数理解析研究所講究録
第 1571 巻 2007 年 5-17

Ford domains of a certain hyperbolic 3-manifold whose boundary consists of a pair of once-punctured tori, II

大阪市立大学数学研究所 秋吉宏尚 (Hirotaka Akiyoshi)
Osaka City University Advanced Mathematical Institute

1 Introduction

This is a sequel to [3]. Our initial problem is the following.

Problem 1.1. Characterize the combinatorial structures of the Ford domains for hyperbolic structures on a 3-manifold which has a pair of punctured tori as boundary.

The Jorgensen theory tells in detail the combinatorial structures of the Ford domains of hyperbolic structures on punctured torus bundles. We expect to understand in detail the hyperbolic structures on manifolds with non-fiber surfaces from the combinatorial structures of Ford domains. Problem 1.1 is the first step to the attempt to fill in the box with "???" in the following table, which tells the relation between analytic and combinatorial aspects of Thurston's Hyperbolization Theorem for Haken manifolds.

	analytic	combinatorial (genus= 1)
fiber surface	double limit theorem	Jorgensen theory
non-fiber surface	fixed point theorem	???

2 A family of 3-manifolds with a pair of punctured tori as boundary

We denote the one-holed torus (resp. once-punctured torus) by T_0 (resp. T). Let γ be an essential simple closed curve on the level surface $T_0 \times \{0\}$ of the product manifold $T_0 \times [-1,1]$, and denote by M_0 the exterior of γ, i.e.,

$M_0 = T_0 \times [-1,1] - \operatorname{Int} N(\gamma)$, where $N(\gamma)$ is a regular neighborhood of γ. For each sign $\epsilon = \pm$, we denote the one-holed torus $T_0 \times \{\epsilon 1\} \subset \partial M_0$ by T_0^ϵ, and denote the natural homeomorphism between T_0 and T_0^ϵ by $\iota^\epsilon : T_0 \to T_0^\epsilon$. We define the *slopes* (= free homotopy classes) μ and λ in $\partial N(\gamma)$ as follows. μ is the meridian slope of γ, i.e., μ is represented by an essential simple closed curve which bounds a disk in $N(\gamma)$, and λ is the slope represented by the intersection of $\partial N(\gamma)$ and the annulus $\gamma \times [0,1]$. Then $\{\mu, \lambda\}$ generates $H_1(\partial N(\gamma))$.

For a pair of coprime integers (p,q), let $M(p,q)$ be the result of Dehn filling on M_0 with slope $p\mu + q\lambda$, i.e., the manifold obtained from M_0 by gluing the solid torus V by an orientation-reversing homeomorphism $\partial V \to \partial N(\gamma) \subset \partial M_0$ so that the meridian of V is identified with a simple closed curve on $\partial N(\gamma)$ of slope $p\mu + q\lambda$. We regard M_0 as a submanifold of $M(p,q)$ by using the canonical embedding. In particular, $M(\pm 1, 0)$ is canonically homeomorphic to the product $T_0 \times [-1,1]$.

Proposition 2.1. *For any pair of coprime integers (p,q), $M(p,q)$ is homeomorphic to the handlebody of genus* 2.

Set $P = \partial T_0 \times [-1,1]$. In contrast to Proposition 2.1, the pair $(M(p,q), P)$ does not necessarily admit a product structure.

Proposition 2.2. *The surfaces $T_0^{\pm}$ is incompressible in $M(p,q)$ if and only if $(p,q) \neq (0, \pm 1)$. In this case, it follows that $(M(p,q), P)$ is an atoroidal Haken pared manifold* in the sense of [12].

By the Thurston's Hyperbolization Theorem for Haken pared manifolds (cf. [12, Theorem 1.43]), we obtain the following corollary.

Corollary 2.3. *For any pair of coprime integers $(p,q) \neq (0, \pm 1)$, $M(p,q)$ admits a complete geometrically finite hyperbolic structure with the parabolic locus P.*

For the rest of this paper, (p,q) denotes a pair of coprime integers distinct from $(0, \pm 1)$.

Definition 2.4. We shall denote by $\mathcal{MP}(p,q)$ the space of geometrically finite (marked) hyperbolic structures on the pared manifold $(M(p,q), P)$ with the parabolic locus P.

By Corollary 2.3, $\mathcal{MP}(p,q)$ is not empty. The following proposition follows from Marden's isomorphism theorem.

Proposition 2.5. *The space $\mathcal{MP}(p,q)$ is isomorphic to the square $\mathcal{T} \times \mathcal{T}$ of the Teichmüller space $\mathcal{T}$ of the punctured torus.*

3 Punctured torus groups

We fix a generator pair (α, β) of $\pi_1(T)$ and denote the commutator $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$ by κ.

Definition 3.1. A representation $\rho : \pi_1(T) \to PSL(2, \mathbb{C})$ is a (marked) *punctured torus group* if it is a faithful discrete representation which maps κ to a parabolic element. Two punctured torus groups are said to be *equivalent* if they are conjugate to each other. Denote the set of equivalence classes of punctured torus groups by $\overline{\mathcal{QF}}$. The set $\overline{\mathcal{QF}}$ is identified with a subset of the $PSL(2, \mathbb{C})$-representation space of $\pi_1(T)$. The topology on $\overline{\mathcal{QF}}$ is induced from this identification.

For any $\rho \in \overline{\mathcal{QF}}$, $\mathbb{H}^3/\rho(\pi_1(T))$ is homeomorphic to $\mathrm{Int}(T_0 \times (-1, 1))$. For each end of $T_0 \times (-1, 1)$, one can define the end invariant of ρ, denoted by $\lambda^\epsilon(\rho)$ ($\epsilon \in \{-, +\}$), as follows. If there is a simply connected component of the domain of discontinuity of $\rho(\pi_1(T))$ corresponding to the end, then its quotient determines a marked conformal structure on T, which is defined to be the end invariant $\lambda^\epsilon(\rho) \in \mathcal{T}$. If the subset of domain of discontinuity of $\rho(\pi_1(T))$ corresponding to the end is the disjoint union of countable family of round disks, then its quotient determines a marked conformal structure on T with node, which is defined to be the end invariant $\lambda^\epsilon(\rho) \in \partial_{\mathbb{Q}}\mathcal{T}$. If the subset of domain of discontinuity of $\rho(\pi_1(T))$ corresponding to the end is empty, then one can define the end invariant $\lambda^\epsilon(\rho) \in \partial\mathcal{T} - \partial_{\mathbb{Q}}\mathcal{T}$ by using a sequence of simple closed geodesics which exits the end. Here $\mathcal{T}$ is compactified via Thurston's compacticfication. Then $\mathcal{T}$ (resp. $\mathcal{T} \cup \partial\mathcal{T}$ and $\partial_{\mathbb{Q}}\mathcal{T}$) is canonically identified with $\mathbb{H}^2$ (resp. $\overline{\mathbb{H}^2}$ and $\partial_{\mathbb{Q}}\mathbb{H}^2 = \widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$). (See [14] for details.)

Definition 3.2. The end invariant map $\boldsymbol{\lambda} : \overline{\mathcal{QF}} \to \overline{\mathbb{H}^2} \times \overline{\mathbb{H}^2} - \mathrm{diag}(\partial\mathbb{H}^2)$ is defined by $\boldsymbol{\lambda}(\rho) = (\lambda^-(\rho), \lambda^+(\rho))$ ($\rho \in \overline{\mathcal{QF}}$).

Theorem 3.3 (Minsky [14]). *The end invariant map $\boldsymbol{\lambda} = (\lambda^-, \lambda^+) : \overline{\mathcal{QF}} \to \overline{\mathbb{H}^2} \times \overline{\mathbb{H}^2} - \mathrm{diag}(\partial\mathbb{H}^2)$ is a bijection and its inverse is a continuous map. In particular, $\overline{\mathcal{QF}}$ is equal to the closure of the quasifuchsian space $\mathcal{QF}$.*

Given an element $\sigma \in \mathcal{MP}(p, q)$, let $\widehat{\rho}_\sigma : \pi_1(M(p, q)) \to PSL(2, \mathbb{C})$ be the holonomy representation. Then let $\iota_\sigma : \pi_1(T) \to PSL(2, \mathbb{C}) \times PSL(2, \mathbb{C})$ be the composition $\iota_\sigma = (\widehat{\rho}_\sigma \circ \iota_*^-, \widehat{\rho}_\sigma \circ \iota_*^+)$. The following proposition follows from Proposition 2.2 and the covering theorem [7].

Proposition 3.4. *The correspondence $\sigma \mapsto \iota_\sigma$ induces an embedding of $\mathcal{MP}(p, q)$ into $\mathcal{QF} \times \mathcal{QF}$.*

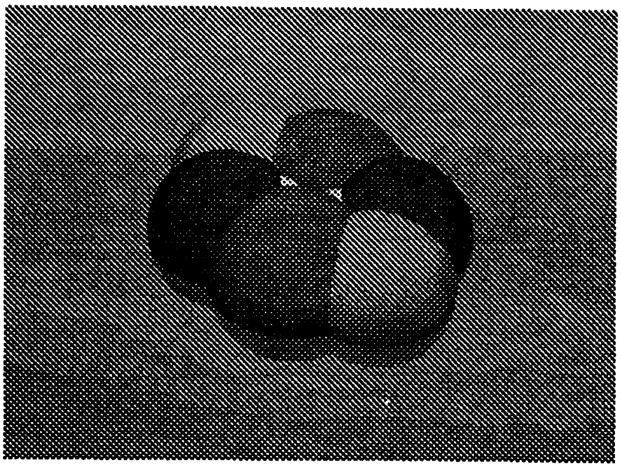

Figure 1: Ford domain of a cyclic Kleinian group

4 Ford domain

In what follows, we use the upper half space model for $\mathbb{H}^3$.

Definition 4.1. For an element γ of $PSL(2,\mathbb{C})$ which does not stabilize ∞, the *isometric hemisphere*, $Ih(\gamma)$, of γ is the set of points in $\mathbb{H}^3$ where γ acts as an isometry with respect to the canonical Euclidean metric on the upper half space. We denote the exterior of $Ih(\gamma)$ by $Eh(\gamma)$.

Definition 4.2. For a Kleinian group Γ, the *Ford domain*, $Ph(\Gamma)$, of Γ is defined by $Ph(\Gamma) = \bigcap_{\gamma\in\Gamma-\Gamma_\infty} Eh(\gamma)$, where Γ_∞ is the stabilizer of ∞ in Γ. For any hyperbolic structure σ on M_0 or $M(p,q)$, the Ford domain for σ is defined to be the Ford domain of the image of a holonomy representation for σ which sends the peripheral element $[\partial T_0 \times \{0\}]$ of the fundamental group to $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

Example 4.3. The Ford domain $Ph(\langle\gamma\rangle)$ of the cyclic Kleinian group $\langle\gamma\rangle$, generated by a loxodromic element $\gamma \in PSL(2,\mathbb{C})$ which does not stabilize ∞, is as depicted in Figures 1 and 2. Every "face" of $Ph(\Gamma)$ is supported by an isometric hemisphere; there are 8 faces in this example. The characterization of combinatorial structures of the Ford domains of cyclic Kleinian groups is given by Jorgensen [10] (cf. [8]).

Let $\mathcal{D} = \{\gamma\langle\infty,0,1\rangle \,|\, \gamma \in PSL(2,\mathbb{Z})\}$ be the Farey tessellation of $\mathbb{H}^2$. The combinatorial structure of the Ford domain of $\rho(\pi_1(T))$ for $\rho \in \overline{\mathcal{QF}}$ is characterized by the extension of Jorgensen's side parameter $\boldsymbol{\nu} = (\nu^-,\nu^+) : \overline{\mathcal{QF}} \to \mathbb{H}^2 \times \mathbb{H}^2 - \mathrm{diag}(\partial\mathbb{H}^2)$. (See [5, Section 4] for detail.)

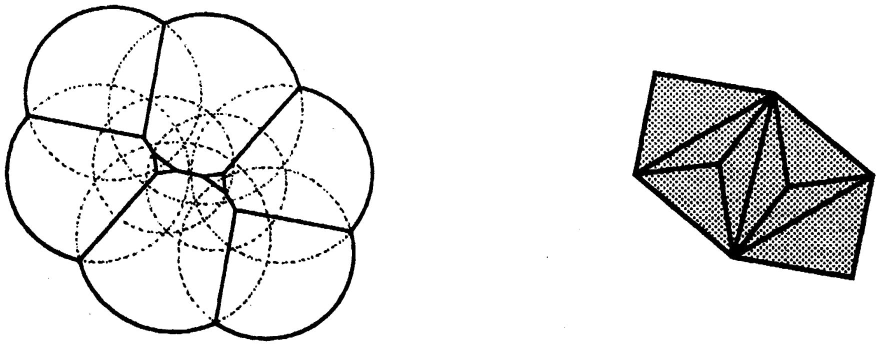

Figure 2: Ford domain of a cyclic Kleinian group and its dual

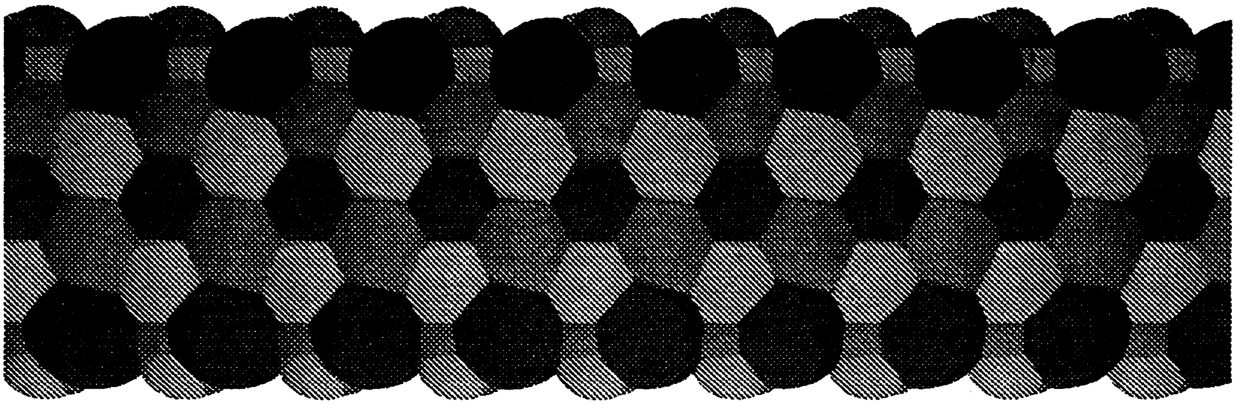

Figure 3: Ford domain of a quasifuchsian punctured torus group

Here we briefly review the idea of the characterization. The combinatorial structures of Ford domains are characterized by using EPH-decomposition introduced in [4].

Definition 4.4. For a hyperbolic structure σ on M_0 or $M(p,q)$, let $\Delta_{\mathbf{E}}(\sigma)$ be the subcomplex of the EPH-decomposition for σ consisting of the Euclidean facets. Let $\Delta_{\mathbf{E},0}(\sigma)$ be the subcomplex of $\Delta_{\mathbf{E}}(\sigma)$ consisting of the facets whose vertices correspond to the parabolic locus P.

By the observation in [4, Section 10], it can be proved that $\Delta_{\mathbf{E},0}(\sigma)$ is dual to the Ford domain for σ.

For any point $\boldsymbol{\nu} \in \overline{\mathbb{H}^2} \times \overline{\mathbb{H}^2} - \mathrm{diag}(\partial\mathbb{H}^2)$, a topological ideal polyhedral complex $\mathrm{Trg}(\boldsymbol{\nu})$ is defined in [5, Section 5]. Then, for any $\rho \in \overline{\mathcal{QF}}$, $\Delta_{\mathbf{E},0}(\rho)$ is isotopic to $\mathrm{Trg}(\boldsymbol{\nu}(\rho))$ in the convex core of $\mathbb{H}^3/\rho(\pi_1(T))$ (see [5, Theorem 5.1]). Figure 3 illustrates the Ford domain of a generic quasifuchsian punctured torus group, and Figure 4 illustrates the intersection of $\Delta_{\mathbf{E}}(\rho)$ and a sufficiently small horosphere centered at ∞, which is the fixed point of the parabolic subgroup corresponding to P by normalization.

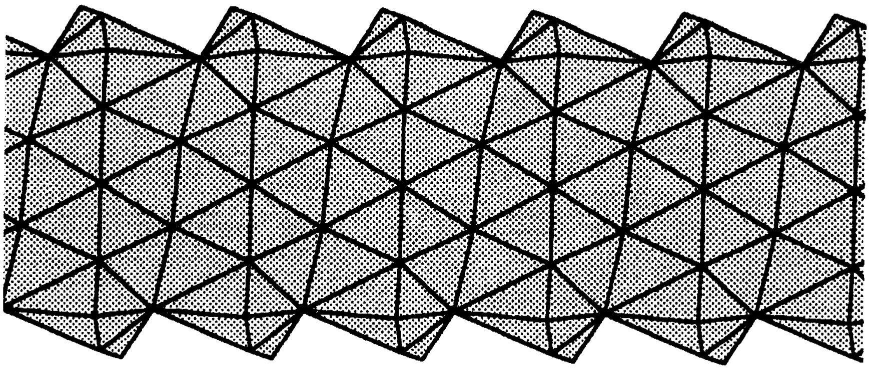

Figure 4: Dual of Ford domain

5 Ford domains for hyperbolic structures in $\mathcal{MP}(p,q)$

To answer Problem 1.1 for the pared manifold $(M(p,q), P)$ with a coprime integers $(p,q) \neq (0, \pm 1)$, we will follow the following program.

(1) Construct a geometrically finite hyperbolic structure, σ_0, on the pared manifold $(M_0, P \cup \partial N(\gamma) \cup N(\alpha^{\pm}))$ with the parabolic locus $P \cup \partial N(\gamma) \cup N(\alpha^{\pm})$, where $N(\alpha^{\pm})$ is the regular neighborhood in $T_0^{\pm}$ of the union of two simple closed curves $\alpha^{\pm} \subset T_0^{\pm}$.

(2) Construct a geometrically finite hyperbolic structure, $\sigma(p,q)$, on the pared manifold $(M(p,q), P \cup N(\alpha^{\pm}))$ in $\partial \mathcal{MP}(p,q)$ by hyperbolic Dehn filling on the structure σ_0.

(3) By using the "geometric continuity" argument, which is used in the Jorgensen theory, characterize the combinatorial structures of Ford domains of the structures in $\mathcal{MP}(p,q)$.

See Figures 5 and 6, which illustrates the Ford domains for σ_0 and $\sigma(3,5)$.

For the step (1), the desired hyperbolic structure, σ_0, is obtained from two copies of the manifold of the double cusp group $\boldsymbol{\lambda}^{-1}(\infty, 1/2)$ by gluing along a pair of boundary components of their convex cores.

Definition 5.1. For each $s \in \widehat{\mathbb{Q}}$ with $0 < s < 1$, let Δ_0^s be the complex obtained from the two copies of the complex $\mathrm{Trg}(\infty, s)$ by gluing them together along the edge with slope ∞ (see Figure 7).

The following proposition follows from [5, Theorem 9.1].

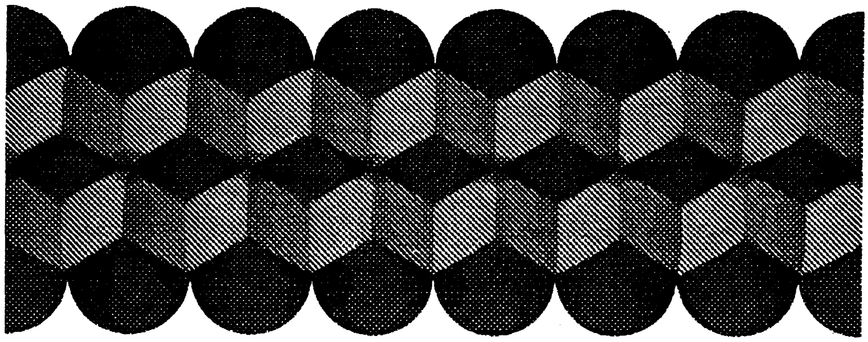

Figure 5: Ford domain for σ_0

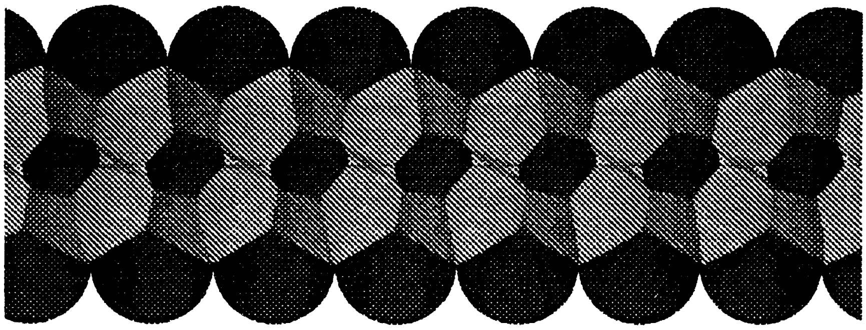

Figure 6: Ford domain for $\sigma(3,5)$

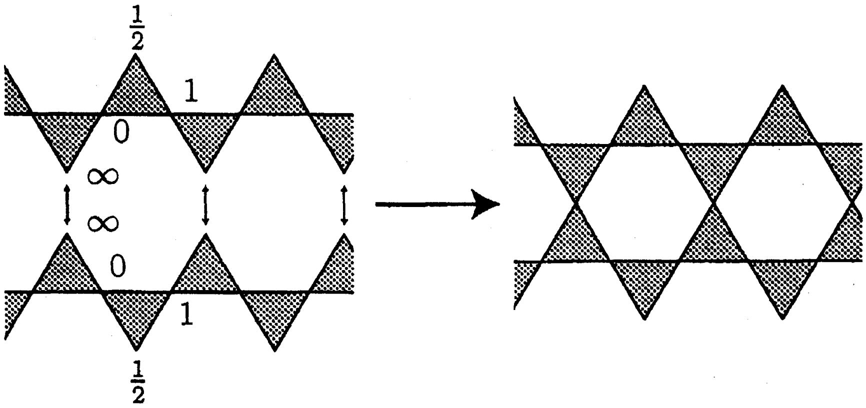

Figure 7: The link of the ideal vertex of $\Delta_0^{1/2}$

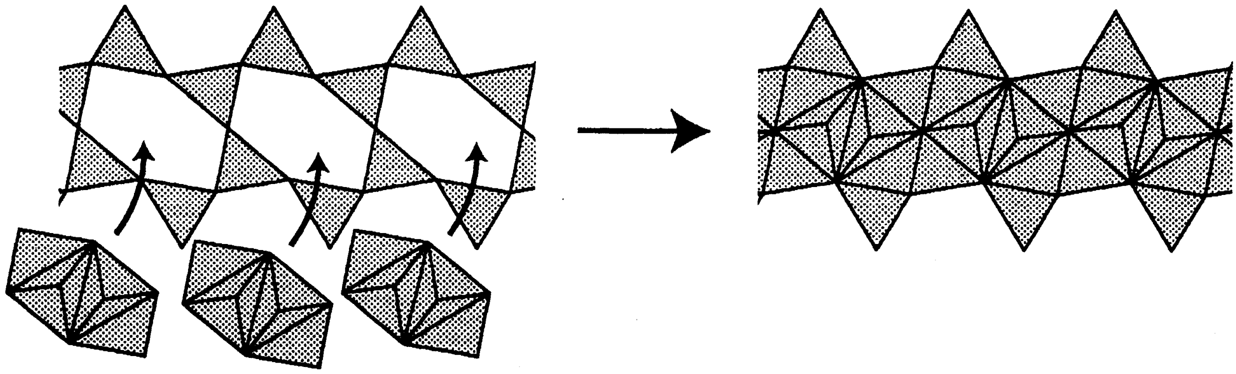

Figure 8: The link of the ideal vertex of $\Delta^{1/2}(p,q)$; this figure illustrates the case $(p,q)=(3,5)$

Proposition 5.2. *The complex $\Delta_{\mathbf{E},0}(\sigma_0)$ is combinatorially equivalent to the complex $\Delta_0^{1/2}$.*

For the step (2), the following Proposition 5.4 is proved by studying the Ford domains after hyperbolic Dehn filling. See [2] for an outline, in which the definition of *layered solid torus* is not correct; it should be modified as follows. The following construction is parallel to the construction of the topological ideal triangulation of the two bridge link complement introduced in [15]. Let σ^+ be the triangle of $\mathcal{D}$ with vertices 0, 1/2 and 1/3. For any coprime integers (p,q), let l be the geodesic in $\mathbb{H}^2$ with endpoints p/q and $s^+ \in \{0, 1/2, 1/3\}$ which intersects the interior of σ^+. Let σ^- be the triangle of $\mathcal{D}$ with vertex p/q whose interior intersects l. Let $\sigma^{-,*}$ be the triangle which shares an edge with σ^- and does not contain p/q. Let s^- be the vertex of $\sigma^{-,*}$ which is not contained in σ^-. We introduce the equivalence relation $\sim_{s^-}$ on the boundary component of $\mathrm{Trg}(s^-,s^+)$ corresponding to $\sigma^{-,*}$ following [15, Section II.2]. Let $V(p,q)$ be the quotient space $\mathrm{Trg}(s^-,s^+)/\sim_{s^-}$. Then $V(p,q)$ is homeomorphic to the solid torus with a point on the boundary removed whenever σ^+ and $\sigma^{-,*}$ do not share an edge. We can see also that the meridian of $V(p,q)$ has slope p/q.

Definition 5.3. Let $\widetilde{V}(p,q)$ be the double cover of $V(p,q)$. For each $s\in\mathbb{Q}$ with $0<s<1$, let $\Delta^s(p,q)$ be the complex obtained by gluing $\widetilde{V}(p,q)$ to Δ_0^s so that the triangle of $\partial\widetilde{V}(p,q)$ with edges of slopes $(0,1/3,1/2)$ and the triangle of $\partial\Delta_0^s$ with edges of slopes $(\infty,0,1)$ are identified. (See Figure 8. See also Figures 10 and 11.)

Proposition 5.4. *For all but finite coprime integers (p,q), the complex $\Delta_{\mathbf{E},0}(\sigma(p,q))$ is combinatorially equivalent to $\Delta^{1/2}(p,q)$ (see Figure 8).*

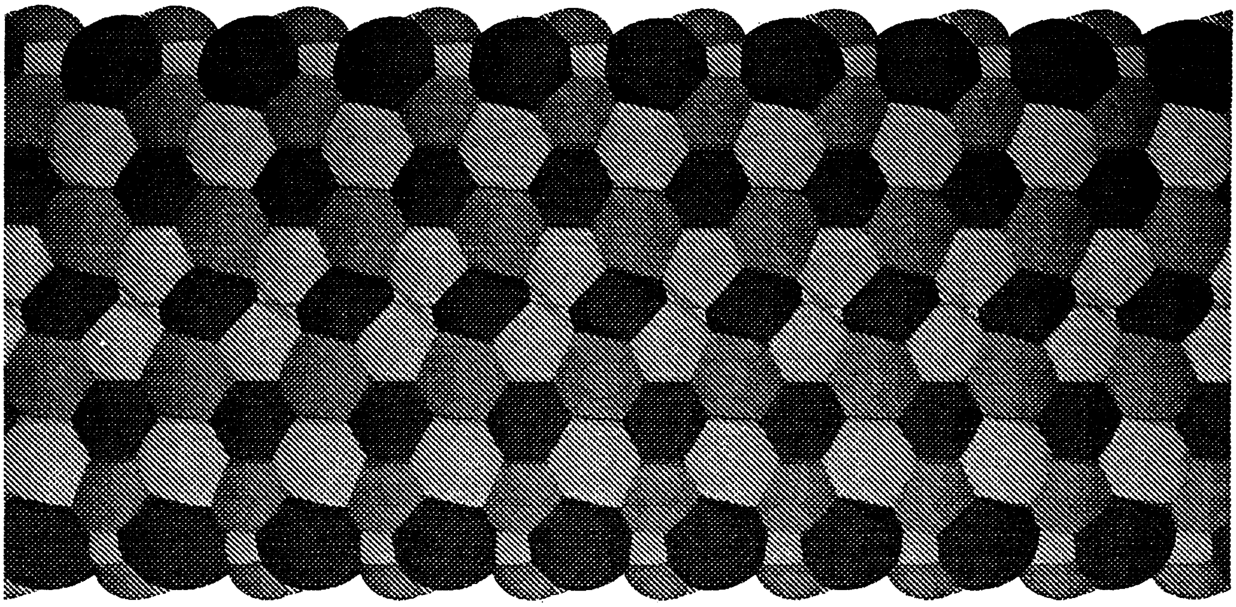

Figure 9: Ford domain for a hyperbolic structure in $\mathcal{J}_{\rm sym}^{\rm thick} \subset \mathcal{MP}_{\rm sym}(p,q)$ for $(p,q)=(3,5)$

Let $\mathcal{MP}_{\rm sym}(p,q)$ be the subspace of $\mathcal{MP}(p,q)$ consisting of the elements whose image (λ^-,λ^+) in $\mathcal{T}\times\mathcal{T}$ by the map defined in Proposition 2.5 satisfies that λ^+ is the mirror image of λ^-. Then $\sigma(p,q)$ is contained in $\partial\mathcal{MP}_{\rm sym}(p,q)$. The symmetry of this kind seems to be useful to carry out the "geometric continuity" argument.

Question 5.5. Is the combinatorial structure of the Ford domain for every hyperbolic structure in $\mathcal{MP}_{\rm sym}(p,q)$ characterized by a way similar to that given in Proposition 5.4?

We say a hyperbolic structure $\sigma \in \mathcal{MP}_{\rm sym}(p,q)$ is *thick good* if $\Delta_{\mathbf{E}}(\sigma)$ is combinatorially equivalent to $\Delta^s(p,q)$ for some $s\in\mathbb{Q}$ with $0<s<1$, and denote the subset of $\mathcal{MP}_{\rm sym}(p,q)$ consisting of the thick good structures by $\mathcal{J}_{\rm sym}^{\rm thick}$. Figure 9 illustrates the Ford domain for a hyperbolic structure contained in $\mathcal{J}_{\rm sym}^{\rm thick} \subset \mathcal{MP}_{\rm sym}(p,q)$ for $(p,q)=(3,5)$. We can draw a conjectural picture of $\mathcal{J}_{\rm sym}^{\rm thick}$ for $(p,q)=(3,5)$ as Figure 12. In the figure, the hyperbolic structures are parametrized by $\mathrm{Tr}\rho(\gamma)$, where ρ is a lift to a $SL(2,\mathbb{C})$-representation of the holonomy representation for the structure. A point in the plane is colored gray if the corresponding representation determines an embedding into $\mathbb{C}$ of a simplicial complex which is supposed to be the dual of the Ford domain and if the radii of the isometric hemispheres corresponding to the vertices of the complex do not exceed 1. The condition on the radii of isometric hemispheres is necessary for the corresponding holonomy representation to be discrete. Those points are colored by two different colors

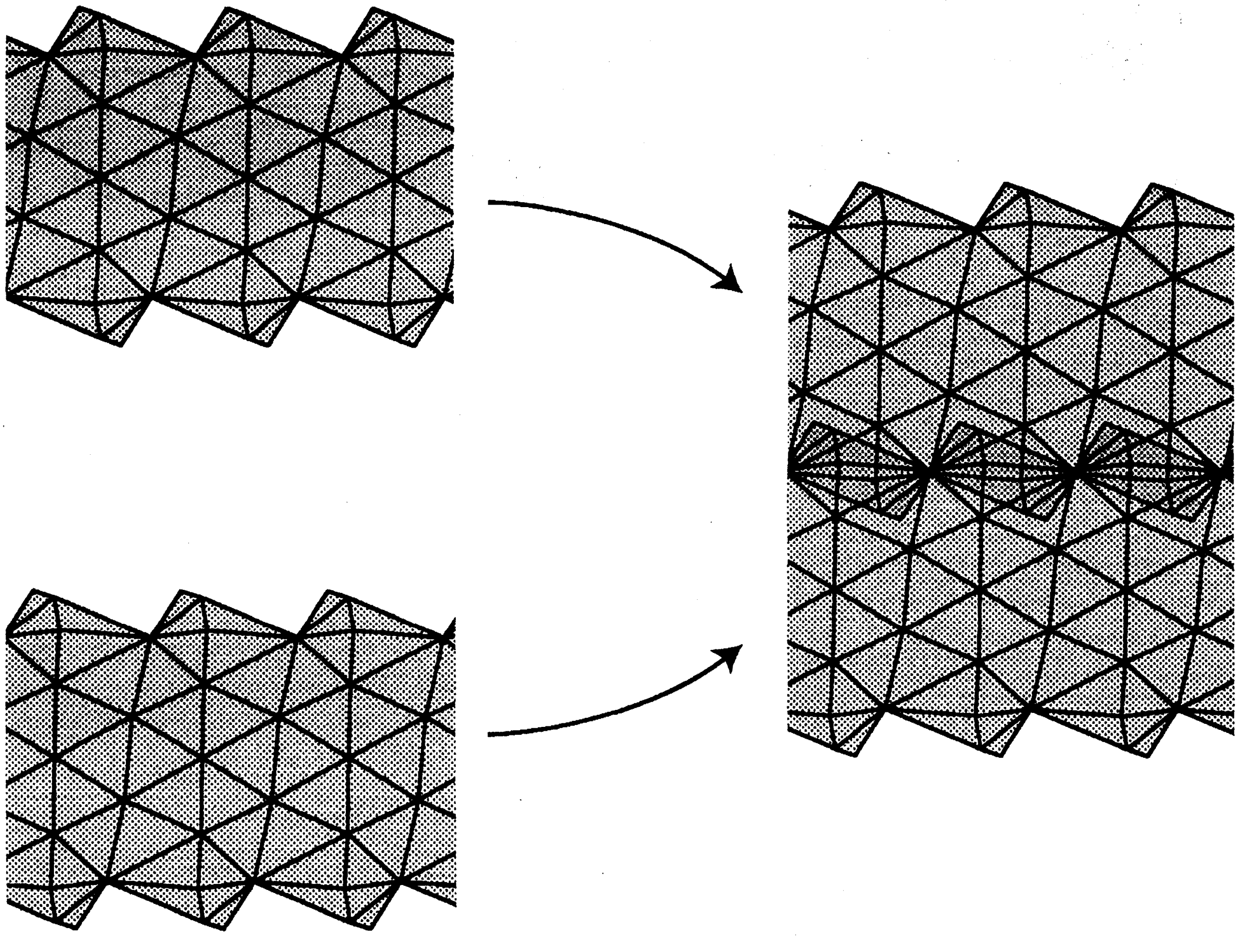

Figure 10: Idea of construction of the dual complex (1): The dual complex is based on two copies of the dual complex for a quasifuchsian punctured torus group

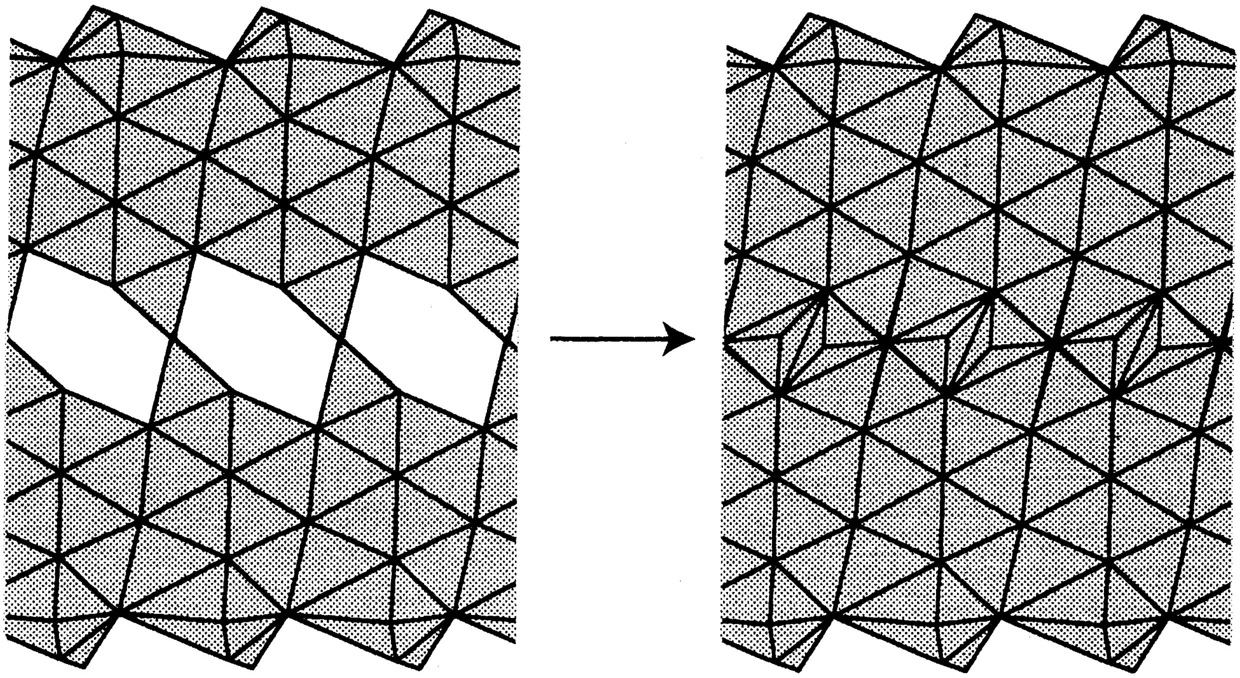

Figure 11: Idea of construction of the dual complex (2): By drilling and filling with the dual complex for a cyclic Kleinian group, we obtain the desired complex

according to the change of combinatorial structures of the Ford domains.

Question 5.6. How is the Ford domain for a hyperbolic structure in $\mathcal{MP}(p,q)-\mathcal{J}^{\mathrm{thick}}_{\mathrm{sym}}$ characterized?

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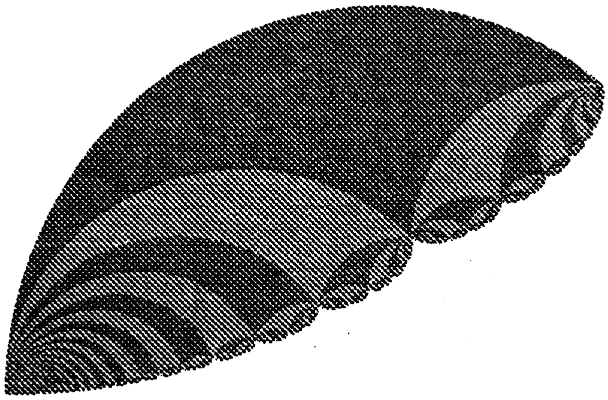

Figure 12: Conjectural picture of $\mathcal{J}_{\mathrm{sym}}$ for $(p, q) = (3, 5)$

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Osaka City University Advanced Mathematical Institute,
Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan
e-mail: akiyoshi@sci.osaka-cu.ac.jp