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Non-existence of free S^1-actions on Kervaire spheres II

Yasuhiko KITADA (北田泰彦)

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ABSTRACT

In this note we shall prove that the Kervaire spheres Σ_K^{4k+1} where $k+1$ is not a power of 2, do not admit any free S^1-actions if k is not divisible by 8. This improves the result obtained at the workshop at RIMS in 2006.

It has been known that differentiable structures affect the existence of group actions on manifolds. In 1971, Brumfiel calculated surgery obstructions of complex projective spaces and obtained results on the existence problem of free S^1-actions on homotopy spheres upto dimension 13 ([1]). Among the homotopy spheres of dimension $2n-1$, those that bound parallelizable manifolds of dimension $2n$ form a subgroup bP_{2n} of all the homotopy spheres Θ_{2n-1}. The homotopy spheres in bP_{2n} have been widely studied by making use of explicit construction either (a) by plumbing of tangent disk bundles of S^n or (b) as an intersection of a Brieskorn variety in $\mathbb{C}^{n+1}$ and the unit sphere centered at the singularity ([3]). Because of this, the elements of bP_{2n} are regarded less exotic than other homotopy spheres. For example, these spheres all admit free actions of finite cyclic groups of any order and as we shall see below the degree of symmetry (the maximal dimension of compact Lie groups that can act effectively) is relatively high. The subgroup bP_{2n} is a finite cyclic subgroup of Θ_{2n-1} and the order of bP_{4k+2} is at most two generated by the Kervaire sphere Σ_K^{4k+1}, which can be described as the subset of $\mathbb{C}^{2k+2}$ satisfying the equations:

$$z_1^d + z_2^2 + \cdots + z_{2k+2}^2 = 0$$
$$|z_1|^2 + |z_2|^2 + \cdots + |z_{2k+2}|^2 = 1,$$

where d is any positive integer such that $d \equiv \pm 3 \mod 8$. From this equation, we can easily see that Σ_K^{4k+1} admits an effective $SO(2) \times O(2k+1)$-action. However Brumfiel's calculation shows that the 9-dimensional Kervaire sphere does not admit free S^1-actions. His calculation is essentially the calculation of the index surgery obstruction and if we proceed to continue similar calculation in higher dimensions, the relation obtained by the vanishing of the surgery obstruction is too lengthy and complicated to draw any meaningful conclusion by human inspection. At the workshop in 2004 at RIMS, we showed by symbolic computation that every Kervaire sphere below dimension 130 does not admit any free S^1 actions based on our computer calculation. From these experiments we conjectured that every Kervaire sphere does not admit any free S^1-actions if $k+1$ is not a power of two. At present we can prove this conjecture affirmatively when the 2-order $\nu_2(k)$ of k is less than 3.

Theorem. The Kervaire sphere of dimension $4k+1$, where $k+1$ is not a power of 2, does not any free S^1-action if k is not divisible by 8.

This slightly improves the result obtained in 2006, where the assumption was "k is not divisible by 4". However we must admit that this is not the best possible result ([2]).

We shall always assume that k is a positive integer such that $k+1$ is not a power of two. In this case it is known that the Kervaire sphere Σ_K^{4k+1} is not diffeomorphic to the standard sphere S^{4k+1}.

1 Surgery Obstruction

We shall translate the statement concerning group actions to the one about surgery obstructions.

Lemma 1. The following two statements are equivalent.
(a) The Kervaire sphere Σ_K^{4k+1} does not admit any free S^1-action.
(b) If the normal map

$$\begin{array}{ccc} \nu_M & \xrightarrow{b} & \xi \\ \downarrow & & \downarrow \\ M^{4k+2} & \xrightarrow{f} & \mathbb{C}P(2k+1) \end{array} \tag{1}$$

has zero $4k$-dimensional surgery obstruction $s_{4k}=0$ for the surgery data

$$f|f^{-1}(\mathbb{C}P(2k)) : f^{-1}(\mathbb{C}P(2k)) \to \mathbb{C}P(2k)$$

obtained by restriction to the codimension 2 subspace, then the $(4k+2)$-dimensional surgery obstruction s_{4k+2} of f must also vanish.

Proof. Let us prove that (a) implies (b). Suppose there exists a normal map $f : M^{4k+2} \to \mathbb{C}P(2k+1)$ such that the surgery obstruction s_{4k+2} of f is nonzero and the restricted surgery problem to $\mathbb{C}P(2k)$ has zero surgery obstruction $s_{4k} = 0$. Then we can perform surgery on $f^{-1}(\mathbb{C}P(2k))$ and within the normal cobordism class we may assume that $X = f^{-1}(\mathbb{C}P(2k)) \to \mathbb{C}P(2k)$ is a homotopy equivalence. The tubular neighborhood N of X is homotopy equivalent to $\mathbb{C}P(2k+1)_0 = \mathbb{C}P(2k+1) - \mathrm{int}D^{4k+2}$ and its boundary ∂N is homotopy equivalent to S^{4k+1}. But the remaining part $W = M - \mathrm{int}(N)$ is a parallelizable manifold and its surgery obstruction for the normal map $W \to D^{4k+2}$ rel. ∂W is nonzero. Therefore W has nonzero Kervaire obstruction and its boundary $\partial W = \partial N$ is the Kervaire sphere. Since ∂N is the total space of an S^1-bundle, this implies that the Kervaire sphere admits a free S^1-action.

Conversely, suppose that (b) holds, but (a) does not hold. If the Kervaire sphere Σ_K^{4k+1} admits a free S^1-action, the quotient space of the S^1-action $X^{4k} = \Sigma^{4k+1}/S^1$ is homotopy equivalent to the complex projective space $\mathbb{C}P(2k)$ and the associated D^2-bundle $N^{4k+2} = (\Sigma_K^{4k+1} \times D^2)/S^1$ is homotopy equivalent to $\mathbb{C}P(2k+1)_0 = (S^{4k+1} \times D^2)/S^1$ where the $S^1 \subset \mathbb{C}$ acts on $S^{4k+1} \subset \mathbb{C}^{2k+1}$ and on $D^2 \subset \mathbb{C}$ by complex number multiplication. Let W^{4k+2} be a smooth parallelizable manifold with $\partial W = \Sigma_K^{4k+1}$ and Kervaire invariant $c(W) = 1$. Then by gluing N and W along the common boundary Σ_K, we obtain a normal map $f : M^{4k+2} = N \cup_{\Sigma_K} W \to \mathbb{C}P(2k+1)$ with an appropriate vector bundle ξ, and its surgery obstruction s_{4k+2} is equal to $c(W) = 1$. Hence we have a normal map f with target space $\mathbb{C}P(2k+1)$ with nonzero Kervaire surgery obstruction, but the codimension 2 surgery problem obtained by restricting the target manifold to $\mathbb{C}P(2k)$ has zero surgery obstruction $s_{4k} = 0$, since $f|X^{4k} : X^{4k} \to \mathbb{C}P(2k)$ is a homotopy equivalence. This contradicts the assumption (b). This completes the proof of Lemma 1.

Our objective of this note is to show that the statement (b) in Lemma 1 is true. To do so, we must deal with all possible vector bundles that appear in (1). We point out the following four items that needs consideration:

Bundle data The stable bundle difference $\zeta = \nu_{\mathbb{C}P(2k+1)} - \xi$ is fiber homotopically trivial, namely it belongs to the kernel of the J-homomorphism $J : \widetilde{KO}(\mathbb{C}P(2k+1)) \to \tilde{J}(\mathbb{C}P(2k+1))$. The generators of the kernel can be expressed by Adams operations in KO-theory. The solution of the Adams conjecture imply that 2-local generators are given by the images of $\psi_{\mathbb{R}}^3 - 1$ ([7], Theorem 11.4.1).

The surgery obstruction s_{4k} in dimension $4k$ In dimension $4k$, the surgery obstruction is given by the index obstruction, which can be computed using Hirzebruch's L classes. However, the exact form of the obstruction gets complicated and requires simplified treatment.

Surgery obstruction s_{4k+2} in dimension $4k+2$ The surgery obstruction s_{4k+2} in dimension $4k+2$ can be dealt with by the results of [4],[5], [6]. In fact, the obstruction s_{4k+2} is equal to the two dimensional obstruction s_2 for the surgery data s_2, which is essentially the 2-dimensional Kervaire class K_2.

Relation of K_2 and the first Pontrjagin class p_1 From the result originally due to Sullivan, the square of K_2 for the bundle data ζ is equal to $p_1(\zeta)/8 \mod 2$ (see [8], 14C). This fact gives us a bridge connecting the integral index obstruction and the mod 2 Kervaire obstruction.

2 Index obstruction in dimension $4k$

The kernel of the 2-local J-homomorphism $J : \widetilde{KO}(\mathbb{C}P(2k+1)) \to \tilde{J}(\mathbb{C}P(2k+1))$ is generated by Image$(\psi_{\mathbb{R}}^q - 1)$ (q odd), where $\psi_{\mathbb{R}}^q$ is the Adams operation in KO-theory and we may take $q = 3$. The additive generators of $\widetilde{KO}(\mathbb{C}P(2k+1))$ are given by ω^j $(1 \le j \le k+1)$ where ω is the realification of the complex virtual vector bundle $\eta_{\mathbb{C}} - 1_{\mathbb{C}}$, where $\eta_{\mathbb{C}}$ is complex Hopf line bundle. The Adams operation $\psi^j_{\mathbb{R}}$ on ω is given by the formula

$$\psi^j_{\mathbb{R}}(\omega) = T_j(\omega) \tag{2}$$

where $T_j(z)$ is a polynomial of degree j characterized by

$$T_j(t + t^{-1} - 2) = t^j + t^{-j} - 2\,. \tag{3}$$

Since the coefficient of z^j in $T_j(z)$ is one, we may consider $T_j(\omega)$ $(1 \le j \le k+1)$ as generators of $\widetilde{KO}(\mathbb{C}P(2k+1))$. However, when restricted on $\mathbb{C}P(2k)$, we have $\omega^{k+1} = 0$ and we may safely discard ω^{k+1} in the actual computation. In our argument, we do not necessarily need to know the kernel of $J : \widetilde{KO}(\mathbb{C}P(2k+1)) \to \tilde{J}(\mathbb{C}P(2k+1))$. Later computation shows that we can ignore odd multiples of elements and we have only to know 2-local generators of the kernel. The 2-local generators of the kernel of J are

$$\zeta_j = (\psi^3_{\mathbb{R}} - 1)\psi^j_{\mathbb{R}}(\omega) \qquad (j = 1, 2, \ldots, k) \tag{4}$$

and an element of the 2-local kernel of the J-homomorphism has the form

$$\zeta = \sum_{j=1}^{k} m_j \zeta_j \tag{5}$$

where m_j belong to $\mathbb{Z}_{(2)}$, the ring of integers localized at 2.

The surgery obstruction s_{4k} of the surgery data (1) when restricted on $\mathbb{C}P(2k)$ is given by

$$8s_{4k} = (\mathrm{Index}(M) - \mathrm{Index}(\mathbb{C}P(2k))) = ((\mathcal{L}(\zeta) - 1)\mathcal{L}(\mathbb{C}P(2k)))\,[\mathbb{C}P(2k)] \tag{6}$$

where $\mathcal{L}$ is the multiplicative class associated to the power series

$$h(x) = \frac{x}{\tanh x} = 1 + \sum_{i \ge 1} \frac{(-1)^{i+1} 2^{2i} B_i}{(2i)!} x^{2i} \tag{7}$$

and B_i is the i-th Bernoulli number characterized by

$$\frac{x}{e^x - 1} = 1 - \frac{1}{2}x + \sum_{i \ge 1} \frac{(-1)^{k-1} B_i}{(2i)!} x^{2i}. \tag{8}$$

Remark that all the coefficients of $h(x)$ belong to $\mathbb{Z}_{(2)}$ the rational numbers with odd denominator because (a) all the denominators of Bernoulli numbers are even but not divisible by four and (b) $\nu_2(m!) < m$ for all integers m. If the total Pontrjagin class of a bundle ξ is given by $p(\xi) = \prod_i (1 + x_i^2)$, $\mathcal{L}(\xi)$ is given by $\prod_i h(x_i)$ and when M is a manifold, we define $\mathcal{L}(M) = \mathcal{L}(\tau_M)$. To calculate the Pontrjagin class of $\psi^j_{\mathbb{R}}(\omega)$, we note that

$$\begin{aligned}\psi^j_{\mathbb{R}}(\omega) \otimes \mathbb{C} &= \psi^j_{\mathbb{C}}(\omega \otimes \mathbb{C}) = \psi^j_{\mathbb{C}}(\eta_{\mathbb{C}} \oplus \bar{\eta}_{\mathbb{C}} - 2_{\mathbb{C}}) \\ &= \psi^j_{\mathbb{C}}(\eta_{\mathbb{C}}) + \psi^j_{\mathbb{C}}(\bar{\eta}_{\mathbb{C}}) - 2_{\mathbb{C}} = \eta^j_{\mathbb{C}} + \bar{\eta}^j_{\mathbb{C}} - 2_{\mathbb{C}},\end{aligned}$$

whose total Chern class is $(1+jx)(1-jx) = 1-j^2x^2$, where x is the generator of $H^2(\mathbb{C}P(2k+1))$. Hence the total Pontrjagin class of $\psi^j_{\mathbb{R}}(\omega)$ is $1+j^2x^2$. For the virtual bundle ζ in (5), we have

$$\mathcal{L}(\zeta) = \prod_{j=1}^{k} \left(\frac{h(3jx)}{h(jx)}\right)^{m_j}. \tag{9}$$

Given a power series $f(x)$ in x, let us express the the coefficient of x^n in $f(x)$ by $(f(x))_n$. The $4k$-dimensional obstruction s_{4k} is given by

$$\left((\mathcal{L}(\zeta)-1)h(x)^{2k+1}\right)_{2k} / 8. \tag{10}$$

To calculate this, we put

$$g(x) = \frac{h(3x)}{h(x)} - 1. \tag{11}$$

Then we have

$$g(x) = \frac{8}{3}x^2 - \frac{8}{3}x^4 + \frac{112}{45}x^6 - \frac{6472}{2835}x^4 + \cdots . \tag{12}$$

Lemma 2. All the coefficients of $g(x)$ are divisible by 8 in $\mathbb{Z}_{(2)}$.
Proof. From the expansion (7), we have

$$\frac{3x}{\tanh 3x} \equiv \frac{x}{\tanh x} \mod 8 \quad \text{in} \quad \mathbb{Z}_{(2)}[[x]].$$

Noting that $x/\tanh x$ is invertible in $\mathbb{Z}_{(2)}[[x]]$, we have

$$\frac{3x}{\tanh 3x}\frac{\tanh x}{x} \equiv 1 \mod 8 \quad \text{in} \quad \mathbb{Z}_{(2)}[[x]].$$

and the assertion follows.

We now calculate the $\mathcal{L}$ class:

$$\begin{aligned}
\mathcal{L}(\zeta) - 1 &= \prod_j (1+g(jx))^{m_j} - 1 \\
&= \prod_j \left(1 + m_j g(jx) + \frac{m_j(m_j-1)}{2}(g(jx))^2 + \cdots\right) - 1 \\
&\equiv \sum_j m_j g(jx) \qquad \mod 64 \\
&\equiv \sum_{j:odd} m_j g(jx) + \frac{8}{3}\sum_{j\equiv 2(4)} m_j (jx)^2 \qquad \mod 64 \\
&\equiv \sum_{j:odd} m_j g(x) + 32 \sum_{j\equiv 2(4)} m_j x^2 \qquad \mod 64.
\end{aligned} \tag{13}$$

From this, we have the $4k$-dimensional surgery obstruction s_{4k}

$$\begin{aligned}
8s_{4k} &= \left((\mathcal{L}(\zeta)-1)h(x)^{2k+1}\right)_{2k} \\
&\equiv \left(\left(\sum_{j:odd} m_j g(x) + 32\sum_{j\equiv 2(4)} m_j x^2\right) h(x)^{2k+1}\right)_{2k} \qquad \text{mod } 64 \\
&\equiv \sum_{j:odd} m_j \left(g(x)h(x)^{2k+1}\right)_{2k} + 32\sum_{j\equiv 2(4)} m_j \left(x^2 h(x)^{2k+1}\right)_{2k} \qquad \text{mod } 64.
\end{aligned} \tag{14}$$

Lemma 3.

(a) $\left(g(x)h(x)^{2k+1}\right)_{2k} = \frac{8}{3}\sum_{i=1}^{k}\left(\frac{-1}{3}\right)^{i-1} = \frac{2(3^k-(-1)^k)}{3^k}$

(b) $\nu_2(3^k-(-1)^k) = \nu_2(k)+2$

(c) $\left(x^2h(x)^{2k+1}\right)_{2k}$ is even if k is divisible by 4.

Proof. (a) Let $\mathrm{Res}_x(F(x))$ denote the residue of $F(x)$ at $x=0$, namely the coefficient of x^{-1} in the Laurent expansion of $F(x)$ around $x=0$. Since

$$\tanh 3x = \frac{3\tanh x + \tanh^3 x}{1+3\tanh^2 x},$$

we have

$$\begin{aligned}
g(x) &= \frac{3\tanh x}{\tanh 3x} - 1 = \frac{3+9\tanh^2 x}{3+\tanh^2 x} - 1 \\
&= \frac{8\tanh^2 x}{3+\tanh^2 x} = \frac{8}{3}\sum_{i\geq 1}\left(\frac{-1}{3}\right)^{i-1}\tanh^{2i} x.
\end{aligned}$$

From this we have

$$\begin{aligned}
\left(g(x)h(x)^{2k+1}\right)_{2k} &= \frac{8}{3}\left(\sum_{i\geq 1}\left(\frac{-1}{3}\right)^{i-1}\tanh^{2i} x\left(\frac{x}{\tanh x}\right)^{2k+1}\right)_{2k} \\
&= \frac{8}{3}\mathrm{Res}_x\left(\sum_{i\geq 1}\left(\frac{-1}{3}\right)^{i-1}\frac{\tanh^{2i} x}{\tanh^{2k+1} x}\right)
\end{aligned}$$

by changing variables $y=\tanh x$,

$$\begin{aligned}
&= \frac{8}{3}\mathrm{Res}_y\left(\sum_{i\geq 1}\left(\frac{-1}{3}\right)^{i-1}\frac{y^{2i}}{y^{2k+1}(1-y^2)}\right) \\
&= \frac{8}{3}\mathrm{Res}_y\left(\sum_{i\geq 1}\left(\frac{-1}{3}\right)^{i-1}\frac{y^{2i}+y^{2i+2}+y^{2i+4}+\cdots}{y^{2k+1}}\right) \\
&= \frac{8}{3}\sum_{i=1}^{k}\left(\frac{-1}{3}\right)^{i-1} = \frac{2(3^k-(-1)^k)}{3^k}.
\end{aligned}$$

Remark: The invariance of residues remains true for formal Laurent series $F(x)$ with finite negative terms provided the formal variable change $x = \phi(y)$, where $\phi(y)$ is a formal power series with $\phi(0) = 0$ and $\phi'(0) \neq 0$:

$$\mathrm{Res}_x(F(x)) = \mathrm{Res}_y(F(\phi(y))\phi'(y)). \tag{15}$$

(b) We use induction on k. Suppose that (b) holds for all k with $k < n$. If n is odd we put $n = 2m+1$. We have $3^n - (-1)^n = 3^{2m+1} + 1 = 3 \cdot 9^m + 1 \equiv 4 \mod 8$. Therefore $\nu_2(3^n - (-1)^n) = 2 = \nu_2(n) + 2$ holds. When n is even, we put $n = 2m$ and we have

$$3^n - (-1)^n = 3^{2m} - 1 = (3^m - (-1)^m)(3^m + (-1)^m).$$

Here we see that $\nu_2(3^m - (-1)^m) = \nu_2(m) + 2$ from the induction assumption and since this factor is divisible by 4 we see that $3^m + (-1)^m$ is an even integer not divisible by 4. Thus we have $\nu_2(3^m + (-1)^m) = 1$ and $\nu_2(3^{2m} - 1) = \nu_2(m) + 3 = \nu_2(n) + 2$. This completes the proof of (b).

(c) Again we use residues. We have

$$\left(x^2 h(x)^{2k+1}\right)_{2k} = \mathrm{Res}_x\left(\frac{x^2}{\tanh^{2k+1} x}\right)$$

by changing to the variable $y = \tanh x$,

$$\begin{aligned}
&= \mathrm{Res}_y\left(\frac{\mathrm{arctanh}^2 y}{y^{2k+1}(1-y^2)}\right) \\
&= \mathrm{Res}_y\left(\frac{(y + y^3/3 + y^5/5 + \cdots)^2}{y^{2k+1}(1-y^2)}\right) \\
&= \left(\frac{(y + y^3/3 + y^5/5 + \cdots)^2}{1-y^2}\right)_{2k} \\
&\equiv \left(\frac{(y + y^3 + y^5 + \cdots)^2}{1-y^2}\right)_{2k} \qquad \mod 2 \\
&= \left(\frac{1}{(1-y^2)^3}\right)_{2k-2} = \left(\frac{1}{(1-z)^3}\right)_{k-1} \\
&= \binom{k+1}{2} = k(k+1)/2.
\end{aligned}$$

Therefore $\left(x^2 h(x)^{2k+1}\right)_{2k}$ is even if k is divisible by 4 and this completes the proof of Lemma 3.

We are now ready to state and prove our key lemma.

Lemma 4. If the $4k$-dimensional surgery obstruction s_{4k} vanished and k is not divisible by 8 then $\sum_{j:odd} m_j$ is even.

Proof. From (14) and Lemma 3, if $s_{4k} = 0$, we see that

$$2^{\nu_2(k)+3} \sum_{j:odd} m_j \tag{16}$$

is divisible by 32. If k is not divisible by 4 then $\sum_{j:odd} m_j$ must be even. When k is divisible by 4 (and not divisible by 8), then again by Lemma 3, we have (16) is divisible by 64 and $\sum_{j:odd} m_j$ must be even. This completes the proof.

3 2-dimensional surgery obstruction

In the normal map (1), let $\zeta = \nu_{\mathbb{C}P(2k+1)} - \xi$, then it can be written (2-locally) $\zeta = \sum_{j=1}^{k} m_j\zeta_j$ where $\zeta_j = (\psi_{\mathbb{R}}^3 - 1)\psi_{\mathbb{R}}^j(\omega)$. The total Pontrjagin class of $\psi_{\mathbb{R}}^m(\omega)$ is given by

(17) $$p(\psi_{\mathbb{R}}^m(\omega)) = 1 + m^2x^2$$

and we have

(18) $$p(\zeta_j) = \frac{1+9j^2x^2}{1+j^2x^2}$$

(19) $$p(\zeta) = \prod_j \left(\frac{1+9j^2x^2}{1+j^2x^2}\right)^{m_j}.$$

For the first Pontrjagin class, we have

(20) $$p_1(\zeta)/8 = \left(\sum_j j^2 m_j\right) x^2.$$

We know that the 2-dimensional surgery obstruction s_2 for $f|f^{-1}(\mathbb{C}P(1))$ is equal to $\sum_j j^2 m_j$ mod 2 since in the complex projective space surgery theory, the mod 2 reduction of $p_1(\zeta)$ coincides with the square of the 2-dimensional Kervaire class for the given normal map (see Wall's book [8, Chap 13.]). And it is known that if $k+1$ is not a power of 2, then $(4k+2)$-dimensional surgery obstruction coincides with 2-dimensional surgery obstruction ([6],[4],[5]). From these facts we get the following Lemma.

Lemma 5. If $\sum_{j:odd} m_j$ is even, then the surgery obstruction s_{4k+2} vanishes.

Proof of Theorem:

Let k is an integer such that $k+1$ is not a power of two and assume that k is not divisible by 8. Then for the surgery problem of $\mathbb{C}P(2k+1)$ with bundle data $\zeta = \sum_j m_j\zeta_j$, if the $4k$-dimensional surgery obstruction s_{4k} vanishes then $\sum_j m_j$ must be even from Lemma 4. Then by Lemma 5, the $(4k+2)$-dimensional surgery obstruction s_{4k+2} should also vanish. In view of Lemma 1, this proves our assertion.

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Yasuhiko KITADA
Department of electrical and computer engineering
Graduate school of engineering
Yokohama National University
E-mail : kitada@mathlab.sci.ynu.ac.jp