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On Variational Formulations of Singular Minimal Subvarieties

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1 Introduction

This article is designed to give a perspective to a series of results in [MY] [GY] by the author and his collaborators in the past several years. We start with the definition of almost-minimal sets. Let a set E be a closed set contained in an open set $U \subset \mathbf{R}^n$, satisfying the finite d-dimensional Hausdorff measure condition;

$$\mathcal{H}^d(E \cap B) < \infty$$

for every compact ball $B \subset U$. To set up a variational formulation, we introduce a class of "competitors" for the set E as follows. Let $\phi_t : U \to U$ be a one-parameter family of continuous functions such that for some compact ball $B \subset U$,

$$\begin{aligned}
&\phi_0(x) = x \text{ for } x \in U \backslash B \\
&\phi_t(B) \subset B \\
&(t, x) \mapsto \phi_t(x) \text{ is continuous in } t \text{ and } x. \\
&\phi_1 \text{ is Lipschitz } (\phi_1 \text{ may not be injective.})
\end{aligned}$$

We consider then the set $\phi_1[E]$, called a deformation of E in B as a competitor in the following variational setting;

Definition For a set E as described above, it is called *almost-minimal* in U, with gauge function h if

$$\mathcal{H}^d(E \backslash F) \leq \mathcal{H}^d(F \backslash E) + h(r) r^d$$

when F is a deformation of E in a compact ball $\overline{B}_r(x) \subset U$.

Note that the above definition is a slight variation of Almgren's (M, ε, δ)-minimal set [Alm], where δ is the size of the support of non-trivial variation, and ε is the growth rate for h, and the *mass* M stands for the two-dimensional Hausdorff measure.

In this note, we consider the special case when $h \equiv 0$ (corresponding to the case $\varepsilon = 0$,) $d = 2$, $n = 3$. Then an almost-minimal set is a minimizer of the functional $\int_E d\mathcal{H}^d$ under topological constraints that allow deformations on small balls $B \subset U$. The following theorem by Jean Taylor [T], whose free boundary regularity is later improved from $C^{1,\alpha}$ to C^ω by Kinderlehrer-Nirenberg-Spruck [KNS], gives a strong restriction on the topological type of the local structure for the almost-minimal set.

Theorem (J. Taylor) *If E is an almost-minimal set of dimension* 2 *in* $\mathbf{R}^3$, *with $h(r) \leq Cr^\alpha$, every point of E has a neighborhood where E is C^1-equivalent to a plane, a Y-singularity, or a T-singularity, with the one-dimensional free boundary being real analytic.*

The purpose of the project undertaken by C.Mese and the author (cf. [MY]) is to describe the set E as a harmonic image of a 2-simplex, where the harmonic map is conformal in a suitable sense. In particular, in this article we restrict ourselves to a question of finding a *local replacement.* Recall that energy minimizing map is a map where *no* local replacement of the given map around a point p would decrease the total energy. Here a local replacement is to satisfy the Dirichlet condition on the boundary of the ball around the point p. Hence the goal is to identify a piece of $(M, 0, \delta)$-minimal set around a singular point p as a locally area minimizing map of a suitable simplicial complex. We remark that in [LM] Lawlor and Morgan considered a similar issue, but the replacement concerned there is restricted to the disc type.

In order to carry out this idea, we need to specify the "energy" functional to be minimized, and the "Teichmüller space" of the simplex, which parameterizes the various conformal structures, compatible with the induced metric of the singular surface. There is a quote from one of Yau's open problem section [Y], which is relevant to the framework of our investigation;

"*How does one give a robust description on irregular geometric objects, especially the singular set of geometric objects defined by variational principles? For example, how regular are the singular sets of* *harmonic maps and minimal varieties? Can one generalize the Douglas-Rado mapping approach to minimal surfaces to the case when the boundary is not a Jordan curve but the one-dimensional skeleton of a complex? One should replace the disk by a simplicial complex with* minimal complexity."

What makes this problem distinct from much of the work on the minimal surface literature is the fact that the "surfaces" concerned here has no *orientation* a-priori, and thus there is no so-called "boundary operator" so often associated to elliptic variational problems, which enables to *integrate by parts*. The variational definition of the almost minimal sets introduced in the beginning is thus an attempt by Almgren [Alm] to overcome the lack of elliptic variational formulations.

As an attempt to get around the obstacles, we set up the following formulation [MY].

2 Douglas-Rado formulation of minimal tri-discs

Let $A_i, \tilde{A}_i$ be the x-axis of Euclidean unit half-discs $\Delta_i, \tilde{\Delta}_i$. We suppress the index for A_i's as we identify them and call them A from now on.) Let $\mathrm{Id}_A : A \longrightarrow \tilde{A}_i$ and Id_{Δ_i} be identity maps. Define the moduli space of conformal structures on the disjoint discs of the three half discs as

$$\mathcal{P} = \{\Phi = (\phi_1, \phi_2, \phi_3)\}$$

where each ϕ_i are quasiconformal maps *smooth up to the boundary*, $\phi_i|_A : A \longrightarrow A_i$ are smooth quasi-symmetric functions, and $\phi_1 = \mathrm{Id}_A$. We let X_Φ be the space $\cup\tilde{\Delta}_i / \sim_\Phi$ where $\sim_\Phi$ defines an identification of $\tilde{A}_i$ and $\tilde{A}_j$ via the gluing map $\phi_j \circ \phi_i^{-1} : \tilde{A}_j \longrightarrow \tilde{A}_i$. Let Y be a real analytic manifold, for example $\mathbf{R}^n$. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be so that $\alpha_i : \tilde{\Delta}_i \longrightarrow Y$ in the function space $W^{1,2} \cap C^0$. For Φ in $\mathcal{P}$, we say that $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ satisfies the Φ-matching condition if

$$\alpha_i \circ \phi_i = \alpha_1 \circ \mathrm{Id}_A.$$

Let $\mathcal{F}(\Phi)$ be the set of α's satisfying the Φ-matching condition. Note that α in $\mathcal{F}(\Phi)$ can be considered as a map defined on the X_Φ whose restriction to each $\tilde{\Delta}_i$ equals to α_i.

Next, define a space of weights for the Dirichlet energy functionals;

$$\mathcal{C} = \{(c_1, c_2, c_3) \in \mathbf{R}^n : c_1 + c_2 + c_3 = 1, c_i \geq 0 \ (i = 1, 2, 3)\}$$

For $c \in \mathcal{C}$, we say that $\alpha \in \mathcal{F}(\Phi)$ is compatible with c if $E(\alpha_i) = 0$ whenever $c_i = 0$. Otherwise we say α is incompatible with c. We define the area of α as $A(\alpha) = \sum_i A(\alpha_i)$ and the c-weighted energy as

$$E_c(\alpha) = \left(\Sigma_{c_i \neq 0} \frac{1}{c_i}(E(\alpha_i))^2\right)^{1/2}$$

if α is compatible with c, otherwise ∞.

In [MY], we introduced the variational problem;

$$\inf_{\Phi \in \mathcal{P}} \inf_{c \in \mathcal{C}} \inf_{\alpha \in \mathcal{F}(\Phi)} E_c(\alpha)$$

and showed that its minimizing element exists when the quasi-conformality induced by the minimizing conformal structure/gluing function Φ is bounded, and that then the energy minimizing map α is the area-minimizer, namely the value of the energy coincides with the 2-dimensional Hausdorff measure of the harmonic image in the ambient space.

Let (Φ, c, α) be the minimizing triple. By a slight modification of the proof of Theorem 9 and Lemma 10 of [MY] we have that

$$c_i = \frac{E(\alpha_i)}{\Sigma^c E(\alpha_i)}$$

and α_i is weakly conformal. The matching condition implies

$$\phi_i = \alpha_i^{-1} \circ \alpha_1 \circ \mathrm{Id}_A : A \to \tilde{A}_i.$$

Define $\beta_i = \alpha_i \circ \phi_i : \Delta_i \to \mathbf{R}^n$. Then β_i is a conformal harmonic map. as α_i is conformal and harmonic, and ϕ_i is an isometry by definition. It was shown [MY] that the map has a continuous free boundary.

3 Restricting the local topological type

We adapt this harmonic map formulation to construct a local replacement for a given $(M, 0, \delta)$-minimal set. In order to gain control over the topological type of the singular surface, we need to choose a scale which is sufficiently small, so that the structure of the tangent cone is reflected on the local structure of the small piece of the surface. We consider the case when the center of the ball is a Y-singularity of the $(M, 0, \delta)$-minimal surface.

Suppose that we have an embedded graph Γ in $\mathbf{R}^n$ which is a union of arcs a_k meeting at vertices q_j, each of which has valence at least two. The *valence* of a vertex q is the number of times q occurs as an endpoint among all of the 1-simplices a_i. Each 1-simplex a_i is assumed to be C^2, and to meet its end points with C^1 smoothness; thus there is a well-defined tangent vector $T_i(q)$ to each 1-simplex a_i at a vertex q, pointing into a_i. At a vertex q of valence d, we consider the *contribution to total curvature* at q:

$$\mathrm{tc}(q) := \sup_{e\in\mathbf{R}^n, \|e\|=1} \left\{ \sum_{a_k : q\in a_k} \left(\frac{\pi}{2} - \angle_q(T_k(q), e) \right) \right\} \tag{1}$$

where $\angle_q(T_k(q), e)$ is the angle between the tangent vector $T_k(q)$ and the direction e. We define the *total curvature* of Γ as

$$\mathcal{C}_{\mathrm{tot}}(\Gamma) := \int_{\Gamma^{\mathrm{reg}}} |\vec{k}|\, ds + \sum_q \{\mathrm{tc}(q) : q \text{ a vertex of } \Gamma\}. \tag{2}$$

where $\vec{k}$ is the geodesic curvature vector of each a_i as a curve in $\mathbf{R}^n$, and $\Gamma^{\mathrm{reg}} = \Gamma\backslash\{\text{vertices}\}$. It should be noted that our definition of total curvature coincides with the standard definition in the case when Γ is a piecewise smooth Jordan curve: the integral of the norm of geodesic curvature vector plus the sum of the exterior angles at the vertices. Namely, in that case, every vertex q of the graph Γ is of valence two, and the supremum in equation (1) is assumed at vectors e lying in the smaller angle between the tangent vectors T_1 and T_2 to Γ, so that $\mathrm{tc}(q)$ is then the exterior angle at q.

The type of surfaces we will consider is the immersed image Σ of a union of three C^2-smooth open discs Σ_i, C^1 up to the piecewise C^1 boundary $\partial\Sigma_i$. We further restrict ourselves here so that the graph Γ is a union of three arcs, each of which is a part of 1-skeleton S of Σ_i. We call such a graph theta-type. S is defined as the union of the piecewise C^1 curves $\partial\Sigma_i$. The class of such strongly stationary surfaces Σ will be denoted by $\mathcal{S}_\Gamma$.

Note that J. Taylor's results imply that for a Y-singularity point p of $(M, 0, \delta)$-minimal surface M, and for a small value $\rho > 0$, the set $M \cap B_\rho(p)$ is a union of three minimal surfaces meeting along a real analytic curve, hence the set $M \cap B_\rho(p)$ is an element of $\mathcal{S}_\Gamma$ with $\Gamma = \partial B_\rho(p) \cap M$, which is a theta graph.

Let $\Gamma \subset \mathbf{R}^3$ be a theta graph, and let $\mathcal{S}_\Gamma$ be the class of singular surfaces containing Γ. We will consider the surfaces $\Sigma \in \mathcal{S}_\Gamma$ satisfying the following property.

Definition [EWW] A rectifiable varifold Σ in $\mathbf{R}^3$ is called *strongly stationary with respect to* Γ if for all smooth $\phi : \mathbf{R}\times\mathbf{R}^n$ with $\phi(0,x) \equiv x$, we have

$$\frac{d}{dt}\Big(\mathrm{Area}(\phi(t,\Sigma)) + \mathrm{Area}(\phi([0,t]\times\Gamma))\Big)\Big|_{t=0} \geq 0.$$

In [GY], a criterion was obtained to restrict the type of singularities belonging to a strongly stationary surfaces in $\mathcal{S}_\Gamma$ with respect to its boundary in terms of the total curvature of the boundary.

Theorem [GY] *Suppose* Γ *is a graph in* $\mathbf{R}^3$ *with* $C_{\mathrm{tot}}(\Gamma) \leq 2\pi C_T$ *where* $C_T \approx 1.8245$ *is the area density of the T-singularity cone, and let* $\Sigma \in \mathcal{S}_\Gamma$ *be embedded as an* $(\mathbf{M}, 0, \delta)$*-minimizing set with respect to* Γ*. Then* Σ *has possibly Y singularities but no other singularities, unless it is a subset of the T stationary cone, with planar faces.*

As the total curvature of the link $\partial B_1(p)\cap Y_p$ where Y_p is the tangent cone of M at p, which is isometric to the Y-singularity cone , is strictly less than $2\pi C_T$, there exists a $\rho_0 > 0$ such that for any $\rho < \rho_0$, the boundary $M \cap \partial B_\rho(p)$ of the surface $M \cap B_\rho(p)$ had total curvature less than $2\pi C_T$ and thus the set $M \cap B_\rho(p)$ is a collection of three minimal surfaces diffeomorphic to the simplicial complex X_{Id}. Note that the boundary $M \cap \partial B_\rho(p)$ as well as the free boundary where the three real analytic surfaces meet are piecewise-real-analytic curves. Furthermore, for ρ chosen sufficiently small, one can have $M \cap B_\rho(p)$ conformal to X_Φ for Φ in $\mathcal{P}(D)$ with a finite dilatation D. We fix such a ρ for the given $p \in M$.

4 Local replacement of a $(M, 0, \delta)$-minimal set by harmonic images

The fact that at a small scale around a Y-singularity point p, an $(M, 0, \delta)$-minimal set is a union of three minimal surfaces with conformal structures with bounded dilatation in the sense of the Douglas-Rado construction as described above, can be interpreted as the local piece $M \cap B_\rho(p)$ of $(M, 0, \delta)$-minimal set M is represented as a competitor of the variational scheme of finding a minimizer for

$$\inf_{\Phi\in\mathcal{P}(D)}\inf_{c\in\mathcal{C}}\inf_{\alpha\in\mathcal{F}(\Phi)} E_c(\alpha)$$

for some finite number $D > 0$. Of course this says in turn that the area of the set $M \cap B_\rho(p)$ is bounded below by the value of inf inf inf above. This is because the triple of the harmonic maps $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ parameterizing the three minimal surfaces are conformal on each open semi-discs Δ_i, and also the weights $c = (c_1, c_2, c_3)$ of E_c energy is chosen so that

$$E_c(\alpha) = \mathcal{H}^2(\alpha[\cup_{i=1}^3 \Delta_i]).$$

as well as the fact that

$$\inf_{\Phi\in\mathcal{P}(D)}\inf_{c\in\mathcal{C}}\inf_{\alpha\in\mathcal{F}(\Phi)} E_c(\alpha) \le E_c(\alpha)$$

5 Open Problems

To achieve the goal of finding a local replacement of a $(M, 0, \delta)$-minimal set by harmonic map images, we need to show the other inequality, namely;

$$\mathcal{H}^2(\alpha[\cup_{i=1}^3 \Delta_i]) \le \inf_{\Phi\in\mathcal{P}(D)}\inf_{c\in\mathcal{C}}\inf_{\alpha\in\mathcal{F}(\Phi)} E_c(\alpha).$$

It should be noted that as was demonstrated in [MY], the space $\mathcal{P}(D)$ is sequentially compact, and thus one can find a minimizing sequence of triples (α, c, Φ), which is convergent. However, the limit thus obtained may have little to do with the $(M, 0, \delta)$-minimal set $M \cap B_\rho(p)$.

At this point one would hope to construct a minimizing sequence in the inf inf inf scheme which approximate the set $M \cap B_\rho(p)$ in a suitable sense, to ensure a convergence of the sets. Then the lower semi-continuity of the energy should be utilized to obtain the desired inequality above.

We make two concluding remarks here. The first is that the lower dimensional analogue ($d = 1, n = 2$,) namely one-dimensional minimal network, in the sense of $(M, 0, \delta)$-minimal sets where *mass* M here is the one-dimensional Hausdorff measure, in $\mathbf{R}^2$ (or more generally in $\mathbf{R}^n$) can be completely characterized locally by the Douglas-Rado formulation [MY] as presented in **Section 2**. The difference from the $d = 2, n = 3$ case is the moduli space of gluing minimal curve is trivial, and thus $\mathcal{P}$ consist of a point.

The second concluding remark is related to the first. The essence of this problem of dealing with singular sets, which are inevitably present from variational viewpoints, is the understanding of the space of conformal structure, or equivalently the space of gluing functions Φ along the singular sets. We expect in order to carry out the convergence of suitably chosen minimizing triples (α, c, Φ), one needs to find a connection between the extrinsic geometry of the minimal sets to the intrinsic/conformal geometry of the induced metrics. The former would involve tools like Reifenberg Approximation Theorem [Re][DT] which would provides a criterion for scale-invariant Hausdorff convergence to a tangent space/cone. The latter would require a good understanding of the relationship between the regularity of the gluing functions and that of the resulting metrics, which would be then enables to use the free boundary PDE's tools as in [KNS].

Those remaining tasks may not be simple, but the author believes that there should be an intense effort to understand the connection between the viewpoint of geometric measure theory, which have resulted in beautiful results such as Taylor's [T], and the approach using nonlinear partial differential equations such as the harmonic map theory.

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