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PRE-BLOCH INVARIANT FOR 3-MANIFOLD WITH HIGHER GENUS BOUNDARY

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ABSTRACT. This is an exposition of the author's paper [3].

1. Introduction

In [5], Neumann and Yang defined the Bloch invariant for oriented hyperbolic 3-manifold with finite volume. The Bloch invariant is defined on the Bloch group and has intimate relation with volume of the manifold and Chern-Simons invariant.

In this exposition, we generalize the Bloch invariants for infinite volume hyperbolic 3-manifolds. Unlike finite volume hyperbolic manifolds, this invariant is not invariant of a hyperbolic manifold. In this case, we essentially need a boundary condition. The boundary condition is given by pants decomposition.

2. Definition of the Bloch invariant

The pre-Bloch group $\mathcal{P}(\mathbb{C})$ is the quotient of the free abelian group generated by $\mathbb{C}-\{0,1\}$ factored by the relation:

$$[x]-[y]+[\frac{y}{x}]-[\frac{1-x^{-1}}{1-y^{-1}}]+[\frac{1-x}{1-y}]=0. \tag{2.1}$$

We denote $[z]$ the class which contains z. Then we define a map

$$\lambda:\mathcal{P}(\mathbb{C})\to\mathbb{C}^*\wedge_{\mathbb{Z}}\mathbb{C}^*,\quad [z]\mapsto 2(z\wedge_{\mathbb{Z}}(1-z)),$$

where we regard $\mathbb{C}^*$ as an abelian group by multiplication. For example, $xy\wedge_{\mathbb{Z}}z=x\wedge_{\mathbb{Z}}z+y\wedge_{\mathbb{Z}}z$.

Theorem 2.1 (Bloch-Wigner, Dupon-Sah [2]).
(2.2)

$$0\to\mathbb{Q}/\mathbb{Z}\xrightarrow{f_1}H_3(\mathrm{PSL}(2,\mathbb{C}),\mathbb{Z})\xrightarrow{f_2}\mathcal{P}(\mathbb{C})\xrightarrow{\lambda}\mathbb{C}^*\wedge_{\mathbb{Z}}\mathbb{C}^*\xrightarrow{f_3}H_2(\mathrm{PSL}(2,\mathbb{C}),\mathbb{Z})\to 0$$

is exact. ($H_i(\mathrm{PSL}(2,\mathbb{C}),\mathbb{Z})$ represents i-th homology of group!)

In the above exact sequence, f_1 is defined by the composition $\mathbb{Q}/\mathbb{Z}\subset H_3(\mathbb{C}^*,\mathbb{Z})\to H_3(\mathrm{PSL}(2,\mathbb{C}),\mathbb{Z})$. f_2 is defined by $[g_1|g_2|g_3]\mapsto[[z:g_1z:g_1g_2z:g_1g_2g_3z]]$ where $[g_1|g_2|g_3]$ is the bar notation and $[z_0:z_1:z_2:z_3]=\dfrac{(z_2-z_1)(z_3-z_0)}{(z_2-z_0)(z_3-z_1)}$. f_3 is the composition $\mathbb{C}^*\wedge_{\mathbb{Z}}\mathbb{C}^*\cong H_2(\mathbb{C}^*,\mathbb{Z})\to H_2(\mathrm{PSL}(2,\mathbb{C}),\mathbb{Z})$.

We define the Bloch group by

$$\mathcal{B}(\mathbb{C})=\mathrm{Ker}(\mathcal{P}(\mathbb{C})\to\mathbb{C}^*\wedge_{\mathbb{Z}}\mathbb{C}^*).$$

Neumann and Yang introduced the Bloch invariant by the following way. An ideal tetrahedron is a geodesic 3-simplex on $\mathbb{H}^3$ with each vertex at infinity $\mathbb{C}P^1$. An oriented ideal tetrahedron with prescribed edge is parametrized by a complex number. Let M be an oriented complete finite volume hyperbolic manifold. Let $M = \Delta(z_1) \cup \cdots \cup \Delta(z_n)$ be an ideal triangulation (we omit the precise definition of ideal triangulation. For example, Thurston's hyperbolic Dehn surgery produces an ideal triangulation. See detail in [5].) Then we define $\beta(M) = \sum_{\nu=1}^{n}[z_\nu]$. The very definition depends on the triangulation and only states that $\beta(M) \in \mathcal{P}(\mathbb{C})$, but Neumann and Yang showed that

Theorem 2.2 (Neumann-Yang [5]). *$\beta(M)$ is an invariant of M and $\beta(M) \in \mathcal{B}(\mathbb{C})$.*

If M is closed, the discrete faithful representation of $\pi_1(M)$ induces $\rho_0 : H_3(M) \to H_3(\mathrm{PSL}(2,\mathbb{C}))$ and $\beta(M)$ is equal to $f_2((\rho_0)_*([M]))$ where $[M]$ is the fundamental cycle of M.

Let M be a compact 3-manifold with torus boundary and ρ be a representation of $\pi_1(M)$ to $\mathrm{PSL}(2,\mathbb{C})$. Neumann and Yang also defined an invariant $\beta(M,\rho)$ in $\mathcal{P}(\mathbb{C})$. If ρ corresponds to the holonomy of hyperbolic Dehn filled manifold, then $\beta(M,\rho)$ is equal to the Bloch invariant of the Dehn filled manifold (in particular $\beta(M,\rho) \in \mathcal{B}(\mathbb{C})$). But in general $\beta(M,\rho)$ takes value in $\mathcal{P}(\mathbb{C})$. Let $(\mathcal{L},\mathcal{M})$ be a set of generators of $H_1(\partial M)$. Let L_0 and M_0 be the derivatives of holonomies along $\mathcal{L}$ and $\mathcal{M}$. Then we have

Theorem 2.3 (Neumann [4]).

$$\lambda(\beta(M,\rho)) = L_0 \wedge_{\mathbb{Z}} M_0.$$

This theorem states that a difference from $\mathcal{B}(\mathbb{C})$ can be expressed in terms of the representation of the boundary of M.

They also defined $\beta(M,\rho)$ for more general manifold. Let M be a compact 3-manifold with boundary S and ρ be a representation of $\pi_1(M)$ to $\mathrm{PSL}(2,\mathbb{C})$. They defined an invariant $\beta(M,\rho)$ if the restriction of ρ to $\partial M = S$ has a fixed point i.e. the restriction of ρ to the boundary is reducible. If the boundary has genus more than 1, this assumption is too strong. In fact the set of reducible surface representations has strictly smaller dimension than the set of all the surface representations.

3. Pants decomposition and ideal triangulation

Let M be a compact oriented 3-manifold. For simplicity, we assume that $S = \partial M$ is a connected surface of genus $g > 1$. The pants decomposition C of S is a maximal set of distinct isotopy classes of disjoint simple closed curves. The number of curves of C is $3g-3$. Let $\mathfrak{o}$ be orientations of the curves of C. The pair $(C,\mathfrak{o})$ defines ideal triangulation of S as follows. $S - C$ is a set of pairs of pants (3-holed spheres). Then each pair of pants P admits ideal triangulation by two ideal triangles so that the ideal vertices of ideal triangles spinning around the boundaries of the pair of pants to the direction $\mathfrak{o}$. Let T be an ideal triangulation which is coincide with the ideal triangulation of the boundary given by $(C,\mathfrak{o})$. We can imagine easily this situation if we attach truncated ideal 3-simplices each other. Truncated vertices make 2-dimensional triangles. When we attach 3-simplices along their faces to each other, then these triangles are attached to each other. These triangles form annuli which are the boundaries of a neighborhood of pants curves C. For each curve $\gamma_k \in C$, we denote these annuli by $A_k (k = 1,\ldots,3g-3)$. We can use more general ideal triangulation, but in this paper we use this definition for simplicity.

4. Representation of $\pi_1(M)$ and pre-Bloch invariant

Let M, C and $\mathfrak{o}$ be as in the last section. Let ρ be a representation of $\pi_1(M)$ to $\mathrm{PSL}(2,\mathbb{C})$. We assume that restriction of ρ to each pair of pants (3-holed sphere) is irreducible and the holonomies around boundaries of pair of pants are hyperbolic elements of $\mathrm{PSL}(2,\mathbb{C})$. Then $(C,\mathfrak{o},\rho)$ determines developing map of S uniquely up to conjugation as follows. Let P be a pair of pants of $S-C$. Let γ_1, γ_2, γ_3 be boundary curves of P. Let g_1, g_2, g_3 be elements of $\pi_1(P)$ which go around γ_1, γ_2, γ_3. Then $\rho(g_i)$ have distinct fixed points in $\mathbb{C}P^1$ by the assumption. Put ideal triangle in $\mathbb{H}^3$ so that ideal vertices are at fixed points of $\rho(g_i)$. Then we can develop this ideal triangle by the action of $\rho(\pi_1(P))$. Do this construction for each pair of pants, we get a developing map of S for ρ. By putting the ideal triangle at another position, we can conjugate the representation ρ.

After constructing developing map of ∂M, we extend it to the developing map of M by using ρ. The developing map defines a complex parameter for each ideal tetrahedron. We denote such complex number by z_ν. Then we define

Definition 4.1. $\beta(M,\rho,C,\mathfrak{o}) = \sum_{i=1}^{n}[z_i] \in \mathcal{P}(\mathbb{C})$.

Proposition 4.2. *$\beta(M,\rho,C,\mathfrak{o})$ only depends on M, ρ, C, $\mathfrak{o}$, and not on the choice of triangulation of M.*

5. Some properties of $\beta(M,\rho,C,\mathfrak{o})$

As in [6], we define the edge relation for each 1-simplex of T which is not facing to ∂M:

$$R_i = \pm\prod_{\nu=1}^{n} z_\nu^{r'_{i,\nu}}(1-z_\nu)^{r''_{i,\nu}} = 1 \quad (i=1,2,\dots,n-3(g-1)).$$

For 1-simplex of T which is facing to ∂M, we define a complex number by multiplication the complex numbers of edges which are adjacent to the 1-simplex:

$$B_i = \pm\prod_{\nu=1}^{n} z_\nu^{b'_{i,\nu}}(1-z_\nu)^{b''_{i,\nu}} \quad (i=1,2,\dots,6(g-1)).$$

Because the number of pairs of pants on $S-C$ is $2g-2$, the number of boundary 1-simplices are $6g-6$. Unlike R_i, B_i is not equal to 1. We call such a 1-simplex *boundary 1-simplex.* B_i measures how bent two ideal triangles at the boundary 1-simplex.

Take a loop h_k on A_k so that h_k is a generator of $H_1(A_k,\mathbb{Z})$. Take a path w_k of $(A_k,\partial A_k)$ so that w_k represents a generator of $H_1(A_k,\partial A_k,\mathbb{Z})$. Then we can define a complex number by multiplication complex parameters as torus boundary case (see, for example, [6].)

$$H_k = \pm\prod_{\nu=1}^{n} z_i^{h'_{k,\nu}}(1-z_i)^{h''_{k,\nu}}, \quad W_k = \pm\prod_{\nu=1}^{n} z_i^{w'_{k,\nu}}(1-z_i)^{w''_{k,\nu}} \quad (k=1,\dots,3(g-1)).$$

We call W_k a *twist parameter* of γ_k for $k=1,\dots,3(g-1)$. H_k depends only on the homology class of h_k and H_k represents the square of eigenvalue of the holonomy around the curve $\gamma_k \in C$. On the other hand W_k is not well-defined. If we deform the endpoint of w_k across the boundary 1-simplex i, then W_k changes to W_kB_i.

Let e_i, e_j be a boundary 1-simplex of a pair of pants P. e_i and e_j are intersect with common pants curve γ_k. Then we can observe that B_iB_j measures how bent ideal triangles on P around γ_k. In fact we can show $B_iB_j = H_k$. Since γ_k has two adjacent pair of pants, we denote the other one by P'. Then P' has two boundary 1-simplex which intersect with γ_k. We denote them by i' and j'. By above observation we have a relation $B_iB_j = H_k = B_{i'}B_{j'}$. So $(B_1, \ldots, B_{6g-6})$ is essentially $(6g-6)-(3g-3) = 3g-3$ dimensional object. Moreover we can represent B_i in terms of H_k's.

(H_k, W_k) reminds us the Fenchel-Nielsen coordinate. Fenchel-Nielsen coordinate define a coordinate of Teichmüller space by length and twist. A length is well-defined for given hyperbolic surface, on the other hand twist is not determined by given hyperbolic surface.

We have a version of Theorem 2.3 for higher genus boundary case.

Theorem 5.1.

$$\lambda(\beta(M, \rho, C, \mathfrak{o})) = \sum_{k=1}^{3(g-1)} H_k \wedge_{\mathbf{Z}} W_k.$$

We remark that W_k is not well defined as we mentioned, but the right hand side of the above equation is well-defined.

6. Volumes of representations

By using invariance of $\beta(M, \rho, C, \mathfrak{o})$, we can define a volume of a representation. We define $Li_2(z) = -\int_0^\infty \frac{\log(1-t)}{t} dt$ and

$$D(z) = \mathrm{Im} Li_2(z) + \log|z| arg(1-z) (z \in \mathbb{C} - \{0, 1\}).$$

For an ideal simplex with complex parameter z of ideal simplex, we can describe its volume by $D(z)$. D satisfies five term relation,

$$D(x) - D(y) + D(y/x) - D(\frac{1-x^{-1}}{1-y^{-1}}) + D(\frac{1-x}{1-y}) = 0.$$

So we have a homomorphism $D : \mathcal{P}(\mathbb{C}) \to \mathbb{R}$. We can define volume of $\beta(M, \rho, C, \mathfrak{o})$ by $D(\beta(M, \rho, C, \mathfrak{o}))$. We next consider the variation of volume in deformation space. Consider smooth family of representations of $\pi_1(M)$ to $\mathrm{PSL}(2, \mathbb{C})$ parametrized by t. Then the derivative of volume is

$$\frac{d\mathrm{Vol}}{dt} = -\frac{1}{2} \sum_{k=1}^{3(g-1)} (\log|W_k| \frac{d\arg(H_k)}{dt} - \log|H_k| \frac{d\arg(W_k)}{dt}).$$

This formula shows that the derivative of volume is written by in terms of the representations of the boundary. We remark that Bonahon proved a variation formula of volume for geometrically finite hyperbolic manifolds (see Theorem 3 of [1]). Bonahon' theorem shows the variation of volume bounded by pleated surface with fixed pleated locus geodesic lamination. In our case, the geodesic lamination is given by ideal triangulation by pants decomposition.

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