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The homotopy approximation of spaces of algebraic maps between algebraic varieties

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1 Introduction.

Let $\mathrm{Map}(X,Y)$ denote the space consisting of all continuous maps $f: X\to Y$. For complex manifolds X and Y, we denote by $\mathrm{Hol}(X,Y)$ the subspace of $\mathrm{Map}(X,Y)$ consisting of all holomorphic maps $f:X\to Y$. It is natural to ask if the two spaces $\mathrm{Hol}(X,Y)$ and $\mathrm{Map}(X,Y)$ are in some topological sense (e.g. homotopy or homology) equivalent. Early examples of this type can be found in the work of Gromov [7] and that of Atiyah-Jones [1]. In many cases of interest, the infinite dimensional space $\mathrm{Hol}(X,Y)$ has a filtration by finite dimensional subspaces, given by some kind of "map degree", and the topology of these finite dimensional spaces approximates the topology of the entire space $\mathrm{Map}(X,Y)$; the approximation becomes more accurate as the degree increases.

Now consider the case $Y=\mathbb{C}\mathrm{P}^n$ and $H_2(X,\mathbb{Z})=\mathbb{Z}$. In this case, the degree of the map $f:X\to\mathbb{C}\mathrm{P}^n$ is d if the induced homomorphism f_* on $H_2(\ ,\mathbb{Z})$ is multiplication by d, and we denote by $\mathrm{Hol}_d^*(X,\mathbb{C}\mathrm{P}^n)$ (resp. $\mathrm{Hol}_d(X,\mathbb{C}\mathrm{P}^n)$) the space consisting of all based holomorphic (resp. holomorphic) maps $f:X\to\mathbb{C}\mathrm{P}^n$ of the degree d. We also denote by $\mathrm{Map}_d^*(X,\mathbb{C}\mathrm{P}^n)$ (resp. $\mathrm{Map}_d(X,\mathbb{C}\mathrm{P}^n)$) the corresponding space of based continuous maps (resp. continuous maps) of degree d.

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As the first such explicit result of this kind, Segal shows the following basic and interesting result.

Theorem 1.1 (G. Segal, [20]). *If M_g is a compact closed Riemann surface of genus g, the inclusions*

$$\begin{cases} i_{d,\mathbb{C}} : \mathrm{Hol}_d^*(M_g, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d^*(M_g, \mathbb{C}\mathrm{P}^n) \\ j_{d,\mathbb{C}} : \mathrm{Hol}_d(M_g, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d(M_g, \mathbb{C}\mathrm{P}^n) \end{cases}$$

are homology equivalences through dimension $(2n-1)(d-2g)-1$ if $g \geq 1$ and homotopy equivalences through dimension $(2n-1)d-1$ if $g = 0$.

Remark. A map $f : X \to Y$ is called *a homology* (*resp. homotopy*) *equivalence through dimension N* if the induced homomorphism

$$f_* : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z}) \quad (\text{resp. } f_* : \pi_k(X) \to \pi_k(Y))$$

is an isomorphism for all $k \leq N$.

Segal conjectured in [20] that this result should generalize to a much larger class of target spaces, such as complex Grassmannians and flag manifolds, and even possibly to higher dimensional source spaces. For example, Boyer, Hurtubise and Milgram [2], and Cohen, Jones and Segal [5] attempted to find the most general target spaces for which the stability theorem holds (c.f. [3], [8], [9], [10], [11], [17]).

There have been however, very few attempts to investigate (as suggested by Segal) the phenomenon of topological stability for source spaces of complex dimension greater than 1. Havlicek [12] considers the space of holomorphic maps from $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$ to complex Grassmanians and Kozlowski and Yamaguchi [14] studies the case of linear maps $\mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$. Recently Mostovoy [19] proved the complete analogue of Segal's theorem for the space of holomorphic maps from $\mathbb{C}\mathrm{P}^m$ to $\mathbb{C}\mathrm{P}^n$.

Theorem 1.2 (J. Mostovoy, [19]). *If $2 \leq m \leq n$ and $d \geq 1$ are integers, the inclusions*

$$\begin{cases} j_{d,\mathbb{C}} : \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \\ i_{d,\mathbb{C}} : \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \end{cases}$$

are homotopy equivalences through dimension $D_{\mathbb{C}}(d;m,n)$ *if* $m < n$, *and homology equivalences through dimension* $D_{\mathbb{C}}(d;m,n)$ *if* $m = n$, *where* $\lfloor x \rfloor$ *denotes the integer part of a real number* x *and the number* $D_{\mathbb{C}}(d;m,n)$ *is given by* $D_{\mathbb{C}}(d;m,n) = (2n-2m+1)\Big(\lfloor \frac{d+1}{2} \rfloor + 1\Big) - 1$.

The main purpose of this note is to study the real analogue of the above two Theorems and we shall explain this in the proceeding section.

2 The real analogues.

In this note we shall consider *real* analogues of the above result. Before we describe our results we need a real analogue of the notion of a holomorphic map.

For $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, let $V \subset \mathbb{K}^n$ be an algebraic subset and U be a (Zariski) open subset of V. A function $f : U \to \mathbb{K}$ is called *a regular function* if it can be written as the quotient of two polynomials $f = g/h$, with $h^{-1}(0) \cap U = \emptyset$. For a subset $W \subset \mathbb{K}^p$, a map $\varphi : U \to W$ is called *a regular map* if its coordinate functions are regular functions. From now on we shall treat the words "regular map" and "algebraic map"as synonyms.

A pre-algebraic vector bundle over a real algebraic variety X means a triple $\xi = (E,p,X)$, such that E is a real algebraic variety, $p : E \to X$ is a regular map, the fiber over each point is a $\mathbb{K}$-vector space and there is a covering of the base X by Zariski open sets over which the the vector bundle E is biregularly isomorphic to the trivial bundle. *An algebraic vector bundle* over X is a pre-algebraic vector bundle which is algebraically isomorphic to a pre-algebraic vector sub-bundle of a trivial bundle.

It is natural to consider analogues of all the above theorems for real algebraic varieties, with holomorphic maps replaced by regular maps. This is indeed what was done independently by Mostovoy [18] and Guest, Kozlowski and Yamaguchi [11], [14] for the case of maps $\mathbb{RP}^1 \to \mathbb{RP}^n$.

For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $1 \leq k < n$, let $\mathbb{G}_{n,k}(\mathbb{K})$ denote the Grassmann manifold consisting of all k dimensional $\mathbb{K}$-linear subspaces in $\mathbb{K}^n$. For a compact affine real algebraic variety X, let $\mathrm{Alg}(X, \mathbb{G}_{n,k}(\mathbb{K}))$ denote the space consisting of all regular maps $f : X \to \mathbb{G}_{n,k}(\mathbb{K})$. Then the first main result of this note is a real analogue of some of results due to Gravesen [6] (c.f. [13]) and it is as follows.

Theorem 2.1 (A. Kozlowski and K. Yamaguchi, [16]). *Let X be a compact affine real algebraic variety, with the property that every topological $\mathbb{K}$-vector bundle of rank k over X is topologically isomorphic to an algebraic $\mathbb{K}$-vector bundle. Then the inclusion $i : \mathrm{Alg}(X, \mathbb{G}_{n,k}(\mathbb{K})) \to \mathrm{Map}(X, \mathbb{G}_{n,k}(\mathbb{K}))$ is a weak homotopy equivalence.*

Remark. Note that the spaces $\mathbb{R}\mathrm{P}^m$ and $\mathbb{C}\mathrm{P}^m$ satisfy the assumption of Theorem 2.1 (where we can take $\mathbb{K}$ to be $\mathbb{R}$ or $\mathbb{C}$ in either case). It is known that, for every compact smooth manifold M, there exists a non-singular real algebraic variety X diffeomorphic to M such that every topological vector bundle over X is isomorphic to a real algebraic one [4].

3 The space $\mathrm{Alg}_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$.

From now on, we shall consider only regular (i.e algebraic) and continuous maps between real projective spaces. In other words, we consider the case $(\mathbb{K}, k) = (\mathbb{R}, 1)$ in Theorem 2.1.

For $1 \leq m < n$ and $\epsilon \in \mathbb{Z}/2 = \pi_0(\mathrm{Map}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n))$, we denote by $\mathrm{Map}_\epsilon(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ the corresponding path component of $\mathrm{Map}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$. We also denote by $\mathrm{Map}^*_\epsilon(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ the subspace of $\mathrm{Map}_\epsilon(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ consisting of all based maps $f : \mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^n$ such that $f(\mathbf{e}_m) = \mathbf{e}_n$, where we take $\mathbf{e}_k = [1 : 0 : \cdots : 0] \in \mathbb{R}\mathrm{P}^k$ as a base point of $\mathbb{R}\mathrm{P}^k$ $(k = m, n)$.

From now on, let $\{z_0, \cdots, z_m\}$ denote the fixed variables. Then a regular map $f : \mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^n$ can always be represented as $f = [f_0 : f_1 : \cdots : f_n]$, such that $f_0, \cdots, f_n \in \mathbb{R}[z_0, z_1, \cdots, z_m]$ are homogenous polynomials of the same degree d with no common *real* root except $\mathbf{0}_{m+1} = (0, \cdots, 0) \in \mathbb{R}^{m+1}$ (but they may have common *complex* roots).

We shall refer to a regular map represented in this way as *an algebraic map of degree d*. We denote by $\mathrm{Alg}_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \subset \mathrm{Map}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ the subspace of consisting of all algebraic maps $f : \mathbb{R}\mathrm{P}^m \to \mathbb{K}\mathrm{P}^n$ of degree d and we also denote by $\mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^m, \mathbb{K}\mathrm{P}^n)$ the corresponding subspace of based maps given by $\mathrm{Alg}_d^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) = \mathrm{Alg}_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \cap \mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$.

For $m \geq 2$ and $g \in \mathrm{Map}_\epsilon^*(\mathbb{R}\mathrm{P}^{m-1}, \mathbb{R}\mathrm{P}^n)$, we denote by $A_d(m,n;g) \subset \mathrm{Alg}_d^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ and $F(m,n;g) \subset \mathrm{Map}_\epsilon^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ the subspaces defined by

$$\begin{cases} A_d(m,n;g) & = \{f \in \mathrm{Alg}_d^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) : f|\mathbb{R}\mathrm{P}^{m-1} = g\}, \\ F(m,n;g) & = \{f \in \mathrm{Map}_\epsilon^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) : f|\mathbb{R}\mathrm{P}^{m-1} = g\}. \end{cases}$$

It is easy to see that there is a homotopy equivalence $F(m,n;g) \simeq \Omega^m \mathbb{R}\mathrm{P}^n$ and that there are inclusions

$$\begin{cases} \mathrm{Alg}_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \subset \mathrm{Map}_{[d]_2}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n), \\ \mathrm{Alg}_d^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \subset \mathrm{Map}_{[d]_2}^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \\ A_d(m,n;g) \subset F(m,n;g) \subset \mathrm{Map}_{[d]_2}^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n), \end{cases}$$

where $[d]_2 \in \mathbb{Z}/2$ denotes the integer d mod 2. From now on, we consider the spaces of algebraic maps $\mathrm{Alg}_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ for $1 \leq m < n$. However, the case $m = 1$ is already well studied by Kozlowski-Yamaguchi and Mostovoy ([14], [18], [22]), and we mainly consider the case $2 \leq m < n$.

First, consider the space $\mathrm{Alg}_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ for the case $d = 1$. Then the second main result of this note is stated as follows.

Theorem 3.1 (The case $\epsilon = 1$, c.f. [23]). *If $m < n$, the inclusion maps*

$$\begin{cases} i_1 : \mathrm{Alg}_1^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \to \mathrm{Map}_1^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \\ j_1 : \mathrm{Alg}_1(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \to \mathrm{Map}_1(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \end{cases}$$

are homotopy equivalences through dimension $2(n-m)-2$.

Corollary 3.2 ([23]). (i) *If $m < n$, there are isomorphisms*

$$\pi_1(\mathrm{Map}_1^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)) \cong \begin{cases} \mathbb{Z} & \text{if } (m,n) = (1,2), \\ 0 & \text{if } 1 \leq m \leq n-2, \\ \mathbb{Z}/2 & \text{if } m = n-1 \geq 2. \end{cases}$$

$$\pi_1(\mathrm{Map}_1(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)) \cong \begin{cases} \mathbb{Z}/2 & \text{if } 1 \leq m \leq n-2, \\ (\mathbb{Z}/2)^2 & \text{if } m = n-1 \text{ and } n \equiv 0,3 \pmod 4, \\ \mathbb{Z}/4 & \text{if } m = n-1 \text{ and } n \equiv 1,2 \pmod 4. \end{cases}$$

(ii) *If $m = n$, there are isomorphisms*

$$\pi_1(\mathrm{Map}_1^*(\mathbb{RP}^n,\mathbb{RP}^n)) \cong \begin{cases} 0 & \textit{if } n=1, \\ \mathbb{Z} & \textit{if } n=2, \\ (\mathbb{Z}/2)^2 & \textit{if } n\geq 3. \end{cases}$$

$$\pi_1(\mathrm{Map}_1(\mathbb{RP}^n,\mathbb{RP}^n)) \cong \begin{cases} \mathbb{Z} & \textit{if } n=1, \\ (\mathbb{Z}/2)^2 & \textit{if } n=2, \\ (\mathbb{Z}/2)^3 & \textit{if } n\geq 3 \textit{ and } n\equiv 0,3 \pmod 4, \\ \mathbb{Z}/4\oplus\mathbb{Z}/2 & \textit{if } n\geq 5 \textit{ and } n\equiv 1,2 \pmod 4). \end{cases}$$

The sketch proof of Theorem 3.1.

Since the proof is analogous, we only consider the based case. The basic idea of the proof is to use the orthogonal group action on $\mathbb{RP}^n$.

Consider the natural map $\alpha_{m,n}: O(n)\to \mathrm{Map}_1^*(\mathbb{RP}^m,\mathbb{RP}^n)$ defined by the matrix multiplication

$$\alpha_{m,n}(A)([x_0:\cdots:x_m]) = [x_0:\cdots:x_m:0:\cdots:0]\cdot \begin{pmatrix} 1 & \mathbf{0}_n \\ {}^t\mathbf{0}_n & A \end{pmatrix}$$

for $([x_0:\cdots:x_m],A)\in \mathbb{RP}^m\times O(n)$, where $\mathbf{0}_n=(0,\cdots,0)\in\mathbb{R}^n$. Since the subgroup $\{E_m\}\times O(n-m)$ is fixed under this map, it induces the map $\overline{\alpha}_{m,n}: V_{n,m}=O(n)/O(n-m)\to \mathrm{Map}_1^*(\mathbb{RP}^m,\mathbb{RP}^n)$ in a natural way, where $V_{n,m}$ denotes the real Stiefel manifold of orthogonal m-frames in $\mathbb{R}^n$ given by $V_{n,m}=O(n)/O(n-m)$. It follows from [[23], Theorem 1.2] that $\overline{\alpha}_{m,n}$ is a homotopy equivalence through dimension $2(n-m)-2$.

However, an easy computation shows that there is a homotopy equivalence $\mathrm{Alg}_1^*(\mathbb{RP}^m,\mathbb{RP}^n)\simeq V_{n,m}$ and we can easily see that the inclusion map i_1 is homotopic to the map $\overline{\alpha}_{m,n}$ (up to homotopy equivalences). This completes the proof. □

Next, consider the space $\mathrm{Alg}_d(\mathbb{RP}^m,\mathbb{RP}^n)$ for $d\geq 2$. For this purpose, we recall several notations.

Let $F(X,r)$ denote the space of ordered configuration space of distinct r points in X defined by $F(X,r)=\{(x_1,\cdots,x_r)\in X^r : x_k\neq x_j \text{ if } j\neq k\}$. The r-th symmetric group S_r acts on $F(X,r)$ in a usual manner and we denote by $C_r(X)$ the unordered configuration space of distinct r-points in X defined by $C_r(X)=F(X,r)/S_r$. Let $\pm\mathbb{Z}$ denote the local system of $F(X,r)$ such that it is locally isomorphic to $\mathbb{Z}$ and changing the sign after odd permutation of the points $x_1, \cdots, x_l \in X$. We use the same notation $\pm\mathbb{Z}$ as the local system on $C_r(X)$ given by its direct image as in [21]. Then the final our main result of this note is as follows.

Theorem 3.3 (A. Kozlowski and K. Yamaguchi, [16]). *Let $2 \leq m \leq n-1$, $g \in \mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^{m-1}, \mathbb{R}\mathrm{P}^n)$ be a fixed algebraic map and let $M(m,n) = 2\lceil \frac{m+1}{n-m} \rceil + 1$, where $\lceil x \rceil = \min\{N \in \mathbb{Z} : N \geq x\}$.*

(i) *If $d \geq M(m,n)$, the inclusion $i'_d : A_d(m,n;g) \to F(m,n;g) \simeq \Omega^m S^n$ is a homotopy equivalence through dimension $D(d;m,n)$ if $m+2 \leq n$, and a homology equivalence through dimension $D(d : m,n)$ if $m+1 = n$, where $D(d;m,n)$ denotes the numbers given by*

$$D(d;m,n) = (n-m)\Big(\lfloor \frac{d+1}{2} \rfloor + 1\Big) - 1.$$

(ii) *If $k \geq 1$, $H_k(A_d(m,n;g), \mathbb{Z})$ contains the subgroup*

$$\bigoplus_{r=1}^{\lfloor \frac{d+1}{2} \rfloor} H_{k-(n-m)r}(C_r(\mathbb{R}^m), (\pm\mathbb{Z})^{\otimes(n-m)}).$$

as a direct summand.

Corollary 3.4 (A. Kozlowski and K. Yamaguchi, [16]). *If $2 \leq m \leq n-1$ and $d \geq M(m,n)$, there is an isomorphism*

$$H_k(A_d(m,n;g), \mathbb{Z}) \cong \bigoplus_{r=1}^{\lfloor \frac{d+1}{2} \rfloor} H_{k-r(n-m)}(C_r(\mathbb{R}^m), (\pm\mathbb{Z})^{\otimes(n-m)})$$

for any integer $1 \leq k \leq D(d;m,n)$.

Theorem 3.5 (A. Kozlowski and K. Yamaguchi, [16]). *If $2 \leq m \leq n-1$, $d = 2d^* \equiv 0 \pmod 2$ and $d^* \geq M(m,n)$, the inclusion maps*

$$\begin{cases} i_d : \mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \to \mathrm{Map}^*_{[d]_2}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \\ j_d : \mathrm{Alg}_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \to \mathrm{Map}_{[d]_2}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \end{cases}$$

are homotopy equivalences through dimension $D(d^*;m,n)$ if $m+2 = n$ and homology equivalences through dimension $D(d^* : m,n)$ if $m+1 = n$, where $D(d^*;m,n) = (n-m)\big(\lfloor \frac{d^*+1}{2} \rfloor + 1\big) - 1$.

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