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A NOTE ON TOTALLY GEODESIC SUBGRAPHS OF THE PANTS GRAPH

(on joint work with Javier Aramayona and Hugo Parlier)

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Abstract

In this proceedings article we discuss joint work with Javier Aramayona and Hugo Parlier, studying the pants graph of a surface after Hatcher and Thurston and the subject of this author's talk. Inspired by a recent theorem of Brock's, the aim of this work is to reinforce the pants graph as a good combinatorial model for the Weil-Petersson metric.

KEYWORDS: Weil-Petersson metric; mapping class group; pants graph.

1 Introduction

When studying a complicated object in low dimensional geometry, such as the Weil-Petersson metric on Teichmüller space, it is sometimes helpful to introduce a combinatorial model to the setting and study this instead. Such models are usually constructed from curves on a given surface. Of course we then sacrifice much of the structure we had to begin with, be it analytic or algebraic, but in return we should expect to have greatly simplified matters.

An excellent example of such a model is the pants graph after Hatcher and Thuston, where an important theorem of Brock's tells us that this is the correct combinatorial model for the Weil-Petersson metric.

2 The pants graph

Let Σ be a compact, connected and orientable surface of genus $g(\Sigma)$ and $|\partial\Sigma|$ boundary components, and refer to as the *mapping class group* $\mathrm{Map}(\Sigma)$ the group of all self-homeomorphisms of Σ up to homotopy. We shall refer to as a *curve* on Σ the free homotopy class of any simple closed loop that neither bounds a disc nor an annulus containing a component of $\partial\Sigma$. (For example, if Σ has negative Euler characteristic, then Σ carries

2000 MSC: 57M50 (primary); 05C12 (secondary) a hyperbolic metric and any curve may then be uniquely represented by a simple closed geodesic. Conversely, every simple closed geodesic determines a curve.) A *multicurve* is then by definition a non-empty set of pairwise distinct and pairwise disjoint curves, and a *pants decomposition* a multicurve maximal subject to inclusion. Every pants decomposition cuts the surface into a disjoint union of 3-holed spheres. Two distinct pants decompositions are said to related by an *elementary move* if they agree on all but a pair of curves, either intersecting once or intersecting twice with zero algebraic intersection.

After Hatcher and Thurston [8], to the surface Σ one may associate a graph $\mathcal{P}(\Sigma)$, the *pants graph*, whose vertices are all the pants decompositions of Σ and any two vertices are connected by an edge if and only if they differ by an elementary move. Since this graph is connected, one may define a path-metric d on $\mathcal{P}(\Sigma)$ by first assigning length 1 to each edge and then regarding the result as a length space. We shall often refer to the pants graph by name or just by $\mathcal{P}$, suppressing the notation for the surface.

The pants graph, with its own geometry, is a fundamental object to study, for it appears in several major topics: Brock [4] revealed deep connections with volumes of hyperbolic 3-manifolds and proved the pants graph is the correct combinatorial model for the Weil-Petersson metric on Teichmüller space, for the two are quasi-isometric and thus share the same large-scale geometry. The mapping class group admits a natural action on the pants graph by isometries, and indeed it is a theorem of Margalit [9] that the isometry group of $(\mathcal{P}, d)$ is almost always isomorphic to the mapping class group. In addition, Masur-Schleimer [11] proved the pants graph of any closed surface of genus at least 3 is one-ended, so that the complementary graph of any bounded set of vertices has exactly one unbounded component. With only a few exceptions the pants graph is not hyperbolic in the sense of Gromov [5], for typically it contains a quasi-isometric copy of the Euclidean plane.

3 Motivation and main results

To appreciate the authors' motivation in writing [1] and [2] requires us to first recall some more background material: It is well known that the Weil-Petersson metric is not complete, for as noted by Wolpert [12] and by Chu [6], the lengths of simple closed geodesics can approach zero in finite time. The completion of the Weil-Petersson metric is in fact characterised by attaching so-called strata [10], and each corresponds bijectively with a multicurve on the surface. Each stratum is the lower dimensional Teichmüller space, or product of such spaces, associated to a noded surface, on which the length of each component of just the corresponding multicurve has degenerated to zero. It is a theorem of Wolpert [13] that each stratum is a totally geodesic subspace of the completed Weil-Petersson metric.

The quasi-isometry defined by Brock [4] extends naturally to the completion of the Weil-Petersson metric, sending each stratum to the subgraph of $\mathcal{P}$ spanned by all vertices containing the corresponding multicurve. It is an immediate consequence that each of these subgraphs of $\mathcal{P}$ is uniformly quasi-convex, so that each such subgraph has a uniform neighbourhood containing every geodesic connecting any two of its vertices. (For any two vertices x and y of a connected graph, by a *geodesic between x and y* we mean a shortest sequence of vertices, beginning with x and ending with y, where any consecutive pair spans an edge.)

The objective of [1] and [2] is to understand to what extent the geometry of the Weil-Petersson metric is replicated in the pants graph. In particular, we seek to establish (or disprove) the full combinatorial analogue of Wolpert's theorem, that, for any multicurve ω, the subgraph $\mathcal{P}_\omega$ spanned by every pants decomposition containing ω be totally geodesic. This remains an intriguing and open problem, owing largely to the demanding combinatorics of the pants graph.

The first results in this vein are recalled here as Theorem 1 and Theorem 2, from [1] and from [2] respectively.

Theorem 1 *Let Σ be a compact, connected and orientable surface containing at least two distinct and disjoint curves, and let ω be a codimension 1 multicurve on Σ, so that ω can be extended to a pants decomposition by adding a single curve. Then, the subgraph $\mathcal{P}_\omega$ of $\mathcal{P}(\Sigma)$ is totally geodesic.*

We remark that for any codimension 1 multicurve ω, the graph $\mathcal{P}_\omega$ is isomorphic to a Farey graph whenever Σ contains at least two distinct and disjoint curves. Indeed, one can show that all Farey subgraphs are thus accounted for – this is precisely Lemma 6 of [1], for instance.

To decipher the statement of the second theorem, recall that a multicurve on Σ is said to be a *2-handle multicurve* only if its complement is the union of 3-holed spheres and two further surfaces, each homeomorphic to either a 1-holed torus or a 4-holed sphere only one boundary component of which is to represent a curve. Similarly, the subgraph $\mathcal{P}_\omega$ of $\mathcal{P}$ spanned by all pants decompositions containing the 2-handle multicurve ω is isomorphic to the product of two Farey graphs.

Theorem 2 *Let Σ be a compact, connected and orientable surface, and denote by ω any 2-handle multicurve on Σ. Then, the subgraph $\mathcal{P}_\omega$ of $\mathcal{P}$ is totally geodesic.*

Considering Wolpert's theorem, Brock's theorem and both Theorem 1 and Theorem 2 above, Aramayona-Parlier-S have asserted the following conjecture.

Conjecture 3 *Let Σ be a compact and orientable surface (possibly disconnected), and let ω be one of its multicurves. Then, the subgraph $\mathcal{P}_\omega$ of $\mathcal{P}$ is totally geodesic.*

4 Applications

Let us indicate two consequences of Theorem 1. First, note that for any hyperbolic self-isometry f of a Farey graph, there exists a bi-infinite geodesic invariant under the action of f^2.

Corollary 4 *Let $f \in \mathrm{Map}(\Sigma)$ be any mapping class leaving invariant a subgraph of $\mathcal{P}(\Sigma)$ isomorphic to a Farey graph, on which it acts as a hyperbolic self-isometry. Then, there exists a bi-infinite geodesic in $\mathcal{P}(\Sigma)$ invariant under the action of f^2.*

We remark that examples of such mapping classes include those whose restriction to the complement of some complexity 1 subsurface Y is the identity and whose restriction to Y is a pseudo-Anosov mapping class. There exists an analogous corollary of Theorem 2, offering an invariant convex plane. It would be of much interest to extend this by, say, finding axes for sufficiently high powers of pure mapping classes made up only of commuting pseudo-Anosov pieces.

Second, let ω be a multicurve on Σ with the property that every complementary component of ω is a surface of complexity 1. Then, the subgraph of $\mathcal{P}(\Sigma)$ spanned by all pants decompositions containing ω is isomorphic to a product of Farey graphs, each totally geodesic by Theorem 1. Considering one bi-infinite geodesic in each Farey graph, we deduce the following. Note, by a *line* in the free abelian group $\mathbb{Z}^r$ we shall mean a coset of any one of the $\mathbb{Z}$-factors.

Corollary 5 *Let r denote the largest integer no greater than $(3g(\Sigma)+|\partial\Sigma|-3)/2$. There exists a quasi-isometric embedding from $\mathbb{Z}^r$, given the L^1-metric, into $\mathcal{P}(\Sigma)$ such that the image of any line is a geodesic.*

Thus, infinitely many of the maximal quasi-flats in $\mathcal{P}$ identified by the Geometric Rank Theorem [5, 3, 7] are convex in their principal directions. Considering the Geometric Rank Theorem in conjunction with Theorem 2, we know that the pants graph of the six-holed sphere, the pants graph of the three-holed torus, and the pants graph of the closed surface of genus two all contain convex maximal flats. Establishing the existence of convex maximal flats for those surfaces Σ with $3g(\Sigma)+|\partial\Sigma|-3 \geq 3$ remains an intriguing open problem.

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