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Bundle Theorem for measure preserving homeomorphisms in 2-Manifolds

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In this report we discuss some bundle theorems for measure preserving homeomorphisms in 2-manifolds and investigate a homotopical relation between the group of measure preserving homeomorphisms and the group of ones with compact support on a noncompact n-manifold.

1. Groups of measure-preserving homeomorphisms

Suppose M is a connected n-manifold without boundary. By $\mathcal{H}(M)$ we denote the group of homeomorphisms of M equipped with the compact-open topology. Below we consider some subgroups of $\mathcal{H}(M)$. For any subgroup $\mathcal{G}$ of $\mathcal{H}(M)$ the notation $\mathcal{G}_0$ denotes the connected component of id_M in $\mathcal{G}$.

A Radon measure on M is a Borel measure μ on M such that $\mu(K) < \infty$ for any compact subset K of M. A Radon measure μ on M is said to be good if $\mu(p) = 0$ for any point $p \in M$ and $\mu(U) > 0$ for any nonempty open subset U of M. Let $\mathcal{B}(M)$ denote the σ-algebra of Borel sets in M and let $\mathcal{M}_g(M)_w$ denote the space of good Radon measure on M equipped with the weak topology.

For $\mu \in \mathcal{M}_g(M)_w$ and $h \in \mathcal{H}(M)$ the induced measure $h_*\mu \in \mathcal{M}_g(M)_w$ is defined by $(h_*\mu)(B) = \mu(h^{-1}(B))$ $(B \in \mathcal{B}(M))$. We say that

(i) h is μ-preserving if $h_*\mu = \mu$ (i.e., $\mu(h(B)) = \mu(B)$ $(B \in \mathcal{B}(M))$),

(ii) h is μ-regular if $h_*\mu$ and μ have same null sets

$$\text{(i.e., } \mu(h(B)) = 0 \ \text{ iff } \ \mu(B) = 0 \ (B \in \mathcal{B}(M))).$$

By $\mathcal{H}(M;\mu)$ and $\mathcal{H}(M;\mu\text{-reg})$ we denote the subgroups of $\mathcal{H}(M)$ consisting of μ-preserving homeomorphisms and μ-regular homeomorphisms of M resp. The group $\mathcal{H}(M)$ acts continuously on $\mathcal{M}_g(M)_w$ by $h \cdot \mu = h_*\mu$ and the subgroup $\mathcal{H}(M;\mu)$ coincides with the stabilizer of $\mu \in \mathcal{M}_g(M)_w$ under this action.

When M is compact, in [3] it is shown that

(1) $\mathcal{H}(M;\mu)$ is a SDR (strong deformation retract) of $\mathcal{H}(M,\mu\text{-reg})$,

(2) $\mathcal{H}(M,\mu\text{-reg})$ is WHD (weak homotopy dense) in $\mathcal{H}(M)$,

(3) the inclusion $\mathcal{H}(M;\mu) \subset \mathcal{H}(M)$ is a WHE (weak homotopy equivalence).

(4) If $n = 1, 2$, then $(\mathcal{H}(M), \mathcal{H}(M;\mu)$ are ℓ_2-manifolds)

(a) $\mathcal{H}(M, \mu\text{-reg})$ is HD (homotopy dense) in $\mathcal{H}(M)$,

(b) $\mathcal{H}(M;\mu)$ is a SDR of $\mathcal{H}(M)$.

Here, a subspace A of a space X is said to be homotopy dense in X if there exists a homotopy $h_t : X \to X$ ($t \in [0,1]$) such that $h_0 = id_M$ and $h_t(X) \subset A$ ($t \in (0,1]$).

When M is noncompact, we need to introduce some notions related to ends of M. We denote by $E = E_M$ the space of ends of M and by $\overline{M} = M \cup E_M$ the end compactification of M. An end $e \in E_M$ is said to be μ-finite if e has a neighborhood U in $\overline{M}$ with $\mu(U \cap M) < \infty$. The symbol E_M^μ denotes the set of μ-finite ends of M. It is seen that $\overline{M}$ is a compact connected metrizable space, M is a dense open subspace of $\overline{M}$, E_M is a 0-dim compact subspace of $\overline{M}$ and E_M^μ is an open subset of E_M.

Every $h \in \mathcal{H}(M)$ has a canonical extension $\overline{h} \in \mathcal{H}(\overline{M})$. If $h \in \mathcal{H}(M)_0$, then $\overline{h}|_{E_M} = id_{E_M}$. We say that

(iii) h is μ-end-regular if h is μ-regular and $\overline{h}(E_M^\mu) = E_M^\mu$.

Let $\mathcal{H}(M;\mu\text{-end-reg})$ denote the subgroup of $\mathcal{H}(M)$ consisting of μ-end-regular homeomorphisms of M. Note that $\mathcal{H}(M;\mu\text{-reg})_0 = \mathcal{H}(M;\mu\text{-end-reg})_0$.

Consider the following subset of $\mathcal{M}_g(M)$:

$$\mathcal{M}_g(M;\mu) = \{\nu \in \mathcal{M}_g(M) \mid \mu \text{ and } \nu \text{ have same total mass, null sets and finite ends}\}$$

This space is equipped with the finite-end weak topology ew. This is the weakest topology such that

$$\Phi_f : \mathcal{M}_g(M;\mu) \longrightarrow \mathbb{R} : \Phi_f(\nu) = \int_M f\, d\nu$$

is continuous for any continuous function $f : M \cup E_M^\mu \to \mathbb{R}$ with compact support.

The group $\mathcal{H}(M;\mu\text{-end-reg})$ acts continuously on $\mathcal{M}_g(M;\mu)_{ew}$ by $h \cdot \nu = h_*\nu$ and the subgroup $\mathcal{H}(M;\mu)$ coincides with the stabilizer of $\mu \in \mathcal{M}_g(M;\mu)_{ew}$ under this action.

In [2, 3] it is shown that

$(1)_1$ the orbit map $\pi : \mathcal{H}(M;\mu\text{-end-reg}) \longrightarrow \mathcal{M}_g(M;\mu)_{ew} : h \longmapsto h_*\mu$ has a continuous section $\sigma : \mathcal{M}_g(M;\mu)_{ew} \longrightarrow \mathcal{H}(M;\mu\text{-end-reg})_0$.

$(1)_2$ (i) $\mathcal{H}(M;\mu\text{-end-reg}) \cong \mathcal{H}(M;\mu) \times \mathcal{M}_g(M;\mu)_{ew}$

(ii) $\mathcal{H}(M;\mu)$ is a SDR of $\mathcal{H}(M;\mu\text{-end-reg})$.

In [9] we have shown that

(2) if $n = 2$, then $(\mathcal{H}(M)_0$ and $\mathcal{H}(M;\mu)_0$ are ℓ_2-manifolds)

(a) $\mathcal{H}(M, \mu\text{-reg})_0$ is HD in $\mathcal{H}(M)_0$, (b) $\mathcal{H}(M;\mu)_0$ is a SDR of $\mathcal{H}(M)_0$.

Let $\mathcal{H}^c(M;\mu)$ denote the subgroup of $\mathcal{H}(M;\mu)$ consisting of μ-preserving homeomorphisms of M with compact support. Let $\mathcal{H}^c(M;\mu)_1^* = \{h \in \mathcal{H}^c(M;\mu) \mid (*)\}$:

(*) $\exists$ a path (an isotopy) $h_t \in \mathcal{H}^c(M;\mu)$ $(t \in [0,1])$ from $h_0 = id_M$ to $h_1 = h$ with a common compact support

To investigate the relation between $\mathcal{H}(M;\mu)$ and $\mathcal{H}^c(M;\mu)$, we use a sort of mass flow homomorphism $J : \mathcal{H}_E(M;\mu) \to V_\mu(M)$.

The homomorphism J is defined as follows: Let $\mathcal{H}_E(M;\mu) = \{h \in \mathcal{H}(M;\mu) \mid \overline{h}|_{E_M} = id_{E_M}\}$ and $\mathcal{B}_c(M) = \{C \in \mathcal{B}(M) \mid \mathrm{Fr}\, C : \text{compact}\}$. For each $h \in \mathcal{H}_E(M;\mu)$ we can define a function J_h by

$$J_h : \mathcal{B}_c(M) \longrightarrow \mathbb{R} \ : \quad J_h(C) = \mu(C - h(C)) - \mu(h(C) - C) \quad (C \in \mathcal{B}_c(M)).$$

The quantity $J_h(C)$ measures the total amount of mass transfered into C by h. The function J_h belongs to the topological vector space $V_\mu(M)$ defined as follows :

$$V_\mu(M) = \{\, a : \mathcal{B}_c(M) \to \mathbb{R} \mid (*)_1, (*)_2, (*)_3, (*)_\mu \,\}$$

$(*)_1$ If $C, D \in \mathcal{B}_c(M)$ and $cl(C - D)$, $cl(D - C)$ are compact, then $a(C) = a(D)$.

$(*)_2$ If $C, D \in \mathcal{B}_c(M)$ and $C \cap D = \emptyset$, then $a(C \cup D) = a(C) + a(D)$.

$(*)_3$ $a(M) = 0$.

$(*)_\mu$ If $C \in \mathcal{B}_c(M)$ and $\mu(C) < \infty$, then $a(C) = 0$.

$V_\mu(M)$ is equipped with the product topology.

The space $V_\mu(M)$ is canonically isomorphic to the space of charges on E_M ([1]). The mass flow homomorphism $J : \mathcal{H}_E(M,\mu) \longrightarrow V_\mu(M) : h \longmapsto J_h$ is a continuous group homomorphism and $\mathcal{H}^c(M;\mu) \subset \mathrm{Ker}\, J$. In [10] we have shown that

$(3)_1$ J has a continuous (non homomorphic) section $\sigma : V_\mu(M) \longrightarrow \mathcal{H}(M,\mu)_0$,

$(3)_2$ (i) $\mathcal{H}_E(M;\mu) \cong \mathrm{Ker}\, J^\mu \times V_\mu(M)$,

(ii) $\mathrm{Ker}\, J$ is a SDR of $\mathcal{H}_E(M;\mu)$.

Let $J_0 : \mathcal{H}(M,\mu)_0 \longrightarrow V_\mu(M)$ denote the restriction of J into $\mathcal{H}(M,\mu)_0$. Since $\mathrm{Im}\,\sigma \subset \mathcal{H}(M,\mu)_0$, the homomorphism J_0 also has the similar properties.

In summary, in $n = 2$ we have obtained the following sequence of groups :

$$n = 2 : \ \mathcal{H}(M)_0 \ \underset{\text{HD}}{\supset} \ \mathcal{H}(M,\mu\text{-reg})_0 \ \underset{\text{SDR}}{\supset} \ \mathcal{H}(M;\mu)_0 \ \underset{\text{SDR}}{\supset} \ \mathrm{Ker}\, J_0 \ \supset \ \mathcal{H}^c(M;\mu)_1^*.$$

It remains to study the relation between the groups $\mathrm{Ker}\, J_0$ and $\mathcal{H}^c(M;\mu)_1^*$. In [7, 8] we have shown that $\mathcal{H}(M)_0$ is an ANR and $\mathcal{H}^c(M)_1^*$ is HD in $\mathcal{H}(M)_0$ for any noncompact connected 2-manifold M. One of main tools in our argument in [7, 8] is a bundle theorem for $\mathcal{H}(M)$ obtained in [6]. To apply the same argument to the groups $\mathrm{Ker}\, J_0$ and $\mathcal{H}^c(M;\mu)_1^*$, we need a bundle theorem for $\mathrm{Ker}\, J_0$. In the next section we discuss this problem.

2. Bundle Theorem for measure preserving homeomorphisms in 2-Manifolds

We begin with a general frame work. Suppose a topological group G acts continuously on a space X. For any point $x_0 \in X$ we have the orbit $Gx_0 \subset X$, the stabilizer G_{x_0} of x_0 and the orbit map $\pi : G \longrightarrow Gx_0 : g \longmapsto gx_0$.

We are concerned with the problems :

$(\#)_1$ Determine whether the orbit map $\pi : G \longrightarrow Gx_0$ is a principal G_{x_0}-bundle or not.

$(\#)_2$ Identify the orbit Gx_0 as a subspace of X (without using the G-action if possible).

It is seen that π is a principal bundle iff π admits a local section around x_0 (i.e., there exists a neighborhood U of x_0 in Gx_0 and a map $s : U \to G$ with $\pi s = \mathrm{inc}_U$). If $\mathrm{Im}\, s$ is contained in a normal subgroup H of G, then we can expect that $Gx_0 = Hx_0$. This situation is described by the next diagram:

$$\begin{array}{ccc} H & \lhd & G \\ s\uparrow & & \downarrow\pi \\ U & \subset & Gx_0 \end{array}$$

This general description can be applied to our situation as follows. Suppose M is a connected n-manifold without boundary and X is a compact subpolyhedron of M. Let $\mathcal{E}(X, M)$ denote the space of embeddings of X into M equipped with the compact open topology and let $\mathcal{E}(X, M)_0$ denote the connected component of the inclusion $i_X : X \to M$ in $\mathcal{E}(X, M)$. The group $\mathcal{H}(M)$ acts continuously on $\mathcal{E}(X, M)$ by the left composition and the orbit map for the inclusion i_X is exactly the restriction map

$$\pi : \mathcal{H}(M) \longrightarrow \mathcal{H}(M)i_X \subset \mathcal{E}(X, M) : h \longmapsto h|_X.$$

More generally, for any subgroup $\mathcal{G}$ of $\mathcal{H}(M)$ we obtain the orbit map $\pi : \mathcal{G} \longrightarrow \mathcal{G}i_X$: $h \longmapsto h|_X$. Bundle theorem for $\mathcal{G}$ is the assertion that the orbit map $\pi : \mathcal{G} \longrightarrow \mathcal{G}i_X$ is a principal $\mathcal{G}_{i_X}$-bundle. This assertion is equivalent to the existence of a local section φ around i_X :

$$\begin{array}{cccc} & \mathcal{H} & \subset & \mathcal{G} \\ & \varphi\uparrow & & \downarrow\pi \\ i_X \in & \mathcal{U} & \subset & \mathcal{G}i_X \end{array} \qquad \varphi(f)|_X = f \quad (f \in \mathcal{U}).$$

The map $\varphi : \mathcal{U} \longrightarrow \mathcal{H}$ assigns to each embedding $f \in \mathcal{G}i_X$ close to i_X its extension to a homeomorphism $\varphi(f) \in \mathcal{H}$.

In the smooth case it is well known that if X is a closed codim 1 submanifold of a smooth manifold M, then the restriction map $\pi : \mathrm{Diff}\, M \longrightarrow \mathrm{Emb}(X, M)$ is a principal bundle over its image (R. Palais (1960)).

In the C^0-case, the corresponding result is still unknown for $n \geq 3$. Below we restrict ourselves to the cases in $n = 2$.

Suppose M is a connected 2-manifold without boundary and X is a compact subpolyhedron of M. For a subset A of M let $\mathcal{H}_A(M) = \{h \in \mathcal{H}(M) \mid h|_A = id_A\}$.

In [6] we have shown that

Theorem 2.1.

(1) The restriction map $\pi : \mathcal{H}(M) \longrightarrow \mathcal{E}(X, M)$ admits a local section $\varphi : \mathcal{U} \to \mathcal{H}^c(M)_1^*$ on a neighborhood $\mathcal{U}$ of i_X in $\mathcal{E}(X, M)$.

(2) The restriction maps $\pi : \mathcal{H}(M) \longrightarrow \mathrm{Im}\,\pi$ and $\pi : \mathcal{H}(M)_0 \longrightarrow \mathcal{E}(X, M)_0$ are principal bundles.

The proof of the assertion (1) is based on the conformal mapping theorem in the complex function theory.

Now we return to the study of groups of measure preserving homeomorphisms. Suppose μ is a good Radon measure on M. Below we discuss bundle theorems for the groups

$$\mathcal{H}(M, \mu\text{-reg}) \supset \mathcal{H}(M; \mu) \supset \mathrm{Ker}\, J^\mu \supset \mathcal{H}^c(M; \mu).$$

2.1. **Bundle Theorem for $\mathcal{H}(M, \mu$-reg).**

Let $\mathcal{E}(X, M; \mu\text{-reg})$ denote the subspace of $\mathcal{E}(X, M)$ consisting of μ-regular embeddings. The group $\mathcal{H}(M; \mu\text{-reg})$ acts on $\mathcal{E}(X, M; \mu\text{-reg})$ by the left composition. Since conformal maps are regular with respect to the Lebesgue measure on the complex plane, a slight modification of the argument used in Theorem 2.1 yields the following conclusion.

Theorem 2.2. For any $f \in \mathcal{E}(X, M)$ and any neighborhood U of $f(X)$ in M there exists a neighborhood $\mathcal{U}$ of f in $\mathcal{E}(X, M)$ and a map $\varphi : \mathcal{U} \longrightarrow \mathcal{H}_{M-U}(M)_0$ such that

(1) $\varphi(g)f = g \quad (g \in \mathcal{U}), \qquad \varphi(f) = id_M$

(2) $\varphi(g) : M - f(X) \cong M - g(X)$ is μ-regular $(g \in \mathcal{U})$

$$\begin{array}{ccccc} \mathcal{H}_{M-U}(M)_0 & \subset & \mathcal{H}(M) & & \\ {}^{\exists}\varphi \uparrow & & \downarrow \pi_f & & \pi_f(h) = hf \\ f \in {}^{\exists}\mathcal{U} & \subset & \mathcal{E}(X, M) & & \end{array}$$

Restriction of the map φ to $\mathcal{E}(X, M; \mu\text{-reg})$ implies the next result.

Theorem 2.2′. For any $f \in \mathcal{E}(X, M; \mu\text{-reg})$ and any neighborhood U of $f(X)$ in M there exists a neighborhood $\mathcal{V}$ of f in $\mathcal{E}(X, M; \mu\text{-reg})$ and a map $\varphi : \mathcal{V} \longrightarrow \mathcal{H}_{M-U}(M; \mu\text{-reg})_0$ such that $\varphi(g)f = g \ \ (g \in \mathcal{V})$ and $\varphi(f) = id_M$.

$$\begin{array}{ccc} \mathcal{H}_{M-U}(M;\mu\text{-reg})_0 & \subset & \mathcal{H}(M;\mu\text{-reg}) \\ {}^{\exists}\varphi \big\uparrow & & \big\downarrow \pi_f \\ f \in {}^{\exists}\mathcal{V} & \subset & \mathcal{E}(X, M;\mu\text{-reg}) \end{array}$$

Corollary 2.1.

(1) The restriction map $\pi : \mathcal{H}(M;\mu\text{-reg}) \longrightarrow \mathcal{H}(M;\mu\text{-reg})i_X$ is a principal $\mathcal{H}_X(M;\mu\text{-reg})$-bundle.

(2) The restriction map $\pi : \mathcal{H}(M;\mu\text{-reg})_0 \longrightarrow \mathcal{E}(X, M;\mu\text{-reg})_0$ is a principal $\mathcal{G}$-bundle for $\mathcal{G} = \mathcal{H}(M;\mu\text{-reg})_0 \cap \mathcal{H}_X(M)$.

2.2. **Bundle Theorem for $\mathcal{H}(M,\mu)$.**

Let $\mathcal{E}(X, M;\mu)$ denote the subspace of $\mathcal{E}(X, M)$ consisting of μ-preserving embeddings. The group $\mathcal{H}(M;\mu)$ acts on $\mathcal{E}(X, M;\mu)$ by the left composition.

Theorem 2.3. The restriction map $\pi : \mathcal{H}(M;\mu) \longrightarrow \mathcal{H}(M;\mu)i_X$ admits a local section $\varphi : \mathcal{V} \longrightarrow \mathcal{H}(M;\mu)_0$ on a neighborhood $\mathcal{V}$ of i_X in $\mathcal{H}(M;\mu)i_X$.

$$\begin{array}{ccc} \mathcal{H}(M;\mu)_0 & \subset & \mathcal{H}(M;\mu) \\ {}^{\exists}\varphi \big\uparrow & & \big\downarrow \pi \\ i_X \in {}^{\exists}\mathcal{V} & \subset & \mathcal{H}(M;\mu)i_X \end{array}$$

Corollary 2.2. The restriction map $\pi : \mathcal{H}(M;\mu) \longrightarrow \mathcal{H}(M;\mu)i_X$ is a principal $\mathcal{H}_X(M;\mu)$-bundle.

2.3. **Bundle Theorem for** $\operatorname{Ker} J \supset \mathcal{H}^c(M,\mu)$.

The condition "$f \in (\operatorname{Ker} J)\, i_X$" means the vanishing of obstruction for extension of f to a μ-preserving homeomorphism with compact support.

Theorem 2.4. For any $f \in (\operatorname{Ker} J)\, i_X$ and any neighborhood U of $f(X)$ in M (which satisfies some additional minor condition) there exists a neighborhood $\mathcal{V}$ of f in $(\operatorname{Ker} J)\, i_X$ and a map $\varphi : \mathcal{V} \longrightarrow \mathcal{H}^c_{M-U}(M;\mu)_0$ such that $\varphi(g)f = g \ \ (g \in \mathcal{V})$ and $\varphi(f) = id_M$.

$$\begin{array}{ccc} \mathcal{H}^c_{M-U}(M;\mu)_0 & \subset & \operatorname{Ker} J^{\mu} \\ {}^{\exists}\varphi \big\uparrow & & \big\downarrow \pi_f \\ f \in {}^{\exists}\mathcal{V} & \subset & \left(\operatorname{Ker} J^{\mu}\right) i_X \end{array}$$

Corollary 2.3. For any subgroup $\mathcal{G}$ with $\operatorname{Ker} J \supset \mathcal{G} \supset \mathcal{H}^c(M;\mu)^*_1$ the restriction map $\pi : \mathcal{G} \longrightarrow \mathcal{G}i_X$ is a principal $\mathcal{G}_{i_X}$-bundle and $\mathcal{G}i_X = \left(\mathcal{H}^c(M;\mu)^*_1\right)i_X$.

As an application of this corollary we obtain the following conclusion.

Theorem 2.5. $\mathcal{H}^c(M;\mu)_1^*$ is HD in Ker J_0.

REFERENCES

[1] S. R. Alpern and V. S. Prasad, Typical dynamics of volume-preserving homeomorphisms, Cambridge Tracts in Mathematics, Cambridge University Press, (2001).

[2] R. Berlanga, Groups of measure-preserving homeomorphisms as deformation retracts, *J. London Math. Soc. (2)* 68 (2003) 241 - 254.

[3] A. Fathi, Structures of the group of homeomorphisms preserving a good measure on a compact manifold, *Ann. scient. Éc. Norm. Sup. (4)* 13 (1980) 45 - 93.

[4] R. Luke and W. K. Mason, The space of homeomorphisms on a compact two - manifold is an absolute neighborhood retract, *Trans. Amer. Math. Soc.,* 164 (1972), 275 - 285.

[5] J. Oxtoby and S. Ulam, Measure preserving homeomorphisms and metrical transitivity, *Ann. of Math.,* 42 (1941) 874 - 920.

[6] T. Yagasaki, Spaces of embeddings of compact polyhedra into 2-manifolds, *Topology Appl.,* 108 (2000) 107 - 122.

[7] T. Yagasaki, Homotopy types of homeomorphism groups of noncompact 2-manifolds, *Topology Appl.,* 108 (2000) 123 - 136.

[8] T. Yagasaki, The groups of PL and Lipschitz homeomorphisms of noncompact 2-manifolds, *Bulletin of the Polish Academy of Sciences, Mathematics,* 51(4) (2003), 445 - 466.

[9] T. Yagasaki, Groups of measure-preserving homeomorphisms of noncompact 2-manifolds, *Topology Appl.,*154 (2007) 1521 - 1531.

[10] T. Yagasaki, Measure-preserving homeomorphisms of noncompact manifolds and mass flow toward ends, arXiv math.GT/0512231.

[11] T. Yagasaki, Bundle theorem for measure-preserving homeomorphisms in 2-manifolds, manuscript.

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