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Index Reduction for Differential-Algebraic Equations by Discrete Optimization Techniques

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1 Introduction

Dynamical systems such as electric circuits, mechanical systems, and chemical plants are often described by differential-algebraic equations (DAEs), which consist of algebraic equations and differential operations. DAEs present numerical and analytical difficulties which do not occur with ordinary differential equations (ODEs).

Several numerical methods have been developed for solving DAEs. For example, Gear [6] proposed the backward difference formulae (BDF), which were implemented in the DASSL code by Petzold (cf. [2]). Hairer and Wanner [9] implemented an implicit Runge-Kutta method in their RADAU5 code.

The *index* concept plays an important role in the analysis of DAEs. The index is a measure of the degree of difficulty in the numerical solution. In general, the higher the index is, the more difficult it is to solve the DAE. While many different concepts exist to assign an index to a DAE such as the *differentiation index* [2, 4, 9], the *perturbation index* [3], and the *tractability index* [20], we focus on the *nilpotency index* in this paper. In the case of linear DAEs with constant coefficients, all these indices are equal [3, 19].

In order to transform a DAE into an alternative form easier to solve, some index reduction methods have been developed [7, 15, 16]. These methods introduce additional variables, which leads to a drawback that the resulting DAE is a larger system than the original one.

This paper focuses on linear DAEs with constant coefficients

$$A_0\boldsymbol{x}(t) + A_1\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = \boldsymbol{f}(t), \tag{1}$$

where A_0 and A_1 are constant matrices, to propose two index reduction methods.

The first one [21], based on the *substitution method*, always reduces by one the index of DAEs in the form of (1) such that A_1 has at most one nonzero entry in each row. This class of DAEs includes the semi-explicit form and most circuit equations (which consist of Kirchhoff's conservation laws and constitutive equations). The substitution method eliminates some variables by replacement to obtain a smaller system than the original one. In contrast to other existing methods [7, 15, 16], it does not introduce any additional variables.

The other one [13] is applicable to DAEs in circuit simulation. The most commonly used analysis method is the modified nodal analysis (MNA). However, the index of the DAE arising from MNA is determined uniquely by the structure of the circuit [20]. Hence there is no room to reduce the index in MNA. Instead, we consider a broader class of analysis method called the *hybrid analysis*. It is famous for the theory of minimizing the size of the *hybrid equations*, i.e., the system of equations to be solved numerically [10, 14, 18]. For linear time-invariant electric circuits, we devise an algorithm for finding an optimal hybrid analysis in which the index of the hybrid equations attains the minimum. The optimal hybrid analysis often results in a DAE with lower index than MNA.

The organization of this paper is as follows. In Section 2, we explain matrix pencils and the definition of the nilpotency index. In Section 3, we propose the index reduction method by the substitution method. Section 4 presents an algorithm for minimizing the index of the hybrid equations. Numerical examples are given in Section 5.

2 DAEs and Matrix Pencils

For a polynomial $a(s)$, we denote the degree of $a(s)$ by $\deg a$, where $\deg 0 = -\infty$ by convention. A polynomial matrix $A(s) = (a_{kl}(s))$ with $\deg a_{kl} \leq 1$ for all (k, l) is called a *matrix pencil.* Obviously, a matrix pencil $A(s)$ can be represented as $A(s) = A_0 + sA_1$ in terms of a pair of constant matrices A_0 and A_1. A matrix pencil $A(s)$ is said to be *regular* if $A(s)$ is square and $\det A(s)$ is a nonvanishing polynomial.

With the use of the Laplace transformation, the DAE in the form of (1) is expressed by the matrix pencil $A(s) = A_0 + sA_1$ as $A(s)\tilde{x}(s) = \tilde{f}(s)$, where s is the variable for the Laplace transform that corresponds to $\mathrm{d}/\mathrm{d}t$, the differentiation with respect to time.

Theorem 2.1 ([2, Theorem 2.3.1]). *The linear DAE with constant coefficients* (1) *is solvable if and only if $A(s)$ is a regular matrix pencil.*

The reader is referred to [2, Definition 2.2.1] for the precise definition of solvability. By Theorem 2.1, we assume that $A(s)$ is a regular matrix pencil throughout this paper. A regular matrix pencil is known to have the *Kronecker canonical form,* which determines the nilpotency index. Let N_μ denote a $\mu \times \mu$ matrix pencil defined by

$$N_\mu = \begin{pmatrix} 1 & s & 0 & \cdots & 0 \\ 0 & 1 & s & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 & s \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

A matrix pencil $A(s)$ is said to be *strictly equivalent* to $\tilde{A}(s)$ if $A(s)$ can be brought into $\tilde{A}(s)$ by an equivalence transformation with nonsingular constant matrices.

Theorem 2.2 ([5, Chapter XII, Theorem 3]). *An $n \times n$ regular matrix pencil $A(s)$ is strictly equivalent to its Kronecker canonical form:*

$$\begin{pmatrix} sI_{\mu_0} + J & O & O & \cdots & O \\ O & N_{\mu_1} & O & \cdots & O \\ O & O & N_{\mu_2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & O \\ O & O & \cdots & O & N_{\mu_b} \end{pmatrix},$$

where

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_b, \quad \mu_0 + \mu_1 + \mu_2 + \cdots + \mu_b = n,$$

and J is a $\mu_0 \times \mu_0$ constant matrix.

The matrices N_{μ_i} $(i = 1, \ldots, b)$ are called the *nilpotent blocks*. The maximum size μ_1 of them is the *nilpotency index*, denoted by $\nu(A)$. It is obvious that ODEs have index zero, and algebraic equations have index one.

We denote by $A[K, L]$ the submatrix of $A(s)$ with row set $K \subseteq R$ and column set $L \subseteq C$, where R and C are the row set and the column set of $A(s)$, respectively. Furthermore, we denote $w(K, L) = \deg\det A[K, L]$, where $w(\emptyset, \emptyset) = 0$ by convention. Then w enjoys the following property.

Lemma 2.3 ([17, pp. 287–289]). *Let $A(s)$ be a matrix pencil with row set R and column set C. For any $(K, L) \in \Lambda$ and $(K', L') \in \Lambda$, where $\Lambda = \{(K, L) \mid |K| = |L|, K \subseteq R, L \subseteq C\}$, both* (VB-1) *and* (VB-2) *below hold:*

(VB-1) *For any $k \in K \setminus K'$, at least one of the following two assertions holds:*

- **(1a)** $\exists l \in L \setminus L' : w(K, L) + w(K', L') \leq w(K \setminus \{k\}, L \setminus \{l\}) + w(K' \cup \{k\}, L' \cup \{l\})$,
- **(1b)** $\exists h \in K' \setminus K : w(K, L) + w(K', L') \leq w(K \setminus \{k\} \cup \{h\}, L) + w(K' \setminus \{h\} \cup \{k\}, L')$.

(VB-2) *For any $l \in L \setminus L'$, at least one of the following two assertions holds:*

- **(2a)** $\exists k \in K \setminus K' : w(K, L) + w(K', L') \leq w(K \setminus \{k\}, L \setminus \{l\}) + w(K' \cup \{k\}, L' \cup \{l\})$,
- **(2b)** $\exists j \in L' \setminus L : w(K, L) + w(K', L') \leq w(K, L \setminus \{l\} \cup \{j\}) + w(K', L' \setminus \{j\} \cup \{l\})$.

Let $\delta_r(A)$ denote the highest degree of a minor of order r in $A(s)$:

$$\delta_r(A) = \max_{K,L}\{w(K, L) \mid |K| = |L| = r, K \subseteq R, L \subseteq C\}.$$

The index $\nu(A)$ is determined from $\delta_r(A)$ as follows.

Theorem 2.4 ([17, Theorem 5.1.8]). *Let $A(s)$ be an $n \times n$ regular matrix pencil. The nilpotency index $\nu(A)$ is given by*

$$\nu(A) = \delta_{n-1}(A) - \delta_n(A) + 1.$$

3 Index Reduction for DAEs by Substitution Method

This section presents our first method of index reduction. In Section 3.1, we introduce the substitution method. Then, in Section 3.2, we show that the method reduces the index exactly by one if A_1 has at most one nonzero entry in each row.

3.1 Substitution Method

In this section, we introduce the substitution method for solving linear DAEs with constant coefficients.

Let $A(s)$ be an $n \times n$ regular matrix pencil with row set R and column set C, and B be a nonsingular constant submatrix of A with row set $P \subset R$ and column set $Q \subset C$. We transform A into $\tilde{A}$ by row operations:

$$A = \begin{pmatrix} B & F \\ G & H \end{pmatrix} \to \tilde{A} = \begin{pmatrix} I & O \\ -GB^{-1} & I \end{pmatrix} \begin{pmatrix} B & F \\ G & H \end{pmatrix} = \begin{pmatrix} B & F \\ O & H - GB^{-1}F \end{pmatrix}, \tag{2}$$

where $F = A[P, C \setminus Q], G = A[R \setminus P, Q]$, and $H = A[R \setminus P, C \setminus Q]$. We denote $H - GB^{-1}F$ by D, which is not necessarily a matrix pencil.

Let $\hat{B}$, $\hat{F}$, $\hat{G}$, $\hat{H}$, and $\hat{D}$ denote the matrices obtained by replacing s with $\mathrm{d}/\mathrm{d}t$ in B, F, G, H, and D, respectively. Consider the DAE

$$\hat{B}\boldsymbol{x}_1(t) + \hat{F}\boldsymbol{x}_2(t) = \boldsymbol{f}_1(t), \tag{3}$$

$$\hat{G}\boldsymbol{x}_1(t) + \hat{H}\boldsymbol{x}_2(t) = \boldsymbol{f}_2(t). \tag{4}$$

By applying the transformation shown in (2), we obtain

$$\hat{B}\boldsymbol{x}_1(t) = \boldsymbol{f}_1(t) - \hat{F}\boldsymbol{x}_2(t), \tag{5}$$

$$\hat{D}\boldsymbol{x}_2(t) = \boldsymbol{f}_2(t) - \hat{G}\hat{B}^{-1}\boldsymbol{f}_1(t). \tag{6}$$

Note that $\hat{B}$ is a constant matrix. The outline of the substitution method is as follows.

Phase 1: Solve the DAE (6) for $\boldsymbol{x}_2(t)$.

Phase 2: Solve the system of linear equations (5) for $\boldsymbol{x}_1(t)$.

In the substitution method, the numerical difficulty is determined by the index $\nu(D)$ of the DAE (6). We show that $\nu(D)$ can be expressed in terms of the degrees of minors in A.

For each $k \in R$ and $l \in C$, let d_{kl} denote the degree of $\det A[R \setminus \{k\}, C \setminus \{l\}]$. Then we have

$$d_{kl} = \deg\det \tilde{A}[R \setminus \{k\}, C \setminus \{l\}], \quad \forall k \in R \setminus P,\ \forall l \in C, \tag{7}$$

because we can transform $\tilde{A}[R \setminus \{k\}, C \setminus \{l\}]$ into $A[R \setminus \{k\}, C \setminus \{l\}]$ by row operations for each $k \in R \setminus P$ and $l \in C$. The index $\nu(D)$ can be rewritten as follows.

Theorem 3.1. *For an $n \times n$ regular matrix pencil $A(s)$, the index of D is given by*

$$\nu(D) = \max_{k,l}\{d_{kl} \mid k \in R \setminus P, l \in C \setminus Q\} - \delta_n(A) + 1. \tag{8}$$

Proof. We denote the size of D by m. By Theorem 2.4, we have $\nu(D) = \delta_{m-1}(D) - \delta_m(D) + 1$. Recall that $\tilde{A} = \begin{pmatrix} B & F \\ O & D \end{pmatrix}$ and that B is a constant matrix. It follows from $\det A = \det \tilde{A}$ that

$$\delta_m(D) = \deg\det D = \deg\det \tilde{A} - \deg\det B = \deg\det A.$$

Moreover, we have

$$\begin{aligned}
\delta_{m-1}(D) &= \max_{K,L}\{\deg\det D[K, L] \mid |K| = |L| = m - 1\} \\
&= \max_{K,L}\{\deg\det \tilde{A}[K, L] \mid |K| = |L| = n - 1, K \supseteq P, L \supseteq Q\} - \deg\det B \\
&= \max_{k,l}\{d_{kl} \mid k \in R \setminus P, l \in C \setminus Q\},
\end{aligned}$$

where the last step is due to (7). Thus we obtain (8). □

3.2 Index Reduction

Let $A(s) = A_0 + sA_1$ be an $n \times n$ regular matrix pencil such that A_1 has at most one nonzero entry in each row. We denote the row set of $A(s)$ by R, and the column set by C. Moreover, we assume that $\nu(A)$ is positive. Let $Q \subseteq C$ be the set of indices such that their column vectors in A_1 are zero vectors. Since $A[R,Q]$ has full column rank by the regularity of $A(s)$, we can find $P \subseteq R$ such that $A[P,Q]$ is regular. Note that because $B = A[P,Q]$ and $G = A[R \setminus P, Q]$ are constant matrices, $D = \tilde{A}[R \setminus P, C \setminus Q]$ is a matrix pencil. We prove that the index of D is one lower than that of A.

Lemma 3.2. *For each $k \in R$ and each $l \in C \setminus Q$, we have $d_{kl} < \delta_{n-1}(A)$.*

Proof. Suppose to the contrary that there exist $k \in R$ and $l \in C \setminus Q$ such that $d_{kl} = \delta_{n-1}(A)$. Let h be a row such that the (h,l) entry of A_1 is nonzero. We put $(K,L) = (\{h\},\{l\})$ and $(K',L') = (R \setminus \{k\}, C \setminus \{l\})$. By (VB-2) in Lemma 2.3, at least one of the following two assertions holds:

(2a) $h = k$, $w(\{h\},\{l\}) + w(R \setminus \{k\}, C \setminus \{l\}) \le w(\emptyset,\emptyset) + w(R,C)$,

(2b) $\exists j \in C \setminus \{l\} : w(\{h\},\{l\}) + w(R \setminus \{k\}, C \setminus \{l\}) \le w(\{h\},\{j\}) + w(R \setminus \{k\}, C \setminus \{j\})$.

Note that $w(\{h\},\{l\}) = 1$ and $w(R \setminus \{k\}, C \setminus \{l\}) = d_{kl} = \delta_{n-1}(A)$.

If (2a) holds, then it follows from $w(\emptyset,\emptyset) = 0$ and $w(R,C) = \delta_n(A)$ that $1+\delta_{n-1}(A) \le \delta_n(A)$, which implies $\nu(A) \le 0$ by Theorem 2.4. This contradicts $\nu(A) > 0$.

On the other hand, if (2b) holds, we have $1 + \delta_{n-1}(A) \le w(\{h\},\{j\}) + d_{kj}$. Since A_1 has at most one nonzero entry in each row, we have $w(\{h\},\{j\}) = 0$. Thus we obtain $1+\delta_{n-1}(A) \le d_{kj}$, which contradicts the definition of $\delta_{n-1}(A)$. □

Theorem 3.3. *The index of $D = \tilde{A}[R \setminus P, C \setminus Q]$ is exactly one lower than that of A.*

Proof. By Theorems 2.4 and 3.1 and Lemma 3.2,

$$\nu(A) - \nu(D) = \delta_{n-1}(A) - \max_{k,l}\{d_{kl} \mid k \in R \setminus P, l \in C \setminus Q\} > 0.$$

We now prove $\nu(D) \ge \nu(A) - 1$. It follows from Lemma 3.2 that there exist $k \in R$ and $l \in Q$ such that $d_{kl} = \delta_{n-1}(A)$.

Suppose that there exist $k \in R \setminus P$ and $l \in Q$ such that $d_{kl} = \delta_{n-1}(A)$. By applying (VB-2) in Lemma 2.3 to (P,Q) and $(R \setminus \{k\}, C \setminus \{l\})$, we have

$$\exists j \in C \setminus Q : w(P,Q) + w(R \setminus \{k\}, C \setminus \{l\}) \le w(P, Q \setminus \{l\} \cup \{j\}) + w(R \setminus \{k\}, C \setminus \{j\}).$$

Note that $w(P,Q) = 0$, because $A[R,Q]$ is a constant matrix. Since A is a matrix pencil and $A[P,Q]$ is a constant matrix, $w(P, Q \setminus \{l\} \cup \{j\}) \le 1$. Therefore, we have $d_{kl} \le d_{kj} + 1$, which implies $\nu(D) \ge d_{kj} - \delta_n(A) + 1 \ge d_{kl} - \delta_n(A) = \nu(A) - 1$ by Theorems 2.4 and 3.1.

We now consider the other case, which means that there exist $k \in P$ and $l \in Q$ such that $d_{kl} = \delta_{n-1}(A)$, and $d_{pq} < \delta_{n-1}(A)$ for any $p \in R \setminus P$ and $q \in Q$. By applying (VB-1) in Lemma 2.3 to (P,Q) and $(R \setminus \{k\}, C \setminus \{l\})$, at least one of the following assertions holds:

(1a) $w(P,Q) + w(R \setminus \{k\}, C \setminus \{l\}) \le w(P \setminus \{k\}, Q \setminus \{l\}) + w(R,C)$,

(1b) $\exists h \in R \setminus P : w(P,Q) + w(R \setminus \{k\}, C \setminus \{l\}) \le w(P \setminus \{k\} \cup \{h\}, Q) + w(R \setminus \{h\}, C \setminus \{l\})$.

Since $A[R,Q]$ is a constant matrix, we have $w(P,Q) = w(P\backslash\{k\}, Q\backslash\{l\}) = w(P\backslash\{k\}\cup\{h\}, Q) = 0$.

If (1a) holds, then we have $d_{kl} \leq \delta_n(A)$. Therefore, $\nu(A) = d_{kl} - \delta_n(A) + 1 \leq 1$ by Theorem 2.4. It follows from the nonnegativity of $\nu(D)$ that $\nu(D) \geq \nu(A) - 1$.

On the other hand, if (1b) holds, we have $d_{kl} \leq d_{hl}$. This contradicts the assumption that $d_{pq} < \delta_{n-1}(A)$ for any $p \in R \setminus P$ and $q \in Q$. □

Theorem 3.3 implies that the index of D is the same for any P with $A[P,Q]$ being a nonsingular constant matrix.

4 Hybrid Analysis

For linear time-invariant electric circuits, we propose a combinatorial algorithm for finding an optimal hybrid analysis in which the index of the DAE to be solved attains the minimum. Our method first finds a *degree matrix*, which is defined by cofactors in the associated polynomial matrix. Then, it makes use of the satisfiability problem for 2-CNF (2SAT). The time complexity of this algorithm is $O(n^6)$, where n is the number of elements in an electric circuit. We can improve the time complexity to $O(n^3)$ under the assumption that the set of nonzero entries coming from the physical parameters is algebraically independent.

We describe the procedure of the hybrid analysis in Section 4.1. Section 4.2 is devoted to a characterization of the index of the DAE to be solved in the hybrid analysis. Section 4.3 presents an index minimization algorithm.

4.1 Hybrid Analysis

In this section, we describe the procedure of the hybrid analysis. We focus on linear time-invariant electric circuits which are composed of resistances, capacitances, inductances, independent/dependent voltage sources, and independent/dependent current sources. For more complicated devices like transistors, there exist equivalent circuits which consist of the previously mentioned devices.

Let $\Gamma = (W,E)$ be a network graph with vertex set W and edge set E. An edge in Γ corresponds to a branch that contains one element in the circuit. We denote the set of edges corresponding to independent voltage sources and independent current sources by E_g and E_h, respectively. We split $E_* := E \setminus (E_g \cup E_h)$ into E_y and E_z, i.e., $E_y \cup E_z = E_*$ and $E_y \cap E_z = \emptyset$. Since the previous works [10, 14, 18] deal with circuits in the frequency domain, the hybrid analysis described therein can choose any partition (E_y, E_z). In order to deal with DAEs in the time domain, however, we need to consider a restricted class of partitions. A partition (E_y, E_z) is called an *admissible partition*, if E_y includes all the capacitances and dependent current sources, and E_z includes all the inductances and dependent voltage sources.

We now explain *circuit equations* for a linear time-invariant electric circuit. Let $\boldsymbol{\xi}$ denote the vector of currents through all branches of the circuit, and $\boldsymbol{\eta}$ the vector of voltages across all branches. We denote the *reduced cutset matrix* by Ψ and the *reduced loop matrix* by Φ. Using *Kirchhoff's current law* (KCL), which states that the sum of currents entering each node is equal to zero, we have $\Psi\boldsymbol{\xi} = \mathbf{0}$. Similarly, using *Kirchhoff's voltage law* (KVL), which states that the sum of voltages in each loop of the network is equal to zero, we have $\Phi\boldsymbol{\eta} = \mathbf{0}$. The physical characteristics of elements determine *constitutive equations*. Given an admissible partition (E_y, E_z), we split ξ and η into

$$\xi = \begin{pmatrix} \xi_g \\ \xi_y \\ \xi_z \\ \xi_h \end{pmatrix} \quad \text{and} \quad \eta = \begin{pmatrix} \eta_g \\ \eta_y \\ \eta_z \\ \eta_h \end{pmatrix},$$

where the subscripts correspond to the partition of E. Circuit equations, which consist of KCL, KVL, and constitutive equations, are described by

$$\left(\begin{array}{cccc|cccc} \multicolumn{4}{c|}{\Psi} & \multicolumn{4}{c}{O} \\ \hline \multicolumn{4}{c|}{O} & \multicolumn{4}{c}{\Phi} \\ \hline O & I & \multicolumn{2}{|c|}{\multirow{2}{*}{Z(s)}} & \multicolumn{2}{c|}{\multirow{2}{*}{Y(s)}} & O & O \\ O & O & \multicolumn{2}{|c|}{} & \multicolumn{2}{c|}{} & I & O \\ \hline O & O & O & O & I & O & O & O \\ O & O & O & I & O & O & O & O \end{array}\right) \begin{pmatrix} \xi_g \\ \xi_y \\ \xi_z \\ \xi_h \\ \hline \eta_g \\ \eta_y \\ \eta_z \\ \eta_h \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ g(s) \\ h(s) \end{pmatrix} \tag{9}$$

after the Laplace transformation. The coefficient matrix $A(s)$ of the circuit equations is a matrix pencil. The row set and the column set of $A(s)$ are denoted by R and C, respectively.

We call a spanning tree T of Γ a *reference tree* if T contains all edges in E_g, no edges in E_h, and as many edges in E_y as possible. Note that T may contain some edges in E_z. The cotree of T is denoted by $\overline{T} = E \setminus T$.

Given an admissible partition (E_y, E_z), we denote the column sets of $A(s)$ corresponding to the current variables and the voltage variables for elements in E_g, E_y, E_z, E_h by I_g, I_y, I_z, I_h, and V_g, V_y, V_z, V_h, respectively. Moreover, given a reference tree T, we denote the column sets of $A(s)$ corresponding to the current variables and the voltage variables for elements in $E_y \cap T$ and $E_y \cap \overline{T}$ by I_y^τ, I_y^λ, and V_y^τ, V_y^λ, respectively. The superscripts τ and λ designate the tree T and the cotree $\overline{T}$. We define I_z^τ, I_z^λ, and V_z^τ, V_z^λ similarly. We also use $I^\tau = I_g \cup I_y^\tau \cup I_z^\tau$ and $V^\lambda = V_y^\lambda \cup V_z^\lambda \cup V_h$ for convenience. The row sets of $A(s)$ corresponding to KCL, KVL, and constitutive equations are denoted by R_I, R_V, and S, respectively.

Given an admissible partition (E_y, E_z) and a reference tree T, we transform $A(s)$ into $A_T(s)$ such that $A_T[R_I, I^\tau] = I$ and $A_T[R_V, V^\lambda] = I$ by row operations in $R_I \cup R_V$. This is possible because $A[R_I, I^\tau]$ and $A[R_V, V^\lambda]$ are nonsingular. Note that R_I and I^τ as well as R_V and V^λ have one-to-one correspondence. The row sets of $A_T(s)$ corresponding to I_g, I_y^τ, I_z^τ, and V_y^λ, V_z^λ, V_h are denoted by R_g, R_y^τ, R_z^τ, and R_y^λ, R_z^λ, R_h, where we have $A_T[K, L] = I$ if $K \subseteq R$ and $L \subseteq C$ have the same superscript and subscript. Similarly, the row sets corresponding to I_y, V_z, V_g, and I_h are denoted by S_y, S_z, S_g, and S_h. Let i_e and v_e denote the column corresponding to the current variable and the voltage variable for an element e. By the definition of a reference tree, $A_T(s)$ has the following property.

Lemma 4.1. *For a reference tree T, we have $A_T[R_z^\tau, I_y^\lambda] = O$ and $A_T[R_y^\lambda, V_z^\tau] = O$.*

Proof. Suppose to the contrary that there exists $e \in E_y \setminus T$ such that $A_T[R_z^\tau, \{i_e\}] \neq \mathbf{0}$. Then the unique cycle in $T \cup \{e\}$ is not contained in $E_y \cup E_g$. Hence, there exists an edge $f \in E_z \cap T$ such that $T \setminus \{f\} \cup \{e\}$ is a tree, which contradicts the assumption that T is a reference tree. Therefore, we have $A_T[R_z^\tau, I_y^\lambda] = O$. Similarly, we also have $A_T[R_y^\lambda, V_z^\tau] = O$. $\square$

Thus $A_T(s)$ is in the form of

$$A_T(s) = \begin{array}{cc} & \begin{array}{cccccccccccc} I_g & I_y^{T} & I_y^{\lambda} & I_z^{T} & I_z^{\lambda} & I_h & V_g & V_y^{T} & V_y^{\lambda} & V_z^{T} & V_z^{\lambda} & V_h \end{array} \\ \begin{array}{c} R_g \\ R_y^{T} \\ R_z^{T} \\ R_y^{\lambda} \\ R_z^{\lambda} \\ R_h \\ S_y \\ \\ S_z \\ \\ S_g \\ S_h \end{array} & \begin{pmatrix} I & O & * & O & * & * & O & O & O & O & O & O \\ O & I & * & O & * & * & O & O & O & O & O & O \\ O & O & O & I & * & * & O & O & O & O & O & O \\ O & O & O & O & O & O & * & * & I & O & O & O \\ O & O & O & O & O & O & * & * & O & * & I & O \\ O & O & O & O & O & O & * & * & O & * & O & I \\ O & I & O & ** & ** & ** & ** & ** & ** & O & O & O \\ O & O & I & ** & ** & ** & ** & ** & ** & O & O & O \\ O & O & O & ** & ** & ** & ** & ** & ** & I & O & O \\ O & O & O & ** & ** & ** & ** & ** & ** & O & I & O \\ O & O & O & O & O & O & I & O & O & O & O & O \\ O & O & O & O & O & I & O & O & O & O & O & O \end{pmatrix} \end{array}, \tag{10}$$

where $*$ means a constant matrix and $**$ means a matrix pencil. We can determine $A_T(s)$ only after being given both an admissible partition (E_y, E_z) and a reference tree T.

We now consider the transformation shown in (2) for $P = R \backslash (R_y^{T} \cup R_z^{\lambda})$ and $Q = C \backslash (I_z^{\lambda} \cup V_y^{T})$. We call the resulting DAE (6) the *hybrid equations*. Let us denote the vectors of currents corresponding to I_g, I_y^{T}, I_y^{λ}, I_z^{T}, I_z^{λ}, I_h by ξ_g, ξ_y^{T}, ξ_y^{λ}, ξ_z^{T}, ξ_z^{λ}, ξ_h, and the vectors of voltages corresponding to V_g, V_y^{T}, V_y^{λ}, V_z^{T}, V_z^{λ}, V_h by η_g, η_y^{T}, η_y^{λ}, η_z^{T}, η_z^{λ}, η_h. The procedure of the hybrid analysis is as follows:

1. The values of ξ_h and η_g are obvious from the equations corresponding to S_h and S_g.
2. Find the values of ξ_z^{λ} and η_y^{T} by solving the *hybrid equations* (6).
3. Compute the values of ξ_z^{T} and η_y^{λ} by substituting the values obtained in Steps 1–2 into the equations corresponding to R_z^{T} and R_y^{λ}.
4. Compute the values of ξ_y^{T}, ξ_y^{λ}, η_z^{T}, and η_z^{λ} by substituting the values obtained in Steps 1–3 into S_y and S_z.
5. Compute the values of ξ_g and η_h by substituting the values obtained in Steps 1–4 into R_g and R_h.

In the case of $E_y = \emptyset$, the above procedure is called the *loop analysis* or the *tieset analysis*. In the case of $E_z = \emptyset$, the procedure is called the *cutset analysis*, which is essentially equivalent to MNA.

In order to ensure that the hybrid equations are a DAE, we require $D = H - GB^{-1}F$ to be a matrix pencil, which is not obviously satisfied because $B = A_T[P, Q]$ is a matrix pencil. Moreover, B needs to be an upper triangular matrix with diagonal ones so that we can compute the values in Steps 3–5 by only substituting the obtained values. The following lemma ensures this for admissible partitions.

Lemma 4.2 ([13, Lemma 3]). *If (E_y, E_z) is an admissible partition, then we can transform B into an upper triangular matrix with diagonal ones by permutations, and D is a matrix pencil.*

Since we only substitute the obtained values in Steps 3–5, the numerical difficulty is determined by the index of the hybrid equations (6).

4.2 Index of Hybrid Equations

In this section, we give a characterization of the index of the hybrid equations. Given an admissible partition (E_y, E_z) and a reference tree T, consider the transformation shown in (2). We now show that $\nu(D)$ can be expressed in terms of the degrees of minors in $A_T(s)$. For each $k \in R$ and $l \in C$, let d_{kl} denote the degree of $\det A_T[R \setminus \{k\}, C \setminus \{l\}]$. Then we have

$$d_{kl} = \deg\det \tilde{A}_T[R \setminus \{k\}, C \setminus \{l\}], \quad \forall k \in R \setminus P,\ \forall l \in C, \tag{11}$$

because we can transform $\tilde{A}_T[R \setminus \{k\}, C \setminus \{l\}]$ into $A_T[R \setminus \{k\}, C \setminus \{l\}]$ by row operations. The index $\nu(D)$ can be rewritten as follows, similarly to Theorem 3.1.

Lemma 4.3. *Given an admissible partition (E_y, E_z) and a reference tree T, the index of D is given by*

$$\nu(D) = \max_{k,l}\{d_{kl} \mid k \in R \setminus P, l \in C \setminus Q\} - \delta_n(A_T) + 1.$$

The index of the hybrid equations has the following property.

Theorem 4.4 ([13, Theorem 7]). *Given an admissible partition (E_y, E_z), the index $\nu(D)$ is the same for any reference tree.*

Theorem 4.4 implies that the index of the hybrid equations is determined only by an admissible partition (E_y, E_z). By Lemma 4.3, the index $\nu(D)$ is determined by the maximum of d_{kl} such that $k \in R \setminus P$ and $l \in C \setminus Q$. However, all the values of d_{kl} are not invariant under row operations on the coefficient matrix $A(s)$ of the circuit equations, while we have to transform $A(s)$ into $A_T(s)$ with respect to an admissible partition (E_y, E_z) and a reference tree T. We now introduce a *degree matrix*, which consists of some invariants under row operations. Let us denote by I_* and V_* the sets corresponding to current and voltage variables for E_*, respectively.

Definition 4.5 (degree matrix). *For each pair of $k \in I_* \cup V_*$ and $l \in I_* \cup V_*$, define*

$$\theta_{kl} = \deg\det \begin{pmatrix} A[R_I \cup R_V, C \setminus \{l\}] & A[R_I \cup R_V, \{k\}] \\ A[S, C \setminus \{l\}] & \mathbf{0} \end{pmatrix}.$$

Then the degree matrix is the matrix $\Theta = (\theta_{kl})$ whose row and column sets are both identical with $I_* \cup V_*$.

Note that the degree matrix is uniquely determined by the circuit, despite $A(s)$ is not unique. By Theorem 4.4, the index of the hybrid equations is expressed in terms of the degree matrix Θ.

Theorem 4.6 ([13, Theorem 11]). *Given an admissible partition (E_y, E_z), we have*

$$\nu(D) = \max_{k,l}\{\theta_{kl} \mid k \in I_y \cup V_z, l \in I_z \cup V_y\} - \delta_n(A) + 1, \tag{12}$$

where A is a coefficient matrix of the circuit equations.

4.3 Index Minimization of Hybrid Equations

Let $\Theta = (\theta_{kl})$ be a degree matrix, where the row set and the column set are identical with $I_* \cup V_*$, and $A(s)$ be a coefficient matrix of the circuit equations. By Theorem 4.6, minimizing the index of the hybrid equations is equivalent to minimizing $\max\{\theta_{kl} \mid k \in I_y \cup V_z, l \in I_z \cup V_y\}$. In this section, we describe how to find an admissible partition (E_y, E_z) which minimizes this maximum value.

Theorem 4.7. *We have $\nu(D) < \alpha - \delta_n(A) + 1$ if and only if an admissible partition (E_y, E_z) satisfies* (i)–(iv) *for any pair of k and l with $\theta_{kl} \geq \alpha$.*

(i) *If $\theta_{kl} \geq \alpha$ for $k = i_e$ and $l = i_f$, then $e \in E_z$ or $f \in E_y$.*

(ii) *If $\theta_{kl} \geq \alpha$ for $k = i_e$ and $l = v_f$, then $e \in E_z$ or $f \in E_z$.*

(iii) *If $\theta_{kl} \geq \alpha$ for $k = v_e$ and $l = i_f$, then $e \in E_y$ or $f \in E_y$.*

(iv) *If $\theta_{kl} \geq \alpha$ for $k = v_e$ and $l = v_f$, then $e \in E_y$ or $f \in E_z$.*

Finding an admissible partition satisfying (i)–(iv) reduces to 2SAT as follows, using the boolean variable u_e to represent $e \in E_z$. First, in order to ensure that (E_y, E_z) is an admissible partition, we impose $u_e = 0$ if the element e is a capacitance or a dependent current source, and we impose $u_e = 1$ if e is an inductance or a dependent voltage source. Next, we rewrite (i) into $u_e \vee \overline{u}_f = 1$, (ii) into $u_e \vee u_f = 1$, (iii) into $\overline{u}_e \vee \overline{u}_f = 1$, and (iv) into $\overline{u}_e \vee u_f = 1$. Thus we obtain the following problem:

2SAT(α) Find u_e for any element e satisfying (1)–(6).

(1) If e is a capacitance or a dependent current source, then $u_e = 0$.

(2) If e is an inductance or a dependent voltage source, then $u_e = 1$.

(3) If $\theta_{kl} \geq \alpha$ for $k = i_e$ and $l = i_f$, then $u_e \vee \overline{u}_f = 1$.

(4) If $\theta_{kl} \geq \alpha$ for $k = i_e$ and $l = v_f$, then $u_e \vee u_f = 1$.

(5) If $\theta_{kl} \geq \alpha$ for $k = v_e$ and $l = i_f$, then $\overline{u}_e \vee \overline{u}_f = 1$.

(6) If $\theta_{kl} \geq \alpha$ for $k = v_e$ and $l = v_f$, then $\overline{u}_e \vee u_f = 1$.

We can solve 2SAT in linear time in the size of literals and clauses [1].

We describe the algorithm for finding an admissible partition which minimizes the index of the hybrid equations.

Algorithm for minimum index hybrid analysis

Step 1: Compute the degree matrix $\Theta = (\theta_{kl})$.

Step 2: Set $E_y \leftarrow \{e \mid e\colon \text{capacitance or dependent current source}\}$, $E_z \leftarrow E_* \setminus E_y$, and $\alpha \leftarrow \max\{\theta_{kl} \mid k \in I_* \cup V_*, l \in I_* \cup V_*\}$.

Step 3: Solve 2SAT(α) to obtain a feasible assignment u_e for $e \in E_*$. If 2SAT(α) is infeasible, then go to Step 5.

Step 4: Set $E_y \leftarrow \{e \mid u_e = 0\}$, $E_z \leftarrow \{e \mid u_e = 1\}$, and $\alpha \leftarrow \alpha - 1$. Go back to Step 3.

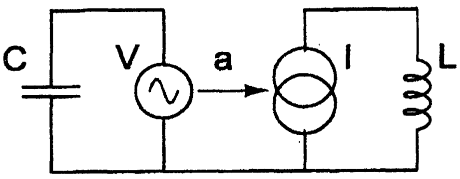

Figure 1: Linear circuit described by circuit equations with index three.

Step 5: Return (E_y, E_z) and α.

Algorithm for minimum index hybrid analysis finds an optimal admissible partition (E_y, E_z) together with the maximum value of α such that 2SAT(α) is infeasible. Therefore, Theorem 4.7 implies that the index of the resulting hybrid equations is $\alpha - \delta_n(A) + 1$ for any reference tree with respect to (E_y, E_z). Instead of the above decremental method, we may adopt the binary search on α.

Finally, we discuss the complexity of our algorithm. Let n be the size of the coefficient matrix of the circuit equations, i.e., the number of elements in the electric circuit is $n/2$. We can compute the degree of the determinant of a $\gamma \times \gamma$ matrix pencil in $O(\gamma^4)$ time [11]. By using this algorithm for n^2 times, a degree matrix can be found in $O(n^6)$ time. Since 2SAT(α) in Step 3 has $O(n)$ literals and $O(n^2)$ clauses, we can solve it in $O(n^2)$ time. Thus the total time complexity of the algorithm is $O(n^6)$.

If one can compute a degree matrix faster, the total time complexity of the algorithm will be better. In [12], we discuss how to compute a degree matrix in $O(n^3)$ time under a genericity assumption that the set of nonzero entries coming from the physical parameters like resistances is algebraically independent, which implies that $A(s)$ is a mixed polynomial matrix [17]. Thus, we improve the time complexity of Algorithm for minimum index hybrid analysis to $O(n^3)$.

If the genericity assumption is not valid, the degree matrix obtained by the improved algorithm may have larger entries than the true values because of unlucky numerical cancellations. Relying on this degree matrix, we may fail to find the minimum index of hybrid equations.

5 Numerical Examples

In this section, we demonstrate the proposed methods in numerical examples. Example 5.1 presents an example of using the substitution method, and Example 5.2 presents an example of applying the hybrid analysis. We use RADAU5 [9] in Matlab as the DAE solver. RADAU5 is an implementation of a fifth order implicit Runge-Kutta method with three stages (RADAU IIA). This is applicable to ODEs and DAEs with index at most three.

Example 5.1 (Electric circuit with index three [8]). Consider a circuit depicted in Figure 1,

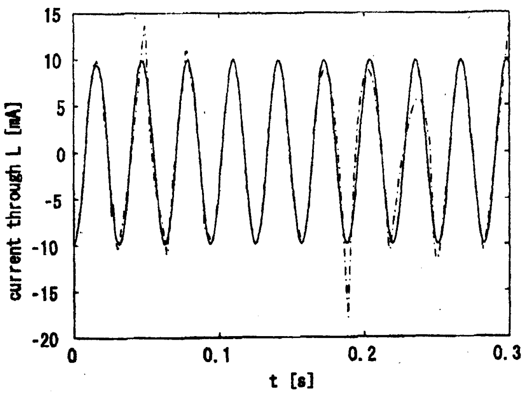

Figure 2: The current through the inductance: numerical solutions of the original DAE (dash-dotted line), the substitution method (solid line), and the exact solution (dotted line).

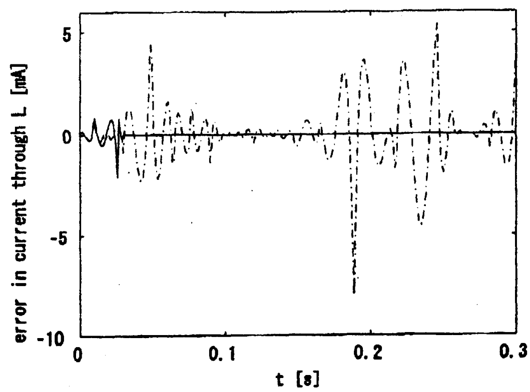

Figure 3: The error in the current through the inductance: the original DAE (dash-dotted line) and the substitution method (solid line).

which is described by the circuit equations with index three:

$$\begin{array}{c|cccccccc} & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 \\ \hline r_1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ r_2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ r_3 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ r_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ r_5 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ r_6 & 0 & -1 & 0 & 0 & 0 & sC & 0 & 0 \\ r_7 & 0 & 0 & 0 & sL & 0 & 0 & 0 & -1 \\ r_8 & a & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array} \begin{pmatrix} \tilde{\xi}_V \\ \tilde{\xi}_C \\ \tilde{\xi}_I \\ \tilde{\xi}_L \\ \tilde{\eta}_V \\ \tilde{\eta}_C \\ \tilde{\eta}_I \\ \tilde{\eta}_L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \tilde{V}(s) \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{13}$$

The modified nodal analysis results in a DAE with index three [8]. However, our method finds

$$X = \{r_1, r_4, r_5, r_6, r_7, r_8\} \quad \text{and} \quad Y = \{c_1, c_2, c_3, c_5, c_7, c_8\},$$

and we obtain

$$D = \begin{pmatrix} 1 & saC \\ 0 & -1 \end{pmatrix},$$

which has index two.

Setting $C = 5[\mu\text{F}], L = 8[\text{mH}], a = 0.99$, and $V(t) = 10\sin(200t)[\text{V}]$, we numerically solve both the original and the resulting DAEs. Figure 2 presents these two numerical solutions and the exact solution, which can be obtained analytically. In Figure 2, the exact solution coincides with the solution of the substitution method. Figure 3 shows the discrepancy of the two numerical solutions from the exact solution. It is observed that the index reduction effectively improves the accuracy of the numerical solution.

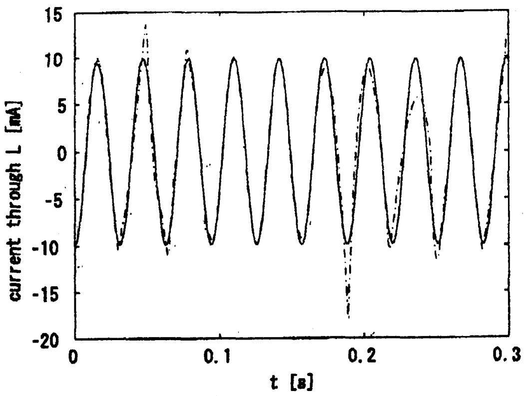

Figure 4: The current through the inductance in Example 5.2: numerical solutions of MNA (dash-dotted line), the hybrid analysis (solid line), and the exact solution (dotted line).

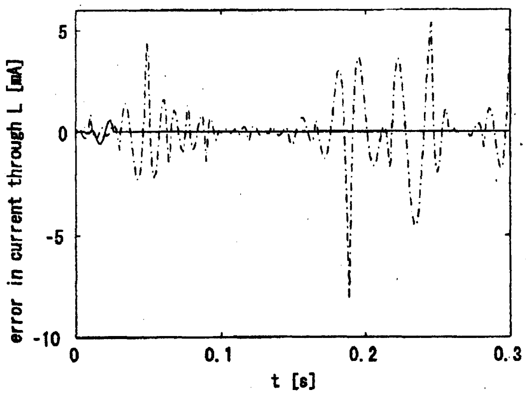

Figure 5: The error in the current through the inductance in Example 5.2: MNA (dash-dotted line) and the hybrid analysis (solid line).

Example 5.2 (Electric circuit with index three [8]). Consider the circuit depicted in Figure 1 again, which is described by the circuit equations (13) with index three. In this example, an admissible partition is uniquely determined and we have

$$E_g = \{V\}, \quad E_h = \emptyset, \quad E_y = \{C, I\}, \quad E_z = \{L\}. \tag{14}$$

By applying the hybrid analysis with respect to the partition (14) and the reference tree $T = \{V, I\}$, we obtain

$$D = \begin{pmatrix} 1 & 0 \\ -sL & 1 \end{pmatrix},$$

which has index two. The hybrid equations are $\tilde{\xi}_L = -saC\tilde{V}(s)$ and $-sL\tilde{\xi}_L + \tilde{\eta}_I = 0$.

Setting the values of C, L, a, and V as given in Example 5.1, we numerically solve both DAEs arising from MNA and the hybrid analysis. Figure 4 presents these two numerical solutions and the exact solution, which can be obtained analytically. In Figure 4, the exact solution coincides with the solution of the hybrid analysis. Figure 5 shows the discrepancy of the two numerical solutions from the exact solution. It is observed that the index reduction effectively improves the accuracy of the numerical solution.

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References

[1] B. Aspvall, M. F. Plass, and R. E. Tarjan: A linear-time algorithm for testing the truth of certain qualified Boolean formulas, *Information Processing Letters*, vol. 8, pp. 121–123, 1979.

[2] K. E. Brenan, S. L. Campbell, and L. R. Petzold: *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*, SIAM, Philadelphia, 2nd edition, 1996.

[3] P. Bujakiewicz: *Maximum Weighted Matching for High Index Differential Algebraic Equations*, Doctor's dissertation, Delft University of Technology, 1994.

[4] S. L. Campbell and C. W. Gear: The index of general nonlinear DAEs, *Numerische Mathematik*, vol. 72, pp. 173–196, 1995.

[5] F. R. Gantmacher: *The Theory of Matrices*, Chelsea, New York, 1959.

[6] C. W. Gear: Simultaneous numerical solution of differential-algebraic equations, *IEEE Transaction on Circuit Theory*, vol. 18, pp. 89–95, 1971.

[7] C. W. Gear: Differential-algebraic equation index transformations, *SIAM Journal on Scientific and Statistical Computing*, vol. 9, pp. 39–47, 1988.

[8] M. Günther and P. Rentrop: The differential-algebraic index concept in electric circuit simulation, *Zeitschrift für angewandte Mathematik und Mechanik*, vol. 76, supplement 1, pp. 91–94, 1996.

[9] E. Hairer and G. Wanner: *Solving Ordinary Differential Equations II*, Springer-Verlag, Berlin, 2nd edition, 1996.

[10] M. Iri: A min-max theorem for the ranks and term-ranks of a class of matrices: An algebraic approach to the problem of the topological degrees of freedom of a network (in Japanese), *Transactions of the Institute of Electronics and Communication Engineers of Japan*, vol. 51A, pp. 180–187, 1968.

[11] S. Iwata: Computing the maximum degree of minors in matrix pencils via combinatorial relaxation, *Algorithmica*, vol. 36, pp. 331–341, 2003.

[12] S. Iwata and M. Takamatsu: Computing degrees of all cofactors in mixed polynomial matrices, *Proceedings of the 5th Hungarian-Japanese Symposium on Discrete Mathematics and Its Applications*, 2007.

[13] S. Iwata and M. Takamatsu: Index minimization of differential-algebraic equations in hybrid analysis for circuit simulation, *Proceedings of the 5th Hungarian-Japanese Symposium on Discrete Mathematics and Its Applications*, 2007.

[14] G. Kishi and Y. Kajitani: Maximally distinct trees in a linear graph (in Japanese), *Transactions of the Institute of Electronics and Communication Engineers of Japan*, vol. 51A, pp. 196–203, 1968.

[15] P. Kunkel and V. Mehrmann: Index reduction for differential-algebraic equations by minimal extension, *Zeitschrift für angewandte Mathematik und Mechanik*, vol. 84, pp. 579–597, 2004.

[16] S. E. Mattsson and G. Söderlind: Index reduction in differential-algebraic equations using dummy derivatives, *SIAM Journal on Scientific Computing*, vol. 14, pp. 677–692, 1993.

[17] K. Murota: *Matrices and Matroids for Systems Analysis*, Springer-Verlag, Berlin, 2000.

[18] T. Ohtsuki, Y. Ishizaki, and H. Watanabe: Network analysis and topological degrees of freedom (in Japanese), *Transactions of the Institute of Electronics and Communication Engineers of Japan*, vol. 51A, pp. 238–245, 1968.

[19] S. Schulz: Four lectures on differential-algebraic equations, Technical Report 497, The University of Auckland, New Zealand, 2003.

[20] D. E. Schwarz and C. Tischendorf: Structural analysis of electric circuits and consequences for MNA, *International Journal of Circuit Theory and Applications*, vol. 28, pp. 131–162, 2000.

[21] M. Takamatsu and S. Iwata: Index reduction for differential-algebraic equations by substitution method, METR 2007-02, Department of Mathematical Informatics, University of Tokyo, 2007.