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Homotopy type of the box complexes of graphs without 4-cycles

筑波大学 数理物質科学研究科 上別府陽 (Akira Kamibeppu)

Institute of Mathematics, University of Tsukuba

A *graph* G is a pair $(V(G), E(G))$, where $V(G)$ is a finite set and $E(G)$ is a family of 2-elements subsets of $V(G)$. We assume that graphs are connected. We follow [3] with respect to the standard notation in graph theory. For a graph G, an abstract simplicial complex $\mathsf{B}(G)$ which is called the box complex of G is introduced by J. Matoušek and G. M. Ziegler in [5]. We define the box complex of a graph following [5].

Let G be a graph and U a subset of $V(G)$. A vertex $v \in V(G)$ which is adjacent to each $u \in U$ is called a *common neighbor* of U in G. The set of all common neighbors of U in G is denoted by $\mathrm{CN}_G(U)$. For convenience, we define $\mathrm{CN}_G(\phi) = V(G)$. For $U_1, U_2 \subseteq V(G)$ such that $U_1 \cap U_2 = \phi$, we define $G[U_1, U_2]$ as the bipartite subgraph of G with

$$V(G[U_1, U_2]) = U_1 \cup U_2 \text{ and } E(G[U_1, U_2]) = \{u_1u_2 \mid u_1 \in U_1,\ u_2 \in U_2,\ u_1u_2 \in E(G)\}.$$

The graph $G[U_1, U_2]$ is said to be *complete* if $u_1u_2 \in E(G)$ for all $u_1 \in U_1$ and $u_2 \in U_2$. For convenience, $G[\phi, U_2]$ and $G[U_1, \phi]$ are also said to be complete.

Let U_1, U_2 be subsets of $V(G)$. The subset $U_1 \uplus U_2$ of $V(G) \times \{1, 2\}$ is defined as

$$U_1 \uplus U_2 := (U_1 \times \{1\}) \cup (U_2 \times \{2\}).$$

For vertices $u_1, u_2 \in V(G)$, $\{u_1\} \uplus \phi$, $\phi \uplus \{u_2\}$, and $\{u_1\} \uplus \{u_2\}$ are simply denoted by $u_1 \uplus \phi$, $\phi \uplus u_2$ and $u_1 \uplus u_2$ respectively.

The *box complex* of a graph G is an abstract simplicial complex with the vertex set $V(G) \times \{1, 2\}$ and the family of simplices

$$\begin{aligned}\mathsf{B}(G) = \{U_1 \uplus U_2 \mid U_1, U_2 \subseteq V(G),\ & U_1 \cap U_2 = \phi, \\ & G[U_1, U_2] \text{ is complete, } \mathrm{CN}_G(U_1) \neq \phi \neq \mathrm{CN}_G(U_2)\}.\end{aligned}$$

An abstract simplex $U_1 \uplus U_2$ and its geometric simplex are denoted by the same symbol $U_1 \uplus U_2$. The simplicial isomorphism $\nu : V(\mathsf{B}(G)) \to V(\mathsf{B}(G))$ is defined by

$$u \uplus \phi \mapsto \phi \uplus u \text{ and } \phi \uplus u \mapsto u \uplus \phi$$

for each $u \in V(G)$. This induces a homeomorphism on $\|\mathsf{B}(G)\|$ satisfying $\nu \circ \nu = \mathrm{id}_{\|\mathsf{B}(G)\|}$. Moreover, we notice that this homeomorphism has no fixed point. In general, a homeomorphism ν on a topological space X satisfying $\nu \circ \nu = \mathrm{id}_X$ is called the $\mathbb{Z}_2$-*action* on X and the pair (X, ν) is called the $\mathbb{Z}_2$-*space*. For two $\mathbb{Z}_2$-spaces (X, ν_X) and (Y, ν_Y), a continuous map $f : X \to Y$ satisfying $f \circ \nu_X = \nu_Y \circ f$ is called a $\mathbb{Z}_2$-map from X to Y. We define the $\mathbb{Z}_2$-*index* of a $\mathbb{Z}_2$-space (X, ν) as

$$\mathrm{ind}_{\mathbb{Z}_2}(X, \nu) := \min\{\, n \mid \text{there is a } \mathbb{Z}_2\text{-map } f : X \to S^n \,\},$$

where $S^n = \{\, x \in \mathbb{R}^{n+1} \mid \|x\| = 1 \,\}$ with the $\mathbb{Z}_2$-action on S^n given by $x \mapsto -x$. If there exists a $\mathbb{Z}_2$-map from X to Y, then we have $\mathrm{ind}_{\mathbb{Z}_2}(X) \leq \mathrm{ind}_{\mathbb{Z}_2}(Y)$.

In [5], J. Matoušek and G. M. Ziegler pointed out the following:

(1) For any graph G,

$$\mathrm{ind}_{\mathbb{Z}_2}(\|\mathsf{B}(G)\|) \le \chi(G) - 2,$$

where $\chi(G)$ is the chromatic number of G.

(2) If a graph G has no 4-cycle, there is a $\mathbb{Z}_2$-retraction of $\|\mathsf{sd}\,\mathsf{B}(G)\|$ onto a 1-dimensional subcomplex $\|\mathsf{L}\|$ of $\|\mathsf{sd}\,\mathsf{B}(G)\|$ defined in [5], p.81, (H1). Then, we have $\mathrm{ind}_{\mathbb{Z}_2}(\|\mathsf{B}(G)\|) \le 1$. This indicates that the difference between $\mathrm{ind}_{\mathbb{Z}_2}(\|\mathsf{B}(G)\|)$ and $\chi(G) - 2$ can be arbitrarily large.

Let $\overline{G}$ be the following 1-dimensional subcomplex of $\mathsf{B}(G)$:

$$\overline{G} := \{u \uplus \phi,\ v \uplus \phi,\ \phi \uplus u,\ \phi \uplus v,\ u \uplus v,\ v \uplus u \mid uv \in E(G)\}.$$

Then, $\|\overline{G}\|$ is the $\mathbb{Z}_2$-space with the restriction of the $\mathbb{Z}_2$-action on $\|\mathsf{B}(G)\|$. This $\mathbb{Z}_2$-action also has no fixed point. The preceding 1-dimensional subcomplex L of $\mathsf{sd}\,\mathsf{B}(G)$ equals to $\mathsf{sd}\,\overline{G}$.

We are interested in the relation between the combinatorics of G and the topology of $\|\mathsf{B}(G)\|$. In what follows, we consider the topology of the box complex of a graph without 4-cycles. Such a box complex has the following two properties:

Lemma 1 ([2], Lemma 4.1). A graph G contains no 4-cycle if and only if for any simplices $U_1 \uplus U_2 \in \mathsf{B}(G)$, we have $|U_1| \le 1$ or $|U_2| \le 1$. For such a graph G, each maximal simplex $U_1 \uplus U_2 \in \mathsf{B}(G)$ satisfies $|U_1| = 1$ or $|U_2| = 1$.

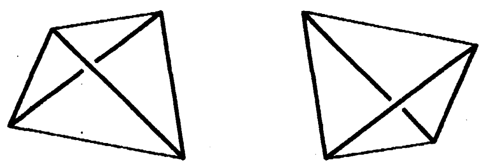
Lemma 2 ([2], Lemma 4.2). Let G be a graph without 4-cycles. For any two distinct maximal simplices of $\mathsf{B}(G)$ with nonempty intersection, the intersection is a simplex of $\overline{G}$.

Let X be a $\mathbb{Z}_2$-space and A a $\mathbb{Z}_2$-subspace of X. A strong deformation retraction $\{f_t\}_{t\in[0,1]}$ of X onto A such that each $f_t : X \to X$ is a $\mathbb{Z}_2$-map is called a strong *$\mathbb{Z}_2$-deformation retraction* of X onto A. Then, we notice that the retraction f_1 of X onto A and the inclusion of A into X are $\mathbb{Z}_2$-maps, so we have $\mathrm{ind}_{\mathbb{Z}_2}(X) = \mathrm{ind}_{\mathbb{Z}_2}(A)$.

Theorem 3 ([2], Theorem 4.3). A graph G contains no 4-cycle if and only if $\|\overline{G}\|$ is a strong $\mathbb{Z}_2$-deformation retract of $\|\mathsf{B}(G)\|$.

Sketch of proof. If a graph G contains a 4-cycle C_4, then $\|\mathsf{B}(C_4)\|$ ($\subseteq \|\mathsf{B}(G)\|$) is the disjoint union of two 3-simplices and $\|\overline{C_4}\|$ is the disjoint union of two circles, each of which is contractible in $\|\mathsf{B}(G)\|$.

$\|\mathsf{B}(C_4)\|$



(The polyhedron $\|\overline{C_4}\|$ is illustrated with —— .)

Figure 1. The box complex $\|\mathsf{B}(C_4)\|$

Suppose that there is a retraction $r : \|\mathsf{B}(G)\| \to \|\overline{G}\|$. We consider the nullhomotopic loop l in $\|\mathsf{B}(G)\|$ which goes around one of two circles of $\|\overline{C_4}\|$. Then, we see that $r \circ l$ is the circle in $\|\overline{G}\|$ which must be nullhomotopic. This is impossible since $\|\overline{G}\|$ is the 1-dimensional complex.

Conversely, we assume that a graph G has no 4-cycle. Then, by Lemma 1, we can divide all maximal simplices of $\mathsf{B}(G)$ into the two sets of simplices

$$B_1 = \{v \uplus U \mid v \uplus U \text{ is maximal}\} \text{ and } B_2 = \{U \uplus v \mid U \uplus v \text{ is maximal}\}.$$

The $\mathbb{Z}_2$-action ν on $\|\mathsf{B}(G)\|$ induces a one-to-one correspondence between B_1 and B_2. For each simplex $v \uplus U \in B_1$, we define a strong deformation retraction $\{f_t^v\}_{t\in[0,1]}$ of $v \uplus U$ onto $K_v^- := \|\overline{G}\| \cap (v \uplus U)$ starting with a collapsing from the free face $\phi \uplus U$ of $v \uplus U$ (see Figure 2):

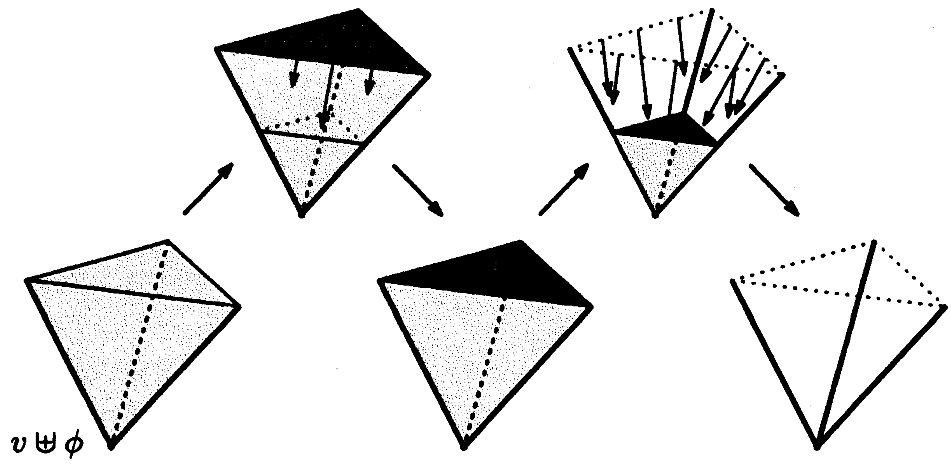


Figure 2. The strong deformation retraction $\{f_t^v\}_{t\in[0,1]}$ of $v \uplus U$ onto K_v^-.

For each simplex $U \uplus v \in B_2$, a strong deformation retraction of $U \uplus v$ onto $K_v^+ := \|\overline{G}\| \cap (U \uplus v)$ is defined as $\{\nu \circ f_t^v \circ \nu\}_{t\in[0,1]}$. Let $X_v = (v \uplus U) \cup (U \uplus v)$, for any $v \in V(G)$. Then, a strong $\mathbb{Z}_2$-deformation retraction F_v of X_v onto $K_v^- \cup K_v^+$ is defined as

$$F_v(x,t) = \begin{cases} f_t^v(x) & \text{if } x \in v \uplus U, \\ \nu \circ f_t^v \circ \nu(x) & \text{if } x \in U \uplus v, \end{cases}$$

where $t \in [0,1]$. Since the homotopies F_u and F_v are stationary on $X_u \cap X_v$ for $u, v \in V(G)$ by Lemma 2, we see that the homotopies $\{F_v \,|\, v \in V(G)\}$ induce a strong $\mathbb{Z}_2$-deformation retraction of $\|\mathsf{B}(G)\|$ onto $\|\overline{G}\|$. □

For (2) above, this theorem shows that $\|\mathsf{L}\|$ is indeed a strong $\mathbb{Z}_2$-deformation retract of $\|\mathsf{B}(G)\|$ if G contains no 4-cycle. The theorem also shows that the converse of this also holds and that we have $\mathrm{ind}_{\mathbb{Z}_2}(\|\mathsf{B}(G)\|) = \mathrm{ind}_{\mathbb{Z}_2}(\|\mathsf{L}\|) = \mathrm{ind}_{\mathbb{Z}_2}(\|\overline{G}\|)$. On the other hand, $\|\overline{G}\|$ is the 1-dimensional complex with the $\mathbb{Z}_2$-action which has no fixed point, so that we have [1] $\mathrm{ind}_{\mathbb{Z}_2}(\|\overline{G}\|) \leq 1$. The homotopy type of $\|\overline{G}\|$ and the $\mathbb{Z}_2$-index of $\|\overline{G}\|$ are determined by the following theorem:

Theorem 4 ([1], Theorem 4.4). Let G be a connected graph with k induced cycles of G.

(1) If G has no cycle of odd length, we have $\|\overline{G}\| \simeq \bigvee_k S^1 \amalg \bigvee_k S^1$ and $\mathrm{ind}_{\mathbb{Z}_2}(\|\overline{G}\|) = 0$.

(2) If G has at least one cycle of odd length, we have $\|\overline{G}\| \simeq \bigvee_{2k-1} S^1$ and $\mathrm{ind}_{\mathbb{Z}_2}(\|\overline{G}\|) = 1$. □

As a conclusion, if a graph G contains no 4-cycle, the homotopy type of $\|\mathsf{B}(G)\|$ and the $\mathbb{Z}_2$-index of $\|\mathsf{B}(G)\|$ is determined by Theorem 3 and 4.

Corollary 5 ([2], Corollary 4.5). Let G be a graph without 4-cycles and k the number of induced cycles of G.

[1] Let $\|\mathsf{K}\|$ be an n-dimensional simplicial complex with a $\mathbb{Z}_2$-action which has no fixed point, then we have $\mathrm{ind}_{\mathbb{Z}_2}(\|\mathsf{K}\|) \leq n$ (see [4], p.96).

(1) If G has no cycle of odd length, we have $\|\mathsf{B}(G)\| \simeq \bigvee_k S^1 \amalg \bigvee_k S^1$ and $\mathrm{ind}_{\mathbb{Z}_2}(\|\mathsf{B}(G)\|) = 0$.
(2) If G has at least one cycle of odd length, we have $\|\mathsf{B}(G)\| \simeq \bigvee_{2k-1} S^1$ and $\mathrm{ind}_{\mathbb{Z}_2}(\|\mathsf{B}(G)\|) = 1$. □

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Institute of Mathematics University of Tsukuba, Tsukuba-shi, Ibaraki 305-8571, Japan
E-mail address: akira04k16@math.tsukuba.ac.jp