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KERNEL RIGIDITY

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This note is an outline of [5]. The main object is the two types of spaces, 0-dimensional compact metric spaces and countable metric spaces . They seem, at first sight, to have no relation to rigidity. Indeed, the well known Cantor-Bendixson theorem says that every space X can be decomposed as

$$X = K \cup S$$

with K a closed subset having no isolated points and S a scattered space. As well known, the kernel K , if not empty, is homeomorphic to the Cantor set in case X is a 0-dimensional compact metric space and to the rationals in case X is a countable metric space. Thus, in either case, X can not be a rigid space with the trivial exceptions of the one point space and the empty set. As for (non-compact and non-countable) 0-dimensional separable metric spaces, a rigid exampe is known. (See Kuratowski [2])

Definition 1. A space X is defined to be *kernel-rigid* if X has a non-empty kernel K and for any homeomorphism $h : X \longrightarrow X$, the restriction $h|K$ is the identity map.

Concerning kernel-rigidity, the two types of spaces, 0-dimensional compact metric spaces and countable metric spaces, have quite different aspects ; the former does not contain a kernel-rigid example (Theorem 2), but the latter does (Theorem 4).

Let us start with basic notations. For a space X let $X^{(0)} = X$ and $X_{(0)}$ the set of the isolated points of $X^{(0)}$. If β is a non-limit ordinal, let $X^{(\beta)} = X^{(\beta-1)} - X_{(\beta-1)}$ and $X_{(\beta)}$ the set of the isolated points of $X^{(\beta)}$, where $\beta - 1$ means the ordinal preceding β. If β is a limit ordinal, let $X^{(\beta)} = \cap_{\gamma<\beta} X^{(\gamma)}$ and $X_{(\beta)}$ the set of the isolated points of $X^{(\beta)}$.

Each $X^{(\beta)}$ is a closed subset of X , and each $X_{(\beta)}$ is a discrete open subset of $X^{(\beta)}$.

The first ordinal α for which $X^{(\alpha)} = X^{(\alpha+1)}$ is called the *length* of the space X and is denoted by leng(X). If leng(X) $= \alpha$ the subset $K = X^{(\alpha)}$ is called the *kernel* of X and $S = X - X^{(\alpha)}$ is called the *scattered part* of X . If $K = \emptyset$ then X is called *scattered* . Clearly the scattered part S of a space X is scattered and satisfies leng(S) = leng(X) .

The following properties are easily checked.

(P-1) If β is an ordinal and U is an open set of X then $X^{(\beta)} \cap U = U^{(\beta)}$ and $X_{(\beta)} \cap U = U_{(\beta)}$.

(P-2) If $\beta \leq \gamma$ then $X_{(\gamma)} \subseteq \overline{X_{(\beta)}}$ so that $\overline{X_{(\gamma)}} \subseteq \overline{X_{(\beta)}}$.

- Non-existence of kernel-rigid 0-dimensional compact metric spaces -

The following notion was used by Knaster and Reichbach [1] to formulate some extension theorems of homeomorphisms.

Definition 2. Let $X = K \cup S$ be a space . The scattered part S is said to be *full-attached* to K if $K \neq \emptyset$ and $K \subseteq \overline{X_{(\beta)}}$ for every $\beta < \text{leng}(X)$.

Theorem 1. (Knaster and Reichbach, see [1 , Corollary 4]) *Let an ordinal α be given and let $X = K \cup S$ be a 0-dimensional compact metric space of length α with S full-attached to K . Then the topological type of X is uniquely determined.*

We thus denote by $F(\alpha)$ the 0-dimensional compact metric space $X = K \cup S$ of length α with S full-attached to K . In particular, $F(0)$ denotes the Cantor set K .

Theorem 2. *There is no 0-dimensional compact metric space which is kernel-rigid.*

Proof. Let $X = K \cup S$ be a 0-dimensional compact metric space with a non-empty kernel K and a scattered part S . Define an ordinal β by

$$\beta = \min\{\gamma \mid K - \overline{X_{(\gamma)}} \neq \emptyset\}$$

and put $D = K - \overline{X_{(\beta)}}$. Since D is a non-empty open set of K we can find two distinct points, say p and q, of D and using 0-dimensionality of X , two disjoint clopen sets U, V of X included in $X - \overline{X_{(\beta)}}$ and satisfying $p \in U$, $q \in V$. Let U be written as $U = K' \cup S'$ with K' the kernel of U and S' the scattered part of U . By (P-1) and the definition of β we have

$$\overline{U_{(\gamma)}} = U \cap \overline{X_{(\gamma)}} \supseteq U \cap K = K'$$

for every $\gamma < \beta$. Note that $\text{leng}(U) = \beta$ because $U \cap X_{(\gamma)} \neq \emptyset$ for every $\gamma < \beta$ and $U \cap X_{(\gamma)} = \emptyset$ for every $\gamma \geq \beta$. Consequently S' is full-attached to K' . Thus $U \approx F(\beta)$ and, analogously, $V \approx F(\beta)$ so that $U \approx V$ by the uniqueness of $F(\beta)$. Taking a homeomorphism $h : U \longrightarrow V$, define a homeomorphism $f : X \longrightarrow X$ by

$$f(x) = \begin{cases} h(x) & \text{if } x \in U \\ h^{-1}(x) & \text{if } x \in V \\ x & \text{if otherwise .} \end{cases}$$

Then f moves the point p , which completes the proof.

- A kernel-rigid countable metric space -

Let α be a non-limit ordinal and n a natural number. The compact countable metric space X satisfying $\text{leng}(X) = \alpha$ and $|X^{(\alpha-1)}| = n$, the uniqueness of which was assured by Mazurkiewicz and Sierpiński [3] , is denoted by $MS(\alpha, n)$.

Definition 3. Let β be an ordinal. A scattered space X is called *β-rigid* if $h(x) = x$ whenever $x \in X_{(\beta)}$ and $h : X \longrightarrow X$ is a homeomorphism. A $\beta-$rigid scattered space X is called *non-trivial* if $|X_{(\beta)}| \geq \aleph_0$.

Proposition 1. *If a Hausdorff scattered space X is β-rigid, then X is γ-rigid for every $\gamma \geq \beta$.*

Remark. Standing on the opposite side we define a scattered space X to be *β-homogeneous* if for any points a, $b \in X_{(\beta)}$ there is a homeomorphim $h : X \longrightarrow X$ sending a to b . A scattered space X is called *rankwise homogeneous* if X is $\beta-$homogeneous for every β .

By the Mazurkiewicz-Sierpiński theorem ([3]), compact countable metric spaces and locally compact countable metric spaces are rankwise homogeneous. There are many other types of rankwise homogeneous scattered countalble metric spaces . (See [4, Table 1])

The following theorem assures the existence of a β-rigid scattered countable metric space in which, furthermore, a sudden shift happens from homogeneity to rigidity.

Theorem 3. *For any ordinal $\beta \geqq \omega$, there exists a non-trivial β-rigid scattered countable metric space X of length $\beta + 1$ which is γ-homogeneous for every $\gamma < \beta$.*

Remark. If $\beta < \omega$ there does not exist a non-trivial β-rigid scattered countable metric space.

Lemma 1. *Let $X = K \cup S$ be a Hausdorff space with a non-empty kernel K . If S is full-attached to K and is β-rigid for some $\beta < leng(X)$, then X is kernel rigid.*

Theorem 4. *There exists a kernel-rigid countable metric space.*

References.

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