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Coefficient conditions for certain univalent functions

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Abstract

For functions $f(z)$ which belong to $\mathcal{T}(\alpha)$, $\mathcal{U}(\alpha)$, and $\mathcal{CC}_{\lambda}(\alpha;g(z))$ in the open unit disk $\mathbb{U}$, some interesting sufficient conditions for coefficient inequalities of $f(z)$ are discussed.

1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

(1.1)
$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$.

Furthermore, let $\mathcal{P}$ be the class of functions $p(z)$ of the form

(1.2)
$$p(z)=1+\sum_{n=1}^{\infty}p_nz^n$$

which are analytic in $\mathbb{U}$.
If $f(z)\in\mathcal{A}$ satisfies

(1.3)
$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)>0 \qquad (z\in\mathbb{U}),$$

then $f(z)$ is said to be starlike in $\mathbb{U}$.
We denote by $\mathcal{S}^*$ all functions $f(z)$ which are starlike in $\mathbb{U}$. Also, $\mathcal{K}$ is said to be the class of convex functions $f(z)$ if $f(z)\in\mathcal{A}$ satisfy

(1.4)
$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>0 \qquad (z\in\mathbb{U}).$$

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We begin with the definitions for the subclasses $\mathcal{T}(\alpha)$, $\mathcal{U}(\alpha)$ and $\mathcal{CC}_\lambda(\alpha; g(z))$ of $\mathcal{A}$.

Definition 1.1 *A function $f(z) \in \mathcal{A}$ belongs to $\mathcal{T}(\alpha)$ if and only if it satisfies*

$$\text{Re}\frac{f(z)}{z} > \alpha \qquad (z \in \mathbb{U}) \tag{1.5}$$

for some α $(0 \leqq \alpha < 1)$.

Definition 1.2 *A function $f(z) \in \mathcal{A}$ is in the class $\mathcal{U}(\alpha)$ if and only if it satisfies*

$$\text{Re} f'(z) > \alpha \qquad (z \in \mathbb{U}) \tag{1.6}$$

for some α $(0 \leqq \alpha < 1)$.

Definition 1.3 (see, for details, [2]) *If $f(z) \in \mathcal{A}$ satisfies*

$$\text{Re } e^{i\lambda}\left(\frac{zf'(z)}{g(z)} - \alpha\right) > 0 \qquad (z \in \mathbb{U}) \tag{1.7}$$

for some α $(0 \leqq \alpha < 1)$, λ $(-\frac{\pi}{2} < \lambda < \frac{\pi}{2})$ and starlike function $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, then $f(z)$ is said to be close-to-convex of order α with respect to a fixed starlike function $g(z)$, and let $\mathcal{CC}_\lambda(\alpha; g(z))$ denote the class of functions $f(z)$ satisfying this condition.

Remark Replacing $g(z)$ by $f(z)$ in (1.7), we say that $f(z)$ is said to be λ-spiral of order α in $\mathbb{U}$, and write $\mathcal{SP}(\lambda, \alpha)$ defined by

$$\mathcal{SP}(\lambda, \alpha) = \left\{ f(z) \in \mathcal{A} \ : \ \text{Re } e^{i\lambda}\left(\frac{zf'(z)}{f(z)} - \alpha\right) > 0 \right\}.$$

We need the following lemmas to prove our results.

Lemma 1.1 (see, [1], [3]) *A function $p(z) \in \mathcal{P}$ satisfies* Re $p(z) > 0$ $(z \in \mathbb{U})$ *if and only if*

$$p(z) \neq \frac{x-1}{x+1} \qquad (z \in \mathbb{U})$$

for all $|x| = 1$.

Lemma 1.2 *A function $f(z) \in \mathcal{A}$ is in $\mathcal{T}(\alpha)$ if and only if*

$$1 + \sum_{n=2}^{\infty} A_n z^{n-1} \neq 0 \tag{1.8}$$

where

$$A_n = \frac{x+1}{2(1-\alpha)} a_n \quad (n \geqq 2).$$

Proof. Putting $p(z) = \dfrac{\frac{f(z)}{z} - \alpha}{1-\alpha}$ for $f(z) \in \mathcal{T}(\alpha)$, we obtain that $p(z) \in \mathcal{P}$, and $\operatorname{Re} p(z) > 0$. Using Lemma 1.1, we have that

$$\frac{\frac{f(z)}{z} - \alpha}{1-\alpha} \neq \frac{x-1}{x+1} \qquad (\text{for all } |x| = 1,\ z \in \mathbb{U}).$$

Then, we need not consider Lemma 1.1 for $z = 0$, because it follows that

$$p(0) = 1 \neq \frac{x-1}{x+1}.$$

This implies that

$$(x+1)f(z) + (1 - 2\alpha - x)z \neq 0. \tag{1.9}$$

It follows that (1.9) is equivalent to

$$(x+1)\left(z + \sum_{n=2}^{\infty} a_n z^n\right) + (1 - 2\alpha - x)z \neq 0$$

or

$$2(1-\alpha)z\left\{1 + \sum_{n=2}^{\infty} \frac{x+1}{2(1-\alpha)} a_n z^{n-1}\right\} \neq 0. \tag{1.10}$$

Dividing the both sides of (1.10) by $2(1-\alpha)z$ $(z \neq 0)$, we know that

$$1 + \sum_{n=2}^{\infty} \frac{x+1}{2(1-\alpha)} a_n z^{n-1} \neq 0.$$

This completes the proof of lemma. □

2 Coefficient conditions for functions in the classes $\mathcal{T}(\alpha)$ and $\mathcal{CC}_\lambda(\alpha; g(z))$

Our result for $f(z)$ to be in $\mathcal{T}(\alpha)$ is contained in

Theorem 2.1 *If $f(z) \in \mathcal{A}$ satisfies the following condition*

$$\sum_{n=2}^{\infty} \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} (-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| \leqq 1 - \alpha$$

for some α $(0 \leqq \alpha < 1)$, $\beta \in \mathbb{R}$ and $\gamma \in \mathbb{R}$, then $f(z) \in \mathcal{T}(\alpha)$.

Proof. Note that

$$(1-z)^{\beta} \neq 0, \ (1+z)^{\gamma} \neq 0 \ (\beta, \ \gamma \in \mathbb{R}; \ z \in \mathbb{U}).$$

Hence if the following inequality

$$\left(1+\sum_{n=2}^{\infty} A_n z^{n-1}\right)(1-z)^{\beta}(1+z)^{\gamma} \neq 0 \tag{2.1}$$

holds true, then we have

$$1+\sum_{n=2}^{\infty} A_n z^{n-1} \neq 0,$$

which is the relation (1.8) of Lemma 1.2. We know that (2.1) is equivalent to

$$1+\sum_{n=2}^{\infty}\left[\sum_{k=1}^{n}\left\{\sum_{j=1}^{k} A_j(-1)^{k-j}\binom{\beta}{k-j}\right\}\binom{\gamma}{n-k}\right] z^{n-1} \neq 0.$$

Therefore, if $f(z) \in \mathcal{A}$ satisfies

$$\sum_{n=2}^{\infty}\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k} A_j(-1)^{k-j}\binom{\beta}{k-j}\right\}\binom{\gamma}{n-k}\right| \leqq 1,$$

that is, that

$$\begin{aligned}
&\frac{1}{2(1-\alpha)}\sum_{n=2}^{\infty}\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(x+1)a_j(-1)^{k-j}\binom{\beta}{k-j}\right\}\binom{\gamma}{n-k}\right| \\
&\leqq \frac{1}{2(1-\alpha)}\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(-1)^{k-j}\binom{\beta}{k-j}a_j\right\}\binom{\gamma}{n-k}\right|\right. \\
&\qquad\left.+|x|\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(-1)^{k-j}\binom{\beta}{k-j}a_j\right\}\binom{\gamma}{n-k}\right|\right] \\
&= \frac{1}{1-\alpha}\sum_{n=2}^{\infty}\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(-1)^{k-j}\binom{\beta}{k-j}a_j\right\}\binom{\gamma}{n-k}\right| \leqq 1,
\end{aligned}$$

then $f(z) \in \mathcal{T}(\alpha)$. This completes the proof of Theorem 2.1. □

Putting $\beta = \gamma = 0$ in Theorem 2.1, we see the following corollary.

Corollary 2.1 *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty}|a_n| \leqq 1-\alpha$$

for some α $(0 \leqq \alpha < 1)$, then $f(z) \in \mathcal{T}(\alpha)$.

Next, we derive the coefficient condition for $f(z)$ to be in the class $\mathcal{U}(\alpha)$.

Theorem 2.2 *If $f(z) \in \mathcal{A}$ satisfies the following condition*

$$\sum_{n=2}^{\infty}\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(-1)^{k-j}\binom{\beta}{k-j}ja_j\right\}\binom{\gamma}{n-k}\right| \leqq 1-\alpha$$

for some α $(0 \leqq \alpha < 1)$, $\beta \in \mathbb{R}$, and $\gamma \in \mathbb{R}$, then $f(z) \in \mathcal{U}(\alpha)$.

Proof. Since $f(z) \in \mathcal{U}(\alpha) \Leftrightarrow zf'(z) \in \mathcal{T}(\alpha)$ and

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \ , \ zf'(z) = z + \sum_{n=2}^{\infty} n a_n z^n,$$

replacing a_j of Theorem 2.1 with ja_j, we prove the theorem. □

Taking $\beta = \gamma = 0$ in Theorem 2.2, we obtain

Corollary 2.2 *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} n|a_n| \leqq 1-\alpha$$

for some α $(0 \leqq \alpha < 1)$, then $f(z) \in \mathcal{U}(\alpha)$.

Lemma 2.1 *A function $f(z) \in \mathcal{A}$ is in $\mathcal{CC}_\lambda(\alpha; g(z))$ if and only if*

$$1 + \sum_{n=2}^{\infty} B_n z^{n-1} \neq 0 \tag{2.2}$$

where

$$B_n = \frac{na_n + (2(1-\alpha)e^{-i\lambda}\cos\lambda - 1)b_n + x(na_n - b_n)}{2(1-\alpha)e^{-i\lambda}\cos\lambda}. \tag{2.3}$$

Proof. Letting $p(z) = \dfrac{e^{i\lambda}\left(\frac{zf'(z)}{g(z)} - \alpha\right) - i(1-\alpha)\sin\lambda}{(1-\alpha)\cos\lambda}$, we see that $p(z) \in \mathcal{P}$ and $\mathrm{Re}\, p(z) > 0$ $(z \in \mathbb{U})$. It follows from Lemma 1.1 that

$$\frac{e^{i\lambda}\left(\dfrac{zf'(z)}{g(z)} - \alpha\right) - i(1-\alpha)\sin\lambda}{(1-\alpha)\cos\lambda} \neq \frac{x-1}{x+1} \qquad (\text{for all } |x| = 1,\ z \in \mathbb{U}). \tag{2.4}$$

Then, we need not consider Lemma 1.1 for $z = 0$, because it follows that

$$p(0) = 1 \neq \frac{x-1}{x+1}.$$

Since (2.4) implies that

$$(x+1)\left\{e^{i\lambda}(zf'(z)-\alpha g(z))-i(1-\alpha)g(z)\sin\lambda\right\}\neq(x-1)(1-\alpha)g(z)\cos\lambda,$$

we obtain that

$$(2.5)\qquad (x+1)e^{i\lambda}zf'(z)-\alpha e^{i\lambda}g(z)-x\alpha e^{i\lambda}g(z)-i(1-\alpha)g(z)\sin\lambda-ix(1-\alpha)g(z)\sin\lambda \neq x(1-\alpha)g(z)\cos\lambda-(1-\alpha)g(z)\cos\lambda.$$

The relation (2.5) is equivalent to

$$(x+1)e^{i\lambda}zf'(z)-\alpha e^{i\lambda}g(z)-x\alpha e^{i\lambda}g(z)-x(1-\alpha)e^{i\lambda}g(z)+(1-\alpha)e^{-i\lambda}g(z)\neq 0$$

that is,

$$(1+x)e^{i\lambda}zf'(z)+(e^{-i\lambda}-xe^{i\lambda}-2\alpha\cos\lambda)g(z)\neq 0.$$

Note that the above relation can be weitten with

$$(x+1)e^{i\lambda}\left(z+\sum_{n=2}^{\infty}na_nz^n\right)+(e^{-i\lambda}-xe^{i\lambda}-2\alpha\cos\lambda)\left(z+\sum_{n=2}^{\infty}b_nz^n\right)\neq 0$$

or

$$(2.6)\qquad 2(1-\alpha)\cos\lambda\ z\left\{1+\sum_{n=2}^{\infty}\frac{n(x+1)a_n+(e^{-2i\lambda}-x-2\alpha e^{-i\lambda}\cos\lambda)b_n}{2(1-\alpha)e^{-i\lambda}\cos\lambda}z^{n-1}\right\}\neq 0.$$

Dividing the both sides of (2.6) by $2(1-\alpha)\cos\lambda\ z$ $(z\neq 0)$ and noting

$$(2.7)\qquad e^{-2i\lambda}=-1+2e^{-i\lambda}\cos\lambda,$$

we know that

$$1+\sum_{n=2}^{\infty}\frac{na_n+(2(1-\alpha)e^{-i\lambda}-1)b_n+x(na_n-b_n)}{2(1-\alpha)e^{-i\lambda}\cos\lambda}z^{n-1}\neq 0.$$

This completes the proof of the lemma. □

Applying Lemma 2.1, we obtain

Theorem 2.3 *If $f(z)\in\mathcal{A}$ satisfies the following condition*

$$\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(ja_j+((1-\alpha)e^{-2i\lambda}-\alpha)b_j)(-1)^{k-j}\binom{\beta}{k-j}\right\}\binom{\gamma}{n-k}\right| + \left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(ja_j-b_j)(-1)^{k-j}\binom{\beta}{k-j}\right\}\binom{\gamma}{n-k}\right|\right]\leqq 2(1-\alpha)\cos\lambda$$

for some α $(0\leqq\alpha<1)$, λ $(-\frac{\pi}{2}<\lambda<\frac{\pi}{2})$, $\beta\in\mathbb{R}$, $\gamma\in\mathbb{R}$ and $g(z)=z+\sum_{n=2}^{\infty}b_nz^n\in\mathcal{S}^*$, then $f(z)\in\mathcal{CC}_\lambda(\alpha;g(z))$.

Proof. Applying the same method of the proof in Theorem 2.1, we know that $f(z)$ belongs to $\mathcal{CC}_\lambda(\alpha; g(z))$ if $f(z) \in \mathcal{A}$ satisfies

$$1 + \sum_{n=2}^{\infty} \left[\sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} B_j (-1)^{k-j} c_{k-j} \right\} d_{n-k} \right] \neq 0$$

where $c_n = \binom{\beta}{n}$, $d_n = \binom{\gamma}{n}$ and B_j is defined by (2.3).

Now, we consider that

$$\begin{aligned}
&\sum_{n=2}^{\infty} \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} B_j (-1)^{k-j} c_{k-j} \right\} d_{n-k} \right| \\
&= \sum_{n=2}^{\infty} \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} \frac{ja_j + (2(1-\alpha)e^{-i\lambda}\cos\lambda - 1)b_j + x(ja_j - b_j)}{2(1-\alpha)e^{-i\lambda}\cos\lambda} (-1)^{k-j} c_{k-j} \right\} d_{n-k} \right| \\
&\leqq \frac{1}{2(1-\alpha)\cos\lambda} \sum_{n=2}^{\infty} \left[\left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} \left(ja_j + (2(1-\alpha)e^{-i\lambda}\cos\lambda - 1)b_j \right)(-1)^{k-j} c_{k-j} \right\} d_{n-k} \right| \right. \\
&\qquad \left. + |x| \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} (ja_j - b_j)(-1)^{k-j} c_{k-j} \right\} d_{n-k} \right| \right] \\
&\leqq 1.
\end{aligned}$$

This implies that if $f(z) \in \mathcal{A}$ satisfies

$$\begin{aligned}
&\sum_{n=2}^{\infty} \left[\left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} \left(ja_j + (2(1-\alpha)e^{-i\lambda}\cos\lambda - 1)b_j \right)(-1)^{k-j} \binom{\beta}{k-j} \right\} \binom{\gamma}{n-k} \right| \right. \\
&\qquad \left. + \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} (ja_j - b_j)(-1)^{k-j} \binom{\beta}{k-j} \right\} \binom{\gamma}{n-k} \right| \right] \leqq 2(1-\alpha)\cos\lambda,
\end{aligned}$$

then $f(z) \in \mathcal{CC}_\lambda(\alpha; g(z))$. This completes the proof of Theorem 2.3. □

Considering $g(z) = f(z)$ in Theorem 2.3 and noting (2.7), we have the following corollary.

Corollary 2.3 (see, [1], Thorem 3) *If $f(z) \in \mathcal{A}$ satisfies the following inequality*

$$\begin{aligned}
&\sum_{n=2}^{\infty} \left[\left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} (j - \alpha + (1-\alpha)e^{-2i\lambda})(-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| \right. \\
&\qquad \left. + \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} (j-1)(-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| \right] \leqq 2(1-\alpha)\cos\lambda
\end{aligned}$$

for some α $(0 \leqq \alpha < 1)$, λ $\left(-\frac{\pi}{2} < \lambda < \frac{\pi}{2}\right)$, $\beta \in \mathbb{R}$ *and* $\gamma \in \mathbb{R}$, *then* $f(z) \in \mathcal{SP}(\lambda, \alpha)$.

Furthermore, setting $\lambda = 0$ in Theorem 2.3, we obtain the following condition for $\mathcal{CC}_0(\alpha; g(z))$.

Corollary 2.4 *If* $f(z) \in \mathcal{A}$ *satisfies the following condition*

$$\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(ja_j + (1-2\alpha)b_j)(-1)^{k-j}\binom{\beta}{k-j}\right\}\binom{\gamma}{n-k}\right|\right.$$
$$\left.+\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(ja_j - b_j)(-1)^{k-j}\binom{\beta}{k-j}\right\}\binom{\gamma}{n-k}\right|\right] \leqq 2(1-\alpha)$$

for some α $(0 \leqq \alpha < 1)$, $\beta \in \mathbb{R}$, $\gamma \in \mathbb{R}$ *and* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*$, *then* $f(z) \in \mathcal{CC}_0(\alpha; g(z))$.

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