal{L}((E)^*, (E)^*)$ is given by

$$a^*(x) : \phi = (f_n)_{n=0}^{\infty} \mapsto (x \hat{\otimes} f_{n-1})_{n=0}^{\infty}, \quad \text{(understanding } f_{-1} = 0\text{)},$$

and is called the *creation operator* associated with x. In particular, for each $t \in \mathbb{R}$ we put

$$a_t = a(\delta_t), \qquad a_t^* = a^*(\delta_t).$$

The pair $\{a_t, a_t^* \,;\, t \in \mathbb{R}\}$ is called the *quantum white noise*, for a survey see e.g., Ji–Obata [4].

3 Quantum Stochastic Gradients

3.1 Classical stochastic gradient

For a suitable ϕ in $\Gamma(H)$ or in a larger space the (classical) stochastic gradient is defined by

$$\nabla\phi(t) = a_t\phi, \qquad t \in \mathbb{R},$$

whenever the map $t \mapsto a_t\phi$ is given a meaning according to a context. Our framework has a significant advantage for a very regular property of the quantum white noise.

Lemma 3.1 *The map $t \mapsto a_t$ is an $\mathcal{L}((E), (E))$-valued rapidly decreasing function, i.e., belong to $\mathcal{S}(\mathbb{R}) \otimes \mathcal{L}((E), (E)) \cong \mathcal{L}((E), \mathcal{S}(\mathbb{R}) \otimes (E)) \cong \mathcal{S}(\mathbb{R}, \mathcal{L}((E), (E)))$.*

As a result, the stochastic gradient

$$\nabla : (E) \to \mathcal{S}(\mathbb{R}) \otimes (E) \cong \mathcal{S}(\mathbb{R}, (E)) \tag{3.1}$$

becomes a continuous linear map. For applications we need to extend the domain of ∇ in (3.1). For $\phi = (f_n) \in (E)^*$ we set

$$\| \phi \|_{\mathbf{D}}^2 = \sum_{n=0}^{\infty} (n+1) n! \, | f_n |_0^2 .$$

Then $\mathbf{D} = \{\phi \in (E)^* \,;\, \|\phi\|_{\mathbf{D}} < \infty\}$ is a subspace of $\Gamma(H)$ and becomes a Hilbert space equipped with the norm $\| \cdot \|_{\mathbf{D}}$. The dual space is identified with $\mathbf{D}^* = \{\Phi \in (E)^* \,;\, \|\Phi\|_{\mathbf{D}^*} < \infty\}$, where

$$\| \Phi \|_{\mathbf{D}^*}^2 = \sum_{n=0}^{\infty} (n+1)^{-1} n! \, | F_n |_0^2, \quad \Phi = (F_n) \in (E)^*. \tag{3.2}$$

Then we have

$$(E) \subset \mathbf{D} \subset \Gamma(H) \subset \mathbf{D}^* \subset (E)^*.$$

Lemma 3.2 *The map* ∇ *in* (3.1) *is extended uniquely to a continuous linear map from* $\mathbf{D}$ *into* $L^2(\mathbb{R}) \otimes \Gamma(H) \cong L^2(\mathbb{R}, \Gamma(H))$. *Denoting the extension by the same symbol, we have*

$$\|\nabla\phi\|_{L^2(\mathbb{R},\Gamma(H))} \le \|\phi\|_{\mathbf{D}}, \qquad \phi \in \mathbf{D}.$$

Proof. For $\phi = (f_n) \in (E)$ we have

$$\begin{aligned}\|\nabla\phi\|^2_{L^2(\mathbb{R},\Gamma(H))} &= \int_{\mathbb{R}} \|a_t\phi\|_0^2\, dt = \sum_{n=0}^{\infty} n! \int_{\mathbb{R}} |(n+1)f_{n+1}(t,\cdot)|_0^2\, dt \\ &= \sum_{n=0}^{\infty}(n+1)(n+1)!\,|f_{n+1}|_0^2 \le \sum_{n=0}^{\infty}(n+1)n!\,|f_n|_0^2 = \|\phi\|^2_{\mathbf{D}},\end{aligned}$$

which proves the assertion. ∎

A further extension is possible.

Lemma 3.3 *The map* ∇ *in* (3.1) *is extended uniquely to a continuous linear map from* $\Gamma(H)$ *into* $L^2(\mathbb{R}) \otimes \mathbf{D}^* \cong L^2(\mathbb{R}, \mathbf{D}^*)$. *Denoting the extension by the same symbol, we have*

$$\|\nabla\phi\|_{L^2(\mathbb{R},\mathbf{D}^*)} \le \|\phi\|_{\Gamma(H)}, \qquad \phi \in \Gamma(H). \tag{3.3}$$

Proof. For $\phi = (f_n) \in (E)$ we have

$$\|\nabla\phi\|^2_{L^2(\mathbb{R},\mathbf{D}^*)} = \int_{\mathbb{R}} \|a_t\phi\|^2_{\mathbf{D}^*}\, dt. \tag{3.4}$$

In view of (3.2) we have

$$\|a_t\phi\|^2_{\mathbf{D}^*} = \sum_{n=0}^{\infty}(n+1)^{-1} n!\,|(n+1)f_{n+1}(t,\cdot)|_0^2 = \sum_{n=0}^{\infty}(n+1)!\,|f_{n+1}(t,\cdot)|_0^2,$$

so (3.4) becomes

$$\|\nabla\phi\|^2_{L^2(\mathbb{R},\mathbf{D}^*)} = \sum_{n=0}^{\infty}(n+1)! \int_{\mathbb{R}} |f_{n+1}(t,\cdot)|_0^2\, dt = \sum_{n=0}^{\infty}(n+1)!\,|f_{n+1}|_0^2 \le \|\phi\|^2_{\Gamma(H)}.$$

This proves (3.3). ∎

The above argument is condensed into the following diagram:

$$\begin{array}{ccccc} (E) & \longrightarrow & \mathbf{D} & \longrightarrow & \Gamma(H) \\ \nabla\big\downarrow & & \nabla\big\downarrow & & \nabla\big\downarrow \\ \mathcal{S}(\mathbb{R}, (E)) & \longrightarrow & L^2(\mathbb{R}, \Gamma(H)) & \longrightarrow & L^2(\mathbb{R}, \mathbf{D}^*), \end{array} \tag{3.5}$$

where the right arrows are continuous injections (inclusions) and the down arrows are continuous linear maps which differ in domains but are denoted by the symbol ∇. We refer to ∇ as the *(classical) stochastic gradient*. The stochastic gradient ∇ with the domain $\mathbf{D}$ appears often in literatures, see e.g., Kuo [7], Nualart [11].

Proposition 3.4 *Let $\zeta \in L^2(\mathbb{R})$. Then we have*

$$a(\zeta)\phi = \int_{\mathbb{R}} \zeta(t)\nabla\phi(t)\,dt, \qquad \phi \in \mathbf{D}, \tag{3.6}$$

and

$$\langle\langle a(\zeta)\phi, \psi\rangle\rangle = \langle\langle \nabla\phi, \zeta \otimes \psi\rangle\rangle, \qquad \phi \in \mathbf{D}, \quad \psi \in \Gamma(H). \tag{3.7}$$

A similar statement is true for $\phi \in \Gamma(H)$.

Proof. Let $\psi \in \Gamma(H)$ and $\zeta \in H$. If $\phi \in (E)$, we have

$$\int_{\mathbb{R}} \zeta(t)\langle\langle \nabla\phi(t), \psi\rangle\rangle\,dt = \int_{\mathbb{R}} \zeta(t)\langle\langle a_t\phi, \psi\rangle\rangle\,dt = \int_{\mathbb{R}} \langle\langle \zeta(t)a_t\phi, \psi\rangle\rangle\,dt = \langle\langle a(\zeta)\phi, \psi\rangle\rangle. \tag{3.8}$$

We know by elementary calculation that $a(\zeta) \in \mathcal{L}(\mathbf{D}, \Gamma(H))$. Hence the right-hand side is continuous in $\phi \in \mathbf{D}$. On the other hand, for $\phi \in \mathbf{D}$ the function $t \mapsto \langle\langle \nabla\phi(t), \psi\rangle\rangle$ belongs to $L^2(\mathbb{R})$ and

$$\int_{\mathbb{R}} \zeta(t)\langle\langle \nabla\phi(t), \psi\rangle\rangle\,dt = \int_{\mathbb{R}} \langle\langle \nabla\phi(t), \zeta(t)\psi\rangle\rangle\,dt = \langle\langle \nabla\phi, \zeta \otimes \psi\rangle\rangle,$$

which is continuous in $\phi \in \mathbf{D}$. Therefore, we see from (3.8) that

$$\int_{\mathbb{R}} \zeta(t)\langle\langle \nabla\phi(t), \psi\rangle\rangle\,dt = \langle\langle a(\zeta)\phi, \psi\rangle\rangle$$

is valid for all $\phi \in \mathbf{D}$, which proves (3.6). During the above discussion (3.7) has been already shown. ∎

3.2 Creation gradient

We first define $\tilde{\nabla}^+$ by compositions of continuous maps as follows:

$$\tilde{\nabla}^+ : \Gamma(H) \otimes (E) \xrightarrow{\nabla \otimes I} L^2(\mathbb{R}, \mathbf{D}^*) \otimes (E) \xrightarrow{\cong} L^2(\mathbb{R}, \mathbf{D}^* \otimes (E)), \tag{3.9}$$

where (3.5) is taken into account. The above isomorphism needs clarification. It follows from a general property of a countably Hilbert nuclear space [12, Proposition 1.3.8] we have

$$L^2(\mathbb{R}, \mathbf{D}^*) \otimes (E) \cong \operatorname{proj\,lim}_{p\to\infty} L^2(\mathbb{R}, \mathbf{D}^*) \otimes \Gamma(E_p),$$
$$\mathbf{D}^* \otimes (E) \cong \operatorname{proj\,lim}_{p\to\infty} \mathbf{D}^* \otimes \Gamma(E_p),$$

where the right hand sides are the Hilbert space tensor products. Taking in mind the isomorphisms:

$$L^2(\mathbb{R}, \mathbf{D}^*) \otimes \Gamma(E_p) \cong (L^2(\mathbb{R}) \otimes \mathbf{D}^*) \otimes \Gamma(E_p) \cong L^2(\mathbb{R}) \otimes (\mathbf{D}^* \otimes \Gamma(E_p)) \cong L^2(\mathbb{R}, \mathbf{D}^* \otimes \Gamma(E_p)),$$

we define

$$L^2(\mathbb{R}, \mathbf{D}^* \otimes (E)) = \operatorname{proj\,lim}_{p\to\infty} L^2(\mathbb{R}, \mathbf{D}^* \otimes \Gamma(E_p)),$$

which justifies the isomorphism in (3.9).

We now define the *creation gradient* ∇^+ by

$$
\begin{array}{ccc}
\mathcal{L}((E)^*, \Gamma(H)) & \xrightarrow{\cong} & \Gamma(H) \otimes (E) \\
{\scriptstyle \nabla^+}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{\nabla}^+} \\
L^2(\mathbb{R}, \mathcal{L}((E)^*, \mathbf{D}^*)) & \xleftarrow{\cong} & L^2(\mathbb{R}, \mathbf{D}^* \otimes (E)).
\end{array}
\tag{3.10}
$$

The above isomorphisms are due to the kernel theorem (recall that (E) is a nuclear space). It is noteworthy that

$$
\begin{aligned}
L^2(\mathbb{R}, \mathcal{L}((E)^*, \mathbf{D}^*)) &\cong L^2(\mathbb{R}, \mathbf{D}^* \otimes (E)) \\
&= \operatorname{proj}\lim_{p\to\infty} L^2(\mathbb{R}, \mathbf{D}^* \otimes \Gamma(E_p)) \\
&= \operatorname{proj}\lim_{p\to\infty} L^2(\mathbb{R}, \mathcal{L}_2(\Gamma(E_{-p}), \mathbf{D}^*)),
\end{aligned}
$$

where $\mathcal{L}_2$ denotes the space of Hilbert–Schmidt operators between Hilbert spaces. It then follows from (3.3) that

$$
\| \nabla^+ \Xi \|_{L^2(\mathbb{R}, \mathcal{L}_2(\Gamma(E_{-p}), \mathbf{D}^*))} \le \| \Xi \|_{\mathcal{L}_2(\Gamma(E_{-p}), \Gamma(H))}. \tag{3.11}
$$

In a similar fashion we can define the creation gradient on different domains. Among others, we note the following:

$$
\begin{aligned}
\nabla^+ : \mathcal{L}((E), \Gamma(H)) &\xrightarrow{\cong} \Gamma(H) \otimes (E)^* \xrightarrow{\nabla \otimes I} L^2(\mathbb{R}, \mathbf{D}^*) \otimes (E)^* \\
&\xrightarrow{\cong} L^2(\mathbb{R}, \mathbf{D}^* \otimes (E)^*) \xrightarrow{\cong} L^2(\mathbb{R}, \mathcal{L}((E), \mathbf{D}^*)),
\end{aligned}
\tag{3.12}
$$

where the last two spaces are defined by

$$
L^2(\mathbb{R}, \mathcal{L}((E), \mathbf{D}^*)) \cong L^2(\mathbb{R}, \mathbf{D}^* \otimes (E)^*) = \operatorname{ind}\lim_{p\to\infty} L^2(\mathbb{R}, \mathbf{D}^* \otimes \Gamma(E_{-p})).
$$

Having defined the creation gradient with two different domains (3.10) and (3.12), we can summarize into the following diagram:

$$
\begin{array}{ccc}
\mathcal{L}((E)^*, \Gamma(H)) & \longrightarrow & \mathcal{L}((E), \Gamma(H)) \\
{\scriptstyle \nabla^+}\big\downarrow & & \big\downarrow{\scriptstyle \nabla^+} \\
L^2(\mathbb{R}, \mathcal{L}((E)^*, \mathbf{D}^*)) & \longrightarrow & L^2(\mathbb{R}, \mathcal{L}((E), \mathbf{D}^*)).
\end{array}
\tag{3.13}
$$

It is also interesting to discuss the creation gradient acting on Hilbert-Schmidt operators. Note that $\mathcal{L}_2(\Gamma(H), \Gamma(H))$ is a subspace of $\mathcal{L}((E), \Gamma(H))$.

Proposition 3.5 *The creation gradient gives rise to a continuous linear map:*

$$
\nabla^+ : \mathcal{L}_2(\Gamma(H), \Gamma(H)) \to L^2(\mathbb{R}, \mathcal{L}_2(\Gamma(H), \mathbf{D}^*)).
$$

Moreover, it holds that

$$
\| \nabla^+ \Xi \|_{L^2(\mathbb{R}, \mathcal{L}_2(\Gamma(H), \mathbf{D}^*))} \le \| \Xi \|_{\mathcal{L}_2(\Gamma(H), \Gamma(H))}, \qquad \Xi \in \mathcal{L}_2(\Gamma(H), \Gamma(H)).
$$

The proof is similar to the argument of (3.10) and (3.12). The next result shows a role of the creation gradient, cf. Proposition 3.4 for the classical case.

Theorem 3.6 *Let $\Xi \in \mathcal{L}((E)^*, \Gamma(H))$. Then, for $\zeta \in H = L^2(\mathbb{R})$ the composition $a(\zeta)\Xi$ is defined as a continuous operator in $\mathcal{L}((E)^*, \mathbf{D}^*)$ and admits the integral expression:*

$$a(\zeta)\Xi = \int_{\mathbb{R}} \zeta(t)\nabla^+\Xi(t)\,dt. \tag{3.14}$$

Similar statements remain valid for $\Xi \in \mathcal{L}((E), \Gamma(H))$ and $\Xi \in \mathcal{L}_2(\Gamma(H), \Gamma(H))$.

Proof. That the composition $a(\zeta)\Xi$ is defined and belongs to $\mathcal{L}((E)^*, \mathbf{D}^*)$ is verified by definition and an elementary norm estimate of annihilation operators.

We show (3.14). Let $\Phi \in (E)^*$ and $\psi \in \mathbf{D}$. Then $t \mapsto \langle\langle \nabla^+\Xi(t)\Phi, \psi\rangle\rangle$ belongs to $L^2(\mathbb{R})$. In fact, choosing $p \geq 0$ such that $\Phi \in \Gamma(E_{-p})$, we see from (3.11) that

$$|\langle\langle \nabla^+\Xi(t)\Phi, \psi\rangle\rangle| \leq \|\nabla^+\Xi(t)\Phi\|_{\mathbf{D}^*}\|\psi\|_{\mathbf{D}} \leq \|\nabla^+\Xi(t)\|_{\mathcal{L}_2(\Gamma(E_{-p}),\mathbf{D}^*)}\|\Phi\|_{-p}\|\psi\|_{\mathbf{D}}.$$

Since $\nabla^+\Xi \in L^2(\mathbb{R}, \mathcal{L}_2(\Gamma(E_{-p}), \mathbf{D}^*))$ by (3.11), we have

$$\int_{\mathbb{R}} |\langle\langle \nabla^+\Xi(t)\Phi, \psi\rangle\rangle|^2 dt \leq \|\Phi\|_{-p}^2\|\psi\|_{\mathbf{D}}^2 \int_{\mathbb{R}} \|\nabla^+\Xi(t)\|^2_{\mathcal{L}_2(\Gamma(E_{-p}),\mathbf{D}^*)}dt < \infty.$$

Then, for any $\zeta \in L^2(\mathbb{R})$ we have

$$\begin{aligned}
\int_{\mathbb{R}} \zeta(t)\langle\langle \nabla^+\Xi(t)\Phi, \psi\rangle\rangle\,dt &= \int_{\mathbb{R}} \langle\langle \tilde{\nabla}^+(\mathcal{K}\Xi)(t), \zeta(t)\psi \otimes \Phi\rangle\rangle\,dt \\
&= \langle\langle (\nabla \otimes I)\mathcal{K}\Xi, \zeta \otimes \psi \otimes \Phi\rangle\rangle \\
&= \langle\langle \mathcal{K}\Xi, \nabla^*(\zeta \otimes \psi) \otimes \Phi\rangle\rangle \\
&= \langle\langle \Xi\Phi, \nabla^*(\zeta \otimes \psi)\rangle\rangle \\
&= \langle\langle \nabla\Xi\Phi, \zeta \otimes \psi\rangle\rangle.
\end{aligned}$$

We see from Proposition 3.4 that the last expression becomes $\langle\langle a(\zeta)\Xi\Phi, \psi\rangle\rangle$. Consequently,

$$\int_{\mathbb{R}} \zeta(t)\langle\langle \nabla^+\Xi(t)\Phi, \psi\rangle\rangle\,dt = \langle\langle a(\zeta)\Xi\Phi, \psi\rangle\rangle,$$

which proves the assertion. ∎

3.3 Annihilation gradient

We define ∇^- by compositions of continuous linear maps as follows:

$$\begin{aligned}
\nabla^- : \mathcal{L}(\mathbf{D}^*, (E)^*) &\xrightarrow{\cong} (E)^* \otimes \mathbf{D} \xrightarrow{I\otimes\nabla} (E)^* \otimes L^2(\mathbb{R}, \Gamma(H)) \\
&\xrightarrow{\cong} \mathcal{L}(L^2(\mathbb{R}, \Gamma(H)), (E)^*) \xrightarrow{\cong} L^2(\mathbb{R}, \mathcal{L}(\Gamma(H), (E)^*)),
\end{aligned}$$

where the last space is defined by

$$L^2(\mathbb{R}, \mathcal{L}(\Gamma(H), (E)^*)) \cong L^2(\mathbb{R}, (E)^* \otimes \Gamma(H)) \cong \operatorname*{ind\,lim}_{p\to\infty} L^2(\mathbb{R}, \Gamma(E_{-p}) \otimes \Gamma(H)).$$

With parallel argument for $\mathcal{L}(\mathbf{D}^*, (E))$ we obtain

$$\begin{array}{ccc} \mathcal{L}(\mathbf{D}^*, (E)) & \longrightarrow & \mathcal{L}(\mathbf{D}^*, (E)^*) \\ {\scriptstyle\nabla^-}\downarrow & & \downarrow{\scriptstyle\nabla^-} \\ L^2(\mathbb{R}, \mathcal{L}(\Gamma(H), (E))) & \longrightarrow & L^2(\mathbb{R}, \mathcal{L}(\Gamma(H), (E)^*)). \end{array} \tag{3.15}$$

We call ∇^- the *annihilation gradient*. As for Hilbert–Schmidt operators,

$$\nabla^- : \mathcal{L}_2(\mathbf{D}^*, \Gamma(H)) \to L^2(\mathbb{R}, \mathcal{L}_2(\Gamma(H), \Gamma(H))$$

becomes a continuous linear map.

Theorem 3.7 *Let $\Xi \in \mathcal{L}(\mathbf{D}^*, (E))$. Then, for $\zeta \in H = L^2(\mathbb{R})$ the composition $\Xi a^*(\zeta)$ is defined as a continuous operator in $\mathcal{L}(\Gamma(H), (E))$ and admits the integral expression:*

$$\Xi a^*(\zeta) = \int_{\mathbb{R}} \zeta(t) \nabla^- \Xi(t)\, dt.$$

Moreover, it holds that

$$\nabla^- \Xi(t) = (\nabla^+ \Xi^*(t))^* \quad \textit{for a.e. } t \in \mathbb{R}.$$

Similar statements remain valid for $\Xi \in \mathcal{L}(\mathbf{D}^*, (E)^*)$ and $\Xi \in \mathcal{L}_2(\mathbf{D}^*, \Gamma(H))$.

The proof of the first half is similar to that of Theorem 3.6. For the second half we need only to note that

$$\Xi a^*(\zeta) = (a(\zeta)\Xi^*)^* = \left(\int_{\mathbb{R}} \zeta(t) \nabla^+ \Xi^*(t)\, dt \right)^* = \int_{\mathbb{R}} \zeta(t) (\nabla^+ \Xi^*(t))^*\, dt.$$

3.4 Conservation gradient

We need the "diagonalized" tensor product $\nabla \oslash \nabla$ of the stochastic gradients. We begin with the following.

Lemma 3.8 *For any $p \ge 0$ and $q > 0$ with $p + q > 5/12$ there exists a constant $C(p, q) > 0$ such that*

$$\sup_{t \in \mathbb{R}} \| \nabla \psi(t) \|_p^2 \le C(p, q) \|\psi\|_{p+q}^2, \qquad \psi \in (E).$$

Proof. We first note that

$$\sup_{t \in \mathbb{R}} |\delta_t|_{-r} < \infty, \qquad r > \frac{5}{12},$$

which is verified by mimicking the argument in Obata [13, Appendix]. Then, for any pair of p, q satisfying the assumption, we have

$$C(p, q) = \max\left\{\rho^{2qn}(n+1) |\delta_t|_{-(p+q)}^2 \; ; \; t \in \mathbb{R}, \; n = 0, 1, 2, \dots \right\} < \infty. \tag{3.16}$$

Now for $\psi = (g_n) \in (E)$ we have $\nabla\psi(t) = a_t\psi$ so that

$$
\begin{aligned}
\| \nabla\psi(t) \|_p^2 &= \sum_{n=0}^{\infty} n! \, |(n+1)\delta_t \otimes_1 g_{n+1} |_p^2 \\
&= \sum_{n=0}^{\infty} n! \, \rho^{2qn} \, |(n+1)\delta_t \otimes_1 g_{n+1} |_{p+q}^2 \\
&\le \sum_{n=0}^{\infty} \rho^{2qn} (n+1) \, |\delta_t|_{-(p+q)}^2 \, (n+1)! \, |g_{n+1}|_{p+q}^2 .
\end{aligned}
$$

Taking (3.16) into account, we obtain

$$
\| \nabla\psi(t) \|_p^2 \le C(p,q) \, \| \psi \|_{p+q}^2 ,
$$

which completes the proof. ∎

For each $\phi \in \mathbf{D}$ and $\psi \in (E)$ we define

$$
[(\nabla \oslash \nabla)\phi \otimes \psi](t) = \nabla\phi(t) \otimes \nabla\psi(t), \qquad \text{for a.e. } t \in \mathbb{R}.
$$

Then, with the help of Lemmas 3.2 and 3.8 one can show easily that

$$
\int_{\mathbb{R}} \| [(\nabla \oslash \nabla)\phi \otimes \psi](t) \|_{\Gamma(H)\otimes\Gamma(E_p)}^2 \, dt \le C(p,q) \, \| \phi \|_{\mathbf{D}}^2 \, \| \psi \|_{p+q}^2 .
$$

We then see that

$$
\nabla \oslash \nabla : \mathbf{D} \otimes (E) \to L^2(\mathbb{R}, \Gamma(H) \otimes (E))
$$

is a continuous linear map. The *conservation gradient* is now defined by compositions of continuous linear maps:

$$
\nabla^0 : \mathcal{L}((E)^*, \mathbf{D}) \xrightarrow{\cong} \mathbf{D} \otimes (E) \xrightarrow{\nabla\oslash\nabla} L^2(\mathbb{R}, \Gamma(H) \otimes (E)) \xrightarrow{\cong} L^2(\mathbb{R}, \mathcal{L}((E)^*, \Gamma(H))).
$$

In a similar manner,

$$
\nabla^0 : \mathcal{L}((E)^*, \Gamma(H)) \xrightarrow{\cong} \Gamma(H) \otimes (E) \xrightarrow{\nabla\oslash\nabla} L^2(\mathbb{R}, \mathbf{D}^* \otimes (E)) \xrightarrow{\cong} L^2(\mathbb{R}, \mathcal{L}((E)^*, \mathbf{D}^*))
$$

becomes also a continuous linear map. Summing up,

$$
\begin{array}{ccc}
\mathcal{L}((E)^*, \mathbf{D}) & \longrightarrow & \mathcal{L}((E)^*, \Gamma(H)) \\
{\scriptstyle \nabla^0}\big\downarrow & & \big\downarrow{\scriptstyle \nabla^0} \\
L^2(\mathbb{R}, \mathcal{L}((E)^*, \Gamma(H))) & \longrightarrow & L^2(\mathbb{R}, \mathcal{L}((E)^*, \mathbf{D}^*)).
\end{array}
\tag{3.17}
$$

4 Quantum Stochastic Integrals

4.1 The Hitsuda–Skorohod integral

The classical stochastic integral of Hitsuda–Skorohod type is defined by means of the adjoint action of the classical stochastic gradient. Let δ denote the adjoint map of the classical stochastic gradient $\nabla : \mathbf{D} \to L^2(\mathbb{R}, \Gamma(H))$, see (3.5). Then,

$$
\delta = \nabla^* : L^2(\mathbb{R}, \Gamma(H)) \to \mathbf{D}^*
$$

becomes a continuous linear map, which is sometimes called the *divergence operator*. By definition it holds that

$$\langle\!\langle \delta(\Psi), \phi \rangle\!\rangle = \int_{\mathbf{R}} \langle\!\langle \Psi(t), \nabla\phi(t) \rangle\!\rangle \, dt, \quad \phi \in \mathbf{D}, \quad \Psi \in L^2(\mathbf{R}, \Gamma(H)). \tag{4.1}$$

Then $\delta(\Psi)$ is called the *Hitsuda–Skorohod integral*.

The quantum stochastic integrals of Hitsuda–Skorohod type are defined in the same spirit, where the quantum stochastic gradients play a role.

4.2 Creation integral

The *creation integral* δ^+ is by definition the adjoint map of the creation gradient ∇^+. From (3.13) we obtain easily the following diagram:

$$\begin{array}{ccc} L^2(\mathbf{R}, \mathcal{L}((E)^*, \mathbf{D})) & \longrightarrow & L^2(\mathbf{R}, \mathcal{L}((E), \mathbf{D})) \\ {\scriptstyle \delta^+}\big\downarrow & & \big\downarrow{\scriptstyle \delta^+} \\ \mathcal{L}((E)^*, \Gamma(H)) & \longrightarrow & \mathcal{L}((E), \Gamma(H)). \end{array}$$

Similarly from Proposition 3.5 we obtain a continuous linear map:

$$\delta^+ : L^2(\mathbf{R}, \mathcal{L}_2(\Gamma(H), \mathbf{D})) \to \mathcal{L}_2(\Gamma(H), \Gamma(H)).$$

The creation integral is expressible in terms of the (classical) Hitsuda–Skorohod integral.

Proposition 4.1 *For $\Xi \in L^2(\mathbf{R}, \mathcal{L}((E), \mathbf{D}))$ it holds that*

$$\delta^+(\Xi)\phi = \delta(\Xi\phi), \qquad \phi \in (E), \tag{4.2}$$

where $\Xi\phi \in L^2(\mathbf{R}, \Gamma(H))$ is defined by $(\Xi\phi)(t) = \Xi(t)\phi$.

PROOF. Taking $\delta^+(\Xi) \in \mathcal{L}((E), \Gamma(H))$ into account, we consider

$$\langle\!\langle \delta^+(\Xi)\phi, \psi \rangle\!\rangle, \qquad \phi \in (E), \quad \psi \in \Gamma(H). \tag{4.3}$$

Let $\tilde{\nabla}^+ : \Gamma(H) \otimes (E) \to L^2(\mathbf{R}, \mathbf{D}^* \otimes (E))$ be the same as in (3.9) and

$$\tilde{\delta}^+ : L^2(\mathbf{R}, \mathbf{D} \otimes (E)^*) \to \Gamma(H) \otimes (E)^*$$

its adjoint operator. Comparing with (3.10) we see that

$$\nabla^+ = \mathcal{K}^{-1} \circ \tilde{\nabla}^+ \circ \mathcal{K}, \qquad \delta^+ = \mathcal{K}^{-1} \circ \tilde{\delta}^+ \circ \mathcal{K}$$

With these notations we calculate (4.3):

$$\begin{aligned} \langle\!\langle \delta^+(\Xi)\phi, \psi \rangle\!\rangle &= \langle\!\langle \mathcal{K}(\delta^+(\Xi)), \psi \otimes \phi \rangle\!\rangle = \langle\!\langle \tilde{\delta}^+(\mathcal{K}(\Xi)), \psi \otimes \phi \rangle\!\rangle \\ &= \langle\!\langle \mathcal{K}\Xi, \tilde{\nabla}^+(\psi \otimes \phi) \rangle\!\rangle = \langle\!\langle \mathcal{K}\Xi, (\nabla\psi) \otimes \phi \rangle\!\rangle \\ &= \int_{\mathbf{R}} \langle\!\langle \mathcal{K}\Xi(t), \nabla\psi(t) \otimes \phi \rangle\!\rangle \, dt = \int_{\mathbf{R}} \langle\!\langle \Xi(t)\phi, \nabla\psi(t) \rangle\!\rangle \, dt. \end{aligned}$$

The last integral is the Hitsuda–Skorohod integral, see (4.1). Thus,

$$\langle\!\langle \delta^+(\Xi)\phi, \psi \rangle\!\rangle = \int_{\mathbf{R}} \langle\!\langle \Xi(t)\phi, \nabla\psi(t) \rangle\!\rangle \, dt = \int_{\mathbf{R}} \langle\!\langle (\Xi\phi)(t), \nabla\psi(t) \rangle\!\rangle \, dt = \langle\!\langle \delta(\Xi\phi), \psi \rangle\!\rangle,$$

which proves (4.2). ∎

4.3 Annihilation Integral

The *annihilation integral* δ^- is by definition the adjoint map of the annihilation gradient ∇^-. In view of (3.15) we obtain

$$\begin{array}{ccc} L^2(\mathbb{R}, \mathcal{L}(\Gamma(H), (E))) & \longrightarrow & L^2(\mathbb{R}, \mathcal{L}(\Gamma(H), (E)^*)) \\ \delta^- \big\downarrow & & \big\downarrow \delta^- \\ \mathcal{L}(\mathbf{D}, (E)) & \longrightarrow & \mathcal{L}(\mathbf{D}, (E)^*). \end{array}$$

For Hilbert–Schmidt operators we have

$$\delta^- : L^2(\mathbb{R}, \mathcal{L}_2(\Gamma(H), \Gamma(H))) \to \mathcal{L}_2(\mathbf{D}, \Gamma(H)).$$

In a similar fashion as in Proposition 4.1 we have the following

Proposition 4.2 *For* $\Xi \in L^2(\mathbb{R}, \mathcal{L}(\Gamma(H), (E)^*))$ *it holds that*

$$\delta^-(\Xi)\phi = \int_{\mathbf{R}} \Xi(t)(\nabla\phi(t))\, dt, \qquad \phi \in \mathbf{D}. \tag{4.4}$$

Remark It is interesting to compare the results in Propositions 4.1 and 4.2 in the following forms:

$$\langle\!\langle \delta^+(\Xi)\phi, \psi \rangle\!\rangle = \int_{\mathbf{R}} \langle\!\langle \Xi(t)\phi, \nabla\psi(t) \rangle\!\rangle\, dt, \tag{4.5}$$

$$\langle\!\langle \delta^-(\Xi)\phi, \psi \rangle\!\rangle = \int_{\mathbf{R}} \langle\!\langle \Xi(t)(\nabla\phi(t)), \psi \rangle\!\rangle\, dt. \tag{4.6}$$

Then one can expect a direct relation between the creation and annihilation integrals, namely,

$$(\delta^-(\Xi))^* = \delta^+(\Xi^*).$$

In fact, the above relation is true for several classes of Ξ. However, for the proof the domains for the creation and annihilation integrals introduced in this note is not sufficient and we need to introduce their complementary domains. The full details will appear in the forthcoming paper Ji–Obata [6].

4.4 Conservation Integral

The *conservation integral* is defined to be the adjoint map of the conservation gradient. From (3.17) we obtain the following diagram:

$$\begin{array}{ccc} L^2(\mathbb{R}, \mathcal{L}((E), \mathbf{D})) & \longrightarrow & L^2(\mathbb{R}, \mathcal{L}((E), \Gamma(H))) \\ \delta^0 \big\downarrow & & \big\downarrow \delta^0 \\ \mathcal{L}((E), \Gamma(H)) & \longrightarrow & \mathcal{L}((E), \mathbf{D}^*). \end{array}$$

In a similar fashion as in Propositions 4.1 and 4.2 we have the following

Proposition 4.3 *For* $\Xi \in L^2(\mathbb{R}, \mathcal{L}((E), \Gamma(H)))$ *it holds that*

$$\delta^0(\Xi)\phi = \delta(\Xi\nabla\phi), \qquad \phi \in (E), \tag{4.7}$$

where $\Xi\nabla\phi \in L^2(\mathbb{R}, \Gamma(H))$ *is defined by* $\Xi\nabla\phi(t) = \Xi(t)(\nabla\phi(t))$.

Remark For comparison with (4.5) and (4.6) we record the following

$$\langle\langle \delta^0(\Xi)\phi, \psi \rangle\rangle = \int_{\mathbf{R}} \langle\langle \Xi(t)\nabla\phi(t), \nabla\psi(t) \rangle\rangle \, dt.$$

Remark The results in Propositions 4.1–4.3 clarify the relation to the works by Belavkin [1] and Lindsay [8]. In their approaches, using the classical stochastic integrals for suitably chosen $\Xi = \{\Xi(t)\}$ and ϕ, the quantum stochastic integrals are defined by the right-hand sides of (4.2), (4.4) and (4.7). Our quantum stochastic integrals are defined directly for $\Xi = \{\Xi(t)\}$. These two approaches yield the same quantum stochastic integrals for a common domain.

Remark In some literatures the Hitsuda–Skorohod integral is denoted by

$$\delta(\Psi) = \int_{\mathbf{R}} \Psi(t)\delta B_t,$$

where $\{B_t\}$ is the standard Brownian motion, see e.g., Nualart [11]. It would be then reasonable to write

$$\delta^+(\Xi) = \int_{\mathbf{R}} \Xi(t)\delta A_t^*, \qquad \delta^-(\Xi) = \int_{\mathbf{R}} \Xi(t)\delta A_t, \qquad \delta^0(\Xi) = \int_{\mathbf{R}} \Xi(t)\delta\Lambda_t^*,$$

where $\{A_t^*\}$, $\{A_t\}$ and $\{\Lambda_t\}$ are the creation, annihilation and conservation processes.

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