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Arithmetic lattices and weak spectral geometry

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Abstract

This note is an expansion of three lectures given at the workshop *Topology, Complex Analysis and Arithmetic of Hyperbolic Spaces* held at Kyoto University in December of 2006 and will appear in the proceedings for this workshop.

Introduction

Our attention in this note will be on the non-exceptional real rank one symmetric spaces arising from the simple Lie groups $\mathrm{SO}(n,1)$, $\mathrm{SU}(n,1)$, and $\mathrm{Sp}(n,1)$ and finite volume quotients of these spaces. These spaces and their quotients are known as *real, complex, and quaternionic hyperbolic n–space* and *real, complex, and quaternionic hyperbolic n–manifolds*, respectively. For these spaces, our aim is 2-fold:

(✠) Provide a description of some of the *arithmetic* quotients of these symmetric spaces.

(✠) Produce interesting examples of closed quotients of these symmetric spaces with regard to various spectral problems.

These two goals are essentially independent, although in general the former is the only means we have for producing examples in general; in particular, to achieve the latter we are forced to consider arithmetic constructions. We shall take a leisurely and loose approach to these goals, providing some background but largely leaving assertions unproven. The reader interested in more detail and rigor is directed to the references provided below.

Organization of the article This note is organized into five sections. In the first section, we briefly recall the definitions of real, complex, and quaternionic hyperbolic n–space. In the second section, we provide a description for constructing certain arithmetic lattices in the associated isometry groups for these spaces. In the third section, we discuss some recent results on isospectral manifolds modelled on these symmetric spaces (and more general symmetric spaces of noncompact type). In the four section, we discuss some recent work on weaker spectral constructions. In the fifth section, we discuss some variants of Sunada's method used to produce the asserted examples from Section 4.

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1 Hyperbolic spaces

For completeness, a short section introducing real, complex, and quaternionic hyperbolic space, their isometry groups, and their orbifold quotients is provided below. The reader should look to [48], [14], and [21] for more thorough treatments of this material.

Notation Throughout, X will denote either $\mathbf{R}, \mathbf{C}$, or $\mathbb{H}$. On X, we have the involution $*$ defined by

$$x^* = \begin{cases} \text{identity}, & X = \mathbf{R} \\ \text{complex conjugation}, & X = \mathbf{C} \\ \text{quaternionic conjugation}, & X = \mathbb{H}. \end{cases}$$

We extend this to a map on matrices

$$*\colon \mathrm{M}(r,s;X) \longrightarrow \mathrm{M}(s,r;X)$$

by applying $*$ to the coefficients of the matrix and then taking its transpose.

The standard model form and the projective model For what follows, we set

$$I_{n,1} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & -1 \end{pmatrix},$$

and call this the *standard form*. More to the point, associated to $I_{n,1}$ is the (bilinear, hermitian, or quaternionic hermitian) form

$$B_{n,1}(x,y) = y^* I_{n,1} x,$$

where $x, y \in X^{n+1}$ are viewed as column vectors. On X^{n+1}, we define the set

$$\mathcal{V} = \left\{ x \in X^{n+1} \;:\; B_{n,1}(x,x) < 0 \right\}.$$

The X–projectivization of $\mathcal{V}$, namely the set of $B_{n,1}$–negative X–lines $\mathcal{L}_X^n$, can be equipped with a metric

$$d([x],[y]) = \cosh^{-1}\left(\frac{1}{2} \frac{B_{n,1}(x,y) B_{n,1}(y,x)}{B_{n,1}(x,x) B_{n,1}(y,y)} \right).$$

The metric space $(\mathcal{L}_X^n, d)$ is called *X–hyperbolic n–space* and we denote this metric space by $\mathbf{H}_X^n$.

Isometry groups Associated to $B_{n,1}$ (or $I_{n,1}$) is the real Lie group

$$\mathrm{SU}(B_{n,1};X) = \left\{ A \in \mathrm{M}(n+1;X) \ : \ I_{n,1}^{-1}A^* I_{n,1} A = I_{n+1} \right\}.$$

The identity component of the associated projective group $\mathrm{PSU}(B_{n,1};X)$ acts on $\mathbb{P}X^n$ and leaves invariant $\mathcal{L}_X^n$. It is a simple matter to see that $\mathrm{PSU}(B_{n,1};X)$ preserves the metric d upon noting that for all $x,y \in X^{n+1}$, the elements of $\mathrm{SU}(B_{n,1};X)$ are precisely those linear transformations A such that

$$B_{n,1}(Ax,Ay) = B_{n,1}(x,y).$$

The group $\mathrm{PSU}(B_{n,1};X)$ is, up to finite index, the full isometry group of the metric space $\mathbf{H}_X^n$. For notational simplicity, we use the traditional notation:

$$\begin{aligned} \mathrm{PSU}(B_{n,1};\mathbf{R}) &= \mathrm{PSO}(n,1) \\ \mathrm{PSU}(B_{n,1};\mathbf{C}) &= \mathrm{PSU}(n,1) \\ \mathrm{PSU}(B_{n,1};\mathbb{H}) &= \mathrm{PSp}(n,1). \end{aligned}$$

Lattices and manifolds Given a torsion free, discrete subgroup Γ of $\mathrm{Isom}(\mathbf{H}_X^n)$, the quotient $\mathbf{H}_X^n/\Gamma$ is Riemannian manifold which is locally isometric to $\mathbf{H}_X^n$. We call such manifolds *X–hyperbolic manifolds*. When $\mathbf{H}_X^n/\Gamma$ has finite volume, we say Γ is a *lattice* and if in addition $\mathbf{H}_X^n/\Gamma$ is compact, we say Γ is *cocompact*. According to the Strong Rigidity Theorem (see [37] and [45]), there is a bijection between the isometry classes of finite volume X–hyperbolic n–manifolds and the $\mathrm{Isom}(\mathbf{H}_X^n)$–conjugacy classes of lattices in $\mathrm{Isom}(\mathbf{H}_X^n)$. Consequently, to understand the former it suffices to understand the latter and we will only be concerned with lattices in $\mathrm{Isom}(\mathbf{H}_X^n)$ up to *wide commensurability*. Recall $\Gamma_1,\Gamma_2 < G$ are commensurable in the wide sense if $[\Gamma_j : g^{-1}\Gamma_1 g \cap \Gamma_2] < \infty$ for some $g \in G$ and $j = 1,2$.

2 Arithmetic constructions

In this section, we introduce a general construction for lattices in $\mathrm{Isom}(\mathbf{H}_X^n)$. Before commencing with this task, we provide an overview on nonarithmetic manifolds. In the case of $\mathbf{H}_{\mathbf{R}}^n$, nonarithmetic lattices exist in every dimension (see [18]). However, in high dimensions, these manifolds are *hybrids* arising from gluing pairs of carefully chosen arithmetic ones along totally geodesic hypersurfaces. In the case of $\mathbf{H}_{\mathbb{H}}^n$, for $n > 1$, every lattice is arithmetic by rigidity theorems of Corlette [10] and Gromov–Schoen [17]. In the case of $\mathbf{H}_{\mathbf{C}}^n$, the story is far less complete. Nonarithmetic lattices are known to exist when $n = 2,3$ by work of Mostow [38] and Deligne–Mostow [11]. In higher dimensions it is unknown whether or not nonarithmetic lattices exist. With this said, we hope that this section will provide those interested but not familiar with arithmetic constructions some basic knowledge on constructing arithmetic lattices. A more detailed introduction can be found in [59].

2.1 Two basic arithmetic examples

The *first* example of an arithmetic lattice is the subgroup $\mathbf{Z}^n \subset \mathbf{R}^n$. The quotient $\mathbf{R}^n/\mathbf{Z}^n$ is the standard flat n–torus (upon equipping $\mathbf{R}^n$ with the geometry induced from the standard inner product). The lattice $\mathbf{Z}^n$ provides us with another example, namely the subgroup $\mathrm{SL}(n;\mathbf{Z}) < \mathrm{SL}(n;\mathbf{R})$ of those elements of $\mathrm{SL}(n;\mathbf{R})$ which preserve $\mathbf{Z}^n$. To be complete, we must say in which sense this is a lattice, and this is done as follows. We can equip $\mathrm{SL}(n;\mathbf{R})$ with a volume form ω which is invariant under both left and right translation in $\mathrm{SL}(n;\mathbf{R})$. For instance, if we select ω_{id} in $\Lambda^{\dim \mathrm{SL}(n;\mathbf{R})} T_{\mathrm{id}} \mathrm{SL}(n;\mathbf{R})$, a volume form on the tangent space of $\mathrm{SL}(n;\mathbf{R})$ at the identity element, we define ω_g in $\Lambda^{\dim(\mathrm{SL}(n;\mathbf{R}))} T_g \mathrm{SL}(n;\mathbf{R})$ to be the image of ω_{id} under the map induced by the isomorphism

$$dR_{g^{-1}} : T_g \mathrm{SL}(n;\mathbf{R}) \longrightarrow T_{\mathrm{id}} \mathrm{SL}(n;\mathbf{R}),$$

where $R_{g^{-1}}$ is the diffeomorphism of $\mathrm{SL}(n;\mathbf{R})$ given by right multiplication by g^{-1}. The volume form ω provides $\mathrm{SL}(n;\mathbf{R})$ with a measure via integration and as it is invariant under $\mathrm{SL}(n;\mathbf{Z})$, descends to a measure on the quotient space $\mathrm{SL}(n;\mathbf{R})/\mathrm{SL}(n;\mathbf{Z})$. It is with respect to this measure that the quotient space $\mathrm{SL}(n;\mathbf{R})/\mathrm{SL}(n;\mathbf{Z})$ has finite volume.

More generally, if G is a locally compact topological group equipped with a right Haar measure μ, for any discrete subgroup Γ of G, the quotient space G/Γ comes equipped with the induced quotient measure.[1] We say Γ is a *lattice* if G/Γ has finite volume with respect to this measure. If in addition G/Γ is compact, we say Γ is a *cocompact lattice*. As $\mathbf{H}^n_X$ is the coset space of $\mathrm{Isom}(\mathbf{H}^n_X)$ modulo a maximal compact subgroup K, this definition and the one given above specific to $\mathrm{Isom}(\mathbf{H}^n_X)$ coincide. This identification on the level of sets is made by using the transitive action of $\mathrm{Isom}(\mathbf{H}^n_X)$ on $\mathbf{H}^n_X$ and the fact that point stabilizers are maximal compact subgroups of $\mathrm{Isom}(\mathbf{H}^n_X)$.

[1] A right Haar measure is a regular Borel measure on G which is invariant under the right action of G on itself. It is well known that every locally compact topological group admits a right Haar measure.

2.2 Lattices arising from forms

We start the generalization of the above pair of examples to $\mathrm{Isom}(\mathbf{H}^n_X)$ with perhaps the most elementary construction based on bilinear, hermitian, and quaternionic hermitian forms over $\mathbf{R}, \mathbf{C}$, and $\mathbb{H}$, respectively. We call this construction the *form construction* (for $X = \mathbf{C}$, we sometimes refer to this as the *first type construction*).

Model forms and the basic examples We say $B \in \mathrm{GL}(n+1;X)$ is *$*$–symmetric* if $B = B^*$ and say that a $*$–symmetric matrix $B \in \mathrm{GL}(n+1;X)$ is a *model form* if B has signature pair $(n,1)$. That is, upon diagonalizing B, all the eigenvalues are real and precisely n of the eigenvalues are positive. For a subring $R \subset X$, we say that B is *R–defined* if B can be conjugated into $\mathrm{GL}(n+1;R)$.

The simplest example of a model form is $I_{n,1}$ which is R–defined for any subring R of X containing $\mathbf{Z}$. Setting

$$\mathcal{O}_X = \begin{cases} \mathbf{Z}, & X = \mathbf{R}, \\ \mathbf{Z}[i], & X = \mathbf{C}, \\ \mathbf{Z}[i,j,k], & X = \mathbb{H}, \end{cases}$$

by work of Borel–Harish-Chandra [4], $\mathrm{PSU}(n,1;\mathcal{O}_X)$ is a lattice in $\mathrm{Isom}(\mathbf{H}^n_X)$. The proof of this takes the only possible route, constructing a finite volume fundamental set for the action of $\mathrm{PSU}(n,1;\mathcal{O}_X)$ on $\mathbf{H}^n_X$. Actually, one constructs a fundamental set for the action of $\mathrm{PSU}(n,1;\mathcal{O}_X)$ on $\mathrm{Isom}(\mathbf{H}^n_X)$ using *reduction theory* (see [44]).

More generally, for any $\mathcal{O}_X$–defined model form B in $\mathrm{GL}(n+1;X)$, we have a real Lie group

$$\mathrm{PSU}(B;X) = \left\{A \in \mathrm{SL}(n+1;X) \ : \ B^{-1}A^*BA = I_{n+1}\right\}$$

with subgroup $\mathrm{PSU}(B;\mathcal{O}_X)$. Selecting a real analytic isomorphism between $\mathrm{PSU}(B;X)$ and $\mathrm{PSU}(n,1;X)$ (one can take this to be conjugation in $\mathrm{GL}(n+1;X)$), the image of$\mathrm{PSU}(B;\mathcal{O}_X)$ is a lattice in $\mathrm{Isom}(\mathbf{H}^n_X)$.

One interesting side note is that for $X = \mathbf{R}$, ranging over all the possible forms B, the above construction produces infinitely many distinct wide commensurability classes of lattices. While for $X = \mathbf{C}$ or $\mathbb{H}$, this produces one wide commensurability class. To produce additional wide commensurability classes over $\mathbf{C}$ and $\mathbb{H}$, one must change the ring $\mathcal{O}_X$.

Using work of Kneser (see [41]), Borel–Harish-Chandra [4], and Mostow–Tamagawa [36], the lattices $\mathrm{PSU}(B,\mathcal{O}_X)$ are noncocompact for all B when $X = \mathbb{H}$, for all B when $X = \mathbf{C}$ and $n > 1$, and for all B when $X = \mathbf{R}$ and $n > 3$. In particular, we have not yet found a construction for producing cocompact lattices in $\mathrm{Isom}(\mathbf{H}^n_X)$.

Cocompact examples in $\mathrm{PSO}(n,1)$ For a finite field extension $k/\mathbf{Q}$, there are, up to the field isomorphisms of $\mathbf{R}$ and $\mathbf{C}$, finitely many embeddings

$$\sigma_1,\ldots,\sigma_{r_1}: k \longrightarrow \mathbf{R}, \quad \tau_1,\ldots,\tau_{r_2}: k \longrightarrow \mathbf{C}$$

where for the latter we insist that $\tau_j(k)$ not be contained in $\mathbf{R}$. For instance, when $k=\mathbf{Q}(\sqrt{2})$, we have a pair of embeddings which we identify with the elements of $\mathrm{Gal}(\mathbf{Q}(\sqrt{2})/\mathbf{Q})$. We say k is *totally real* if $r_2=0$ and *totally imaginary* if $r_1=0$. Moreover, given a totally real extension F of $\mathbf{Q}$, by adjoining $\sqrt{-d}$ to F where $d\in\mathbf{N}$ is square-free, we (generically) obtain a totally imaginary quadratic extension E/F of F. We call the pair E/F a *CM field* (CM stands for complex multiplication). Finally, $\mathcal{O}_k$ shall denote the ring of algebraic k–integers.

For a totally real field k, fix an embedding $\sigma_1: k\longrightarrow\mathbf{R}$. For a k–defined model form $B\in\mathrm{GL}(n+1;k)$, for each $\sigma_j\neq\sigma_1$, we obtain a new form ${}^{\sigma_j}B$ with signature pair (p_j,q_j) by applying σ_j to the matrix B. We say that B is *admissible* if

$$(p_j,q_j)=(n+1,0)$$

for all $j\neq 1$. Again, work of Borel–Harish-Chandra implies that $\mathrm{PSO}(B;\mathcal{O}_k)$ is a lattice in $\mathrm{Isom}(\mathbf{H}^n_{\mathbf{R}})$.

Example. For $k=\mathbf{Q}(\sqrt{2})$, we can take B to be

$$B=\begin{pmatrix}1&0&\ldots&0&0\\0&1&\ldots&0&0\\\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\ldots&1&0\\0&0&\ldots&0&-\sqrt{2}\end{pmatrix}.$$

For the nontrivial Galois involution σ, the resulting form

$${}^{\sigma}B=\begin{pmatrix}1&0&\ldots&0&0\\0&1&\ldots&0&0\\\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\ldots&1&0\\0&0&\ldots&0&\sqrt{2}\end{pmatrix}$$

is positive definite as required.

Indeed, for any totally real field F, the Weak Approximation Theorem allows for the selection of $\alpha_1,\ldots,\alpha_{n+1}\in\mathcal{O}_F$ such that

$$B=\mathrm{diag}(\alpha_1,\ldots,\alpha_{n+1})$$

is admissible. It follows from Borel–Harish-Chandra and Mostow–Tamagawa that these lattices $\mathrm{PSO}(B;\mathcal{O}_F)$ are cocompact for any $F\neq\mathbf{Q}$.

Cocompact examples in $\mathrm{PSU}(n,1)$ For $X = \mathbf{C}$, we can take a CM field E/F and select a model form B defined over F which is admissible. Viewing B instead as a hermitian matrix and taking instead the associated group $\mathrm{PSU}(B;\mathbf{C})$, the subgroup $\mathrm{PSU}(B;\mathcal{O}_E)$ is a lattice in $\mathrm{PSU}(n,1)$ and is cocompact so long as $F \neq \mathbf{Q}$.

Cocompact examples in $\mathrm{PSp}(n,1)$ To produce lattices in $\mathrm{PSp}(n,1)$, we need some new algebraic objects. For a totally real field F and $\alpha,\beta \in F$, we define

$$A_{\alpha,\beta} = \left(\frac{\alpha,\beta}{F}\right)$$

to be the 4–dimensional F–algebra spanned by $1,x,y,xy$ (as a F–vector space) with multiplication given by

$$x^2 = \alpha, \quad y^2 = \beta, \quad xy = -yx, \quad \lambda x = x\lambda, \quad \lambda y = y\lambda$$

for all $\lambda \in F$. The algebra $A_{\alpha,\beta}$ is called a *F–quaternion algebra.* For each embedding σ_j of F into $\mathbf{R}$, we obtain a new algebra

$${}^{\sigma_j}A \otimes_F \mathbf{R} = \left(\frac{\sigma_j(\alpha),\sigma_j(\beta)}{\mathbf{R}}\right),$$

and according to a theorem of Wedderburn (see [43]),

$${}^{\sigma_j}A \otimes_F \mathbf{R} \cong \mathbb{H} \text{ or } \mathrm{M}(2;\mathbf{R}).$$

We require A have the property that ${}^{\sigma_j}A \otimes_F \mathbf{R} \cong \mathbb{H}$ for all j. For a model for $B \in \mathrm{GL}(n+1;A)$, we say B is admissible as before if the signature pair for all $j \neq 1$ is $(n+1,0)$. Taking $\alpha,\beta \in \mathcal{O}_F$, we have the subring $\mathcal{O} = \mathcal{O}_F[1,x,y,xy]$, $\mathrm{SU}(B;\mathcal{O})$ is a lattice in $\mathrm{PSp}(n,1)$ by work of Borel–Harish-Chandra.

Remark. Up to wide commensurability, one can take B to reside in $\mathrm{GL}(n+1;F)$ (indeed, B can be assumed to be diagonal with coefficients in $\mathcal{O}_F$).

2.3 Arithmetic constructions in general

In this short subsection, we give a quick overview on arithmetic lattices in $\mathrm{Isom}(\mathbf{H}^n_X)$. In particular, we mention how typical the above examples are and when there exist additional constructions of arithmetic lattices.

In $\mathrm{PSp}(n,1)$ The lattices constructed above yield all arithmetic lattices in $\mathrm{PSp}(n,1)$ up to wide commensurability so long as $n \neq 1$. In this exceptional case, there is an isometry between $\mathbf{H}^4_{\mathbf{R}}$ and $\mathbf{H}^1_{\mathbb{H}}$.

In $\mathrm{PSO}(n,1)$ For $n+1$ odd, this produces all the arithmetic lattices in $\mathrm{PSO}(n,1)$. For n odd and not equal to 3 or 7, there is but one other construction in $\mathrm{PSO}(n,1)$ which utilizes quaternion algebras. The case of $n=3$ is exceptional due to a local isomorphism between $\mathrm{SO}(3,1)$ and $\mathrm{SL}(2;\mathbf{C})$. In the case $n=7$, there is another arithmetic construction coming from triality algebras. This construction is possible due to an unusually large symmetry group for the associated Dynkin diagram for the associated complex simple Lie group.

In $\mathrm{PSU}(n,1)$ For $X=\mathbf{C}$, each pair $r,d \in \mathbf{N}$ such that $rd=n+1$ has an associated arithmetic construction. The pair $r=n+1$ and $d=1$ is the one given above and produces the arithmetic lattices of first type. For the pair $r=1$ and $d=n+1$, the construction utilizes cyclic division algebras A over CM fields E/F equipped with an involution of second kind. Essentially nothing is known about the associated complex hyperbolic manifolds produced by these lattices (see [29] and [55] for some recent work on these lattices); perhaps the deepest result is the vanishing of first cohomology for congruence covers of the associated arithmetic manifolds (see [51]). One such example is Mumford's fake $\mathbf{CP}^2$ [39] (see also [47]), which has the same rational homology as $\mathbf{CP}^2$. For brevity, we have chosen to omit a detailed description of these constructions and refer the reader to our preliminary manuscript [31] on this topic.

2.4 Why care about arithmetic and nonarithmetic constructions?

It is natural to ask why one should care about arithmetic and nonarithmetic constructions. Or more to the point, why arithmetic constructions produce amenable examples for geometers to work with. Here is a loose summary of "properties" typical arithmetic and nonarithmetic lattices and manifolds possess:

Arithmetic

- Predictable nature of group elements; see for instance Cooper–Long–Reid [9].
- Predictable nature of totally geodesic submanifolds and geodesics; see for instance Maclachlan–Reid [26].
- Symmetry; see for instance Farb–Weinberger [12] and Cooper–Long–Reid [9].
- Downside; it is difficult to find an explicit description like a group presentation (see [25]).

<u>Nonarithmetic</u>:

- Margulis dichotomy (see below or [28], [59]); see for instance Step 3 below.
- Usually "explicitly" constructed.
- Downside; Substructures (like totally geodesic submanifolds) are more mysterious.
- Downside; For most symmetric spaces, only arithmetic constructions are possible (see [28] or [59]).

3 Spectral geometry

Associated to any Riemannian n–manifold M are several sets which encode some portion of the geometry and topology of M. Perhaps the most natural (from a geometric viewpoint) of these sets is the *geodesic length spectrum* consisting of the lengths of the closed geodesics γ on M, where each length is counted with multiplicity. We denote this set by $\mathcal{L}(M)$. We could instead insist that the geodesics be primitive or simple and this produces the *primitive geodesic length spectrum* and *simple geodesic length spectrum* which we denote by $\mathcal{L}_p(M), \mathcal{L}_s(M)$, respectively. If we forget the multiplicities of these sets, we call the resulting set the (primitive or simple) *geodesic length set* and denote it by $L(M)$ (resp., $L_p(M), L_s(M)$).

Another natural set to associate to M is the spectrum of the Laplace–Beltrami operator acting on the Hilbert space $\mathrm{L}^2(M)$ of square integrable functions of M. More generally, this operator acts on the Hilbert space of square integrable p–forms and we denote the spectra for this operator on these spaces by $\mathcal{E}(M)$ and $\mathcal{E}_p(M)$. Given a pair of isometric Riemannian n–manifolds M,N, we have equality among the spectra for the pair. The so-called inverse problem asks if the converse holds.

Question (Inverse Problem). *If $\mathcal{E}(M) = \mathcal{E}(N)$, are M and N isometric?*

In 1964, Milnor [33] answered this question in the negative by producing a pair of nonisometric flat 16-tori with equal eigenvalue spectra. Since Milnor's article, many additional examples have been given. Most notably for us is a construction due to Sunada [56] which is purely algebraic. For brevity alone, we shall speak in detail only on this construction and refer the reader to the survey [15] for an detailed overview on the subject of isospectral constructions.

3.1 Sunada's method

Sunada's construction (which itself was inspired by number theory; see [42]) utilizes the following group theoretic concept.

Definition 1. *For a finite group G, we call a pair of subgroup H,K* ***almost conjugate*** *if for each G–conjugacy class $[g]$, we have the equality*

$$|H \cap [g]| = |K \cap [g]|.$$

For a Riemannian n–manifold M whose fundamental group $\pi_1(M)$ surjects G, it is an easy exercise to verify $\mathcal{L}(M_H) = \mathcal{L}(M_K)$ for the metric covers corresponding to the pullbacks of H and K. That these covers also have equal eigenvalue spectra follows from the equivalence of almost conjugacy with the following condition.

(♠) For every finite dimensional complex representation

$$\rho : G \longrightarrow \mathrm{GL}(n;\mathbf{C})$$

we have the equality

$$\dim \mathrm{Fix}(\rho(H)) = \dim \mathrm{Fix}(\rho(K)).$$

Given the equivalence of Definition 1 and (♠), it is not difficult to prove that $\mathcal{E}(M_H) = \mathcal{E}(M_K)$.

3.2 Using Sunada's method

Using known examples of almost conjugate pairs H,K, Sunada [56] produced many new examples of isospectral, nonisometric hyperbolic 2–manifolds. Since then, examples of isospectral hyperbolic n–manifolds for every n were found (see [3], [7], [49], [27], [57]). For complex and quaternionic hyperbolic manifolds, Spatzier [53] (see also [54]) found examples so long as the dimension is sufficiently high. Recently, we completed his work [30], finding examples in every dimension. Both of these constructions utilize Sunada's method. Indeed, the work involved in applying Sunada's method is showing the manifolds are nonisometric. The main tool we use for this is recent work of Belolipetsky–Lubotzky [2]. Briefly, the main points of our construction are:

(Step 1) Find families of finite groups N_j with r_j pairwise almost conjugate, nonconjugate subgroups $\{H_{j,k}\}_{k=1}^{r_j}$. Important here is that r_j tends to infinity as a function of j.

(Step 2) For a manifold M, find surjective homomorphisms $\pi_1(M) \twoheadrightarrow N_j$.

(Step 3) Find bounds on the number of ways a given cover can be isometric to another cover of M associated to the pullbacks of $H_{j,k}$ under the surjections of $\pi_1(M)$ to N_j.

It is worth noting that our approach was inspired by the approach taken by Belolipetsky and Lubotzky [1] in the resolution of the inverse Galois problem for isometry groups of closed hyperbolic n–manifolds. As a somewhat lengthy side note, we describe this philosophy employed in [1].

One approach to the inverse Galois problem for Riemann surfaces is as follows (see [16] for rigorous treatment, [22] for the inverse Galois problem for hyperbolic 3–manifolds, and [24] for the general case of the trivial group). Using the largeness of surface groups, given a finite group G, one can find a surjective homomorphism $\pi_1(\Sigma_g) \twoheadrightarrow G$. For each hyperbolic structure on $\pi_1(\Sigma_g)$, one obtains a hyperbolic structure on the cover corresponding to the pullback of the trivial group under the surjection of $\pi_1(\Sigma_g)$ onto G. In particular, these hyperbolic structures always have G as a subgroup of their isometry groups; this provides an embedding of the Teichmüller space of Σ_g into the Teichmüller space of the cover. Loosely, when the hyperbolic structure possesses more symmetry than G, these structures sit on an embedded copy of the Teichmüller space of a smaller surface. In particular, generic structures on the image of Teich(Σ_g) have precisely G for their isometry group.

Belolipetsky–Lubotzky [1] proceed in a similar manner to produce hyperbolic n–manifolds with isometry group G. The real and obvious sticking point is the lack of a Teichmüller space due to Strong Rigidity. The variational method in their approach becomes discrete; they produce t covers of a large, nonarithmetic manifold and by a counting argument show that some (generically) of these covers must have precisely G for their isometry group.

Sunada [56] (see also [5]) takes a similar approach for symmetry groups but instead to produce isospectral, nonisometric Riemann surfaces. Via largeness of $\pi_1(\Sigma_g)$, one is afforded surjective homomorphisms $\pi_1(\Sigma_g) \twoheadrightarrow G$, where G possesses an almost conjugate pair H,K. This produces two copies the Teichmüller space for Σ_g in the Teichmüller space of the surface corresponding to H (or equivalent K). By selecting a hyperbolic metric on Σ_g with trivial isometry group, which by the same reasoning above, occurs generically, the lifted metrics on the covers corresponding to the pullbacks of H,K are nonisometric (and by Sunada's theorem, isospectral).

With this view, our approach in [30] was to replace the continuous variational approach of Sunada with a discrete variational approach as done by Belolipetsky–

Lubotzky. The first two steps aim to produce large families of isospectral covers while the third step replaces the Baire category argument used in a continuous variational approach.

3.3 A sketch of how to achieve the basic steps

For completeness, we provide a sketch of how the three steps to our approach are achieved.

Step One The starting point for our approach (aside from Sunada's paper [56]) is a paper of Brooks, Gornet, and Gustafson [5].

For any field k, we define the *3–dimensional Heisenberg group over k* to be

$$\mathfrak{N}_3(k) = \left\{ \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \; : x,y,t \in k \right\}.$$

Via the inclusion of $\mathrm{GL}(3;k)$ into $\mathrm{GL}(n+3;k)$ into the upper three by three block, we may view $\mathfrak{N}_3(k)$ as a subgroup of $\mathrm{GL}(n+3;k)$ for all $n \geq 0$. The *horizontal subgroup*

$$H(k) = \left\{ \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \; : x \in k \right\}$$

and twists of it will produce the sought after $H_{j,k}$. Specifically, for a finite field $\mathbb{F}_q$ with $q = p^n$, Brooks–Gornet–Gustafson [5] found large (depending on p and n) collections of pairwise almost conjugate, nonconjugate subgroups of the finite groups $\mathfrak{N}_3(\mathbb{F}_q)$ by "twisting" the horizontal subgroup $H(\mathbb{F}_q)$ by certain maps $f\colon \mathbb{F}_q \longrightarrow \mathbb{F}_q$. Recall that $\mathbb{F}_q$ is simultaneously an n–dimensional $\mathbb{F}_p$–vector space and a 1–dimensional $\mathbb{F}_q$–vector space. The set of $\mathbb{F}_p$–linear endomorphisms is a $\mathbb{F}_p$–vector space with the $\mathbb{F}_q$–linear endomorphisms sitting as an $\mathbb{F}_p$–linear subspace. Upon selecting an $\mathbb{F}_p$–basis, the former may be identified with $\mathrm{M}(n;\mathbb{F}_p)$ and the latter with $\mathbb{F}_q$. The quotient $\mathbb{F}_p$–vector space $\mathrm{AL}(\mathbb{F}_q)$ of $\mathrm{M}(n;\mathbb{F}_p)$ by $\mathbb{F}_q$ will be called the *space of twist maps*. For simplicity in what follows, we fix a splitting

$$\mathrm{M}(n;\mathbb{F}_q) = \mathbb{F}_q \oplus \mathrm{AL}(\mathbb{F}_q)$$

which exists by elementary linear algebra.

Given a $\mathbb{F}_p$–linear endomorphism f of $\mathbb{F}_q$, we define the *f–twisted horizontal subgroup* ${}^f H(\mathbb{F}_q)$ to be

$$ {}^f H(\mathbb{F}_q) = \left\{ \begin{pmatrix} 1 & x & f(x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : x \in \mathbb{F}_q \right\}. $$

The following lemma is due to Brooks–Gornet–Gustafson [5].

Lemma 3.1. *For any pair of $\mathbb{F}_p$–linear endomorphisms f, g, the subgroups ${}^f H(\mathbb{F}_q)$ and ${}^g H(\mathbb{F}_q)$ are almost conjugate in $\mathfrak{N}_3(\mathbb{F}_q)$ and conjugate in $\mathfrak{N}_3(\mathbb{F}_q)$ if and only if $f - g \in \mathbb{F}_q$.*

An immediate consequence of Lemma 3.1 is the existence of $p^{n(n-1)}$ pairwise almost conjugate, nonconjugate subgroups $\{ {}^f H(\mathbb{F}_q) \}_{f \in \mathrm{AL}(\mathbb{F}_q)}$ of $\mathfrak{N}_3(\mathbb{F}_q)$.

Step Two The resolution of Step 2 is on the one hand a formal matter, appealing to well known results from number theory and the structure theory of algebraic groups. On the other hand, it is the most technical step in our approach. For this reason, we have opted to omit a lengthy discussion of how this is achieved. The main points are:

- The Strong Approximation Theorem (see [40] and [58]).
- Existence of algebraic F–forms $\mathbf{G}$ of the complexification of model semisimple group G with certain properties; for instance $\mathbf{G}$ is an inner form and F has certain desired properties.
- Ensuring that the groups $\mathbf{G}$ contain Heisenberg groups.

Step Three The resolution of Step 3 splits naturally into two cases. Having achieved Steps 1 and 2 for a manifold M, we split our considerations into two cases depending on whether or not M is arithmetic. In the case M is nonarithmetic, extremely good bounds on the number of ways finite covers of M can be isometric are obtained from deep work of Margulis [28]. In the case M is arithmetic, we appeal to work of Belolipetsky–Lubotzky [2].

4 What do the multiplicities see?

Though the isometry type of a manifold is not preserved under isospectrality, certain quantities like volume and dimension are when passing between isospectral manifolds. One basic question that can be asked is:

Question. *How much geometric information is encoded in the multiplicities? For example, is volume an invariant of the spectral set without multiplicities?*

In [52], Schmutz produced infinitely many pairs of finite covers of the modular quotient $\mathbf{H}^2_{\mathbf{R}}/\mathrm{PSL}(2;\mathbf{Z})$ with identical geodesic length sets but with different volume (thus producing a negative answer to the second part of the above question). The proof utilized the structure of $\mathrm{PSL}(2;\mathbf{Z})$, using some elementary matrix calculations; in particular, it is not a method which appears to be easy to generalize to other settings or even other lattices in $\mathrm{PSL}(2;\mathbf{R})$. Recently, with Leininger, Neumann, and Reid [23], we investigated this question and specifically the question of how abundant such examples are. Here are some of our results:

Theorem 4.1 ([23]). *Let M be a closed X–hyperbolic n–manifold. Then there exists an infinite family of finite covers (M_j,N_j) of M such that*

(1) $L_p(M_j) = L_p(N_j)$,

(2) $\mathrm{vol}(M_j)/\mathrm{vol}(N_j)$ *is unbounded as a function of j.*

Theorem 4.2 ([23]). *Let M be a closed X–hyperbolic n–manifold. Then there exists an infinite family of finite covers (M_j,N_j) of M such that*

(1) $E(M_j) = E(N_j)$,

(2) $\mathrm{vol}(M_j)/\mathrm{vol}(N_j)$ *is unbounded as a function of j.*

These results are achieved with variations of Sunada's method. Below, we briefly describe the group theoretic conditions.

It is not too difficult to show that two Riemannian manifolds with identical geodesic length spectra do indeed have identical primitive geodesic length spectra. Moreover, for compact locally symmetric manifolds, the eigenvalue spectrum is known to determine the primitive geodesic length spectrum, at least up to multiplication by rational numbers (see [46]). For negatively curved manifolds, there is a even stronger relations between the eigenvalue and primitive geodesic length spectra (see [13]), and for Riemannian surfaces, one can recover each from the other (see [19], [20], [6]). It might then come as a surprise that these implications typically fail upon forgetting the multiplicities. Specifically, in [23], we construct examples (typically Riemann surfaces) with the following properties:

- $L(M_1) = L(M_2)$ but $L_p(M_1) \neq L_p(M_2)$.
- $L(M_1) = L(M_2)$ but $E(M_1) \neq E(M_2)$.

- $E(M_1) = E(M_2)$ but $L_p(M_1) \neq L_p(M_2)$.
- $E(M_1) = E(M_2)$ but $L(M_1) \neq L(M_2)$.

Of course, one always has the implication that when $L_p(M_1) = L_p(M_2)$, then $L(M_1) = L(M_2)$. Thus the only remaining relation is whether or not the equality $L_p(M_1) = L_p(M_2)$ implies the equality $E(M_1) = E(M_2)$. There seems to be no reason to expect this to either hold or fail.

Using examples constructed in [8], one can produce examples of closed hyperbolic 3–manifolds M_1, M_2 with $L_s(M_1) = L_s(M_2)$ with arbitrarily large volume gap. However, this does not address how much geometric content is encoded in the simple length set as the manifolds M_j, $j = 1, 2$, have the remarkable property that any manifold commensurable to M_j possesses only simple closed geodesics. Heuristically, one expects closed geodesics on an X–hyperbolic n–manifolds to be simple generically, so long as the manifold is not a Riemannian surface.[2] This leads us to a pair of questions which we view as fundamental:

Question. *Do there exist distinct Riemann surfaces X_1, X_2 such that $L_s(X_1) = L_s(X_2)$?*

Question. *Do there exist distinct Riemann surfaces X_1, X_2 such that $\mathcal{L}_s(X_1) = \mathcal{L}_s(X_2)$?*

In the latter case, it is not immediately obvious that X_1, X_2 are homeomorphic. However, using known asymptotic upper and lower bounds on the number of simple closed geodesics on a Riemann surface ([34], [35]), it follows that $X_1 \cong X_2$, topologically. One reason to perhaps expect more geometric content in the simple length spectrum is the fact that one can determine the Riemann surface knowing only the length of a special finite collection of closed curves on the surface. Nevertheless, it seems too early to conjecture simple length spectral rigidity for Riemann surfaces.

To the author's knowledge, equality of simple geodesic length sets is not known to imply that the surfaces are topologically equivalent. Rivin [50] has conjectured that the multiplicities in the simple geodesic length spectrum are bounded (independent of the hyperbolic structure); the multiplicity is known to be one for a generic surface by a straightforward Baire category argument (see for instance [32]). If Rivin's conjecture holds, then the simple geodesic length set would determine the topological type by again appealing to the asymptotic growth rate of simple closed

[2] At present, it is unknown whether or not every finite volume hyperbolic n–manifold possesses infinitely many simple closed geodesics up to free homotopy.

geodesics. Indeed, we only require that the multiplicities in the spectrum be relatively small in comparison to the number of simple closed curves.

Remark. For flat tori, despite the fact that simple multiplicity need not be bounded (see [32]), one can find linear bounds on the simple multiplicities as a function of length. Indeed, one can make a coarse geometric argument using the isoparametric inequality for $\mathbf{R}^2$ to see this. It seems plausible that even if Rivin's conjecture is false that one might be able to produce polynomial bounds on the simple multiplicity as a function of length.

Finally, for length, primitive, and simple geodesic length sets, the number of pairwise distinct surfaces of genus g which can be pairwise length, primitive, or simple geodesic length equivalent is finite. Indeed, by continuity of length such a set is discrete in the moduli space of genus g curves and contained in a compact set of $\mathcal{M}_g$ by Mumford's compactness criterion.

5 Using symmetry in spectral constructions: Sunada's method and some variants

In the next three subsections, the associated Sunada-type group theoretic condition will be given for length, eigenvalue, and primitive length set equivalence.

5.1 Elementwise conjugate

Our first definition is motivated from Definition 1.

Definition 2. *Given a group G (not necessarily finite) and a pair of subgroups $H,K < G$, we say H,K are **elementwise conjugate** if*

$$\bigcup_{g\in G} g^{-1}Hg = \bigcup_{g\in G} g^{-1}Kg.$$

If G is finite, this is equivalent to:

(♣) *for all G–conjugacy classes $[g]$,*

$$H\cap[g]\neq\emptyset \text{ if and only if } K\cap[g]\neq\emptyset.$$

The following is one of the main examples used in [23] to produce manifolds with equal geodesic length sets (for instance examples of closed hyperbolic n–manifolds in every dimension).

Example. Let p be an odd prime, $\mathbb{F}_p$ the (unique) finite field with p elements, $G = \mathbb{F}_p^n \rtimes \mathrm{SL}(n;\mathbb{F}_p)$, $H = W$, and $K = V$, where $W, V \subset \mathbb{F}_p^n$ are non-trivial $\mathbb{F}_p$–subspaces. The inclusion of $\mathbb{F}_p^n$ into G provides us with a pair of subgroups H, K in G. The transitivity of the action of $\mathrm{SL}(n;\mathbb{F}_p)$ on the set of $\mathbb{F}_p$–lines in $\mathbb{F}_p^n$ is enough to imply that H, K are elementwise conjugate in G.

5.2 Fixed point equivalent

Our next definition is motivated by (♠).

Definition 3. *We say subgroups H and K of a finite group G are* ***fixed point equivalent*** *if for any finite dimensional complex representation ρ of G, the restriction $\rho|_H$ has a nontrivial fixed vector if and only if $\rho|_K$ does.*

It is not true that Definitions 2 and 3 are equivalent unlike the equivalence of Definition 1 and (♠). This is the first indication that relationships upon forgetting multiplicities could be more subtle.

The elementwise conjugate examples above also produce fixed point equivalent pairs with a slightly different condition on the subspaces V, W.

Example. With $G = \mathbb{F}_p^n \rtimes \mathrm{SL}(n;\mathbb{F}_p)$, if $H = W$, and $K = V$, where $W, V \subset \mathbb{F}_p^n$ are proper $\mathbb{F}_p$–subspaces, then H, K are fixed point equivalent subspaces of G. The proof of this uses standard results from character theory in tandem with an elementary counting argument.

5.3 Primitive pairs

Our final group theoretic concept does not fit into the general pattern taken with the previous two. Nevertheless, this condition does produce manifolds with equal primitive geodesic length sets.

Definition 4 (Primitive). *We shall call a subgroup H of G* ***primitive in*** *G if the following holds:*

(a) All non-trivial cyclic subgroups of H have the same order p (necessarily prime).

(b) $\bigcap_{g \in G} g^{-1}Hg = \{1\}$.

As before, primitive pairs can be found in $\mathbb{F}_p^n \rtimes \mathrm{SL}(n;\mathbb{F}_p)$.

Example. Setting G as before to be the affine group $\mathbb{F}_p^n \rtimes \mathrm{SL}(n;\mathbb{F}_p)$, if $H = W$, and $K = V$, where $W, V \subset \mathbb{F}_p^n$ are proper, nontrivial $\mathbb{F}_p$–subspaces, then H, K are primitive and elementwise conjugate; (a) is trivial to verify while (b) again follows from the transitivity of the action of $\mathrm{SL}(n;\mathbb{F}_p)$ on the set of $\mathbb{F}_p$–lines.

5.4 A variant of Sunada's theorem

One of the main results of [23] is the following variation on Sunada's theorem.

Theorem 5.1. *Let M be a Riemannian manifold, G a group, and H and K elementwise conjugate subgroups of G.*

(1) If $\pi_1(M)$ admits a homomorphism onto G, then $L(M_H) = L(M_K)$ for the covers M_H and M_K associated to the pullback subgroups of H and K.

(2) If, in addition, H and K are primitive in G and $\pi_1(M)$ has the property that any pair of distinct maximal cyclic subgroups of Γ intersect trivially, then $L_p(M_H) = L_p(M_K)$.

(3) If instead H and K are fixed point equivalent, then $E(M_H) = E(M_K)$.

The reader will note that on top of being less natural in regard to the associated group theoretic condition, the production of primitive geodesic length equivalent manifolds also requires conditions on the fundamental group $\pi_1(M)$ of the Riemannian manifold. The condition on maximal cyclic subgroups required in our proof is likely not needed (that some condition is required is seen from examples in [23]).

5.5 The existence of weak spectrally equivalent covers

To prove our results in the generality stated above (i.e., for *any* closed hyperbolic n–manifold), one can typically work with the examples of pairs H,K given above. In dimensions 3,4 however, other examples are required. These pairs are similar to those given above being subgroups A_1,A_2 of a fixed abelian p–group A which in turn is embedded in a semidirect product $A \rtimes \theta$ for some $\theta < \mathrm{Aut}(A)$. The virtual surjection of $\pi_1(M)$ onto groups of this form follows from the Strong Approximation and Cebotarev Density Theorems. The lion's share of the work is in showing that these pairs A_1,A_2 are primitive, elementwise conjugate, and eigenvalue equivalent.

These methods also work to produce covers over any closed X–hyperbolic n–manifold. In addition, one can also produce arbitrarily long towers of covers

$$M_r \longrightarrow M_{r-1} \longrightarrow \ldots \longrightarrow M_2 \longrightarrow M_1 \longrightarrow M$$

such that each pair M_j, M_k is length, primitive length, or eigenvalue equivalent. These methods also work more generally for locally symmetric manifolds of noncompact type.

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