数理解析研究所講究録
第 1579 巻 2008 年 25-36



# Some subordination criteria for analytic functions

Kazuo Kuroki and Shigeyoshi Owa

## 1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n,$$

which are analytic in the open unit disk

$$\mathbb{U}=\{z:z\in\mathbb{C}\quad and\quad |z|<1\}.$$

For functions $f(z)$ and $g(z)$ in the class $\mathcal{A}$, we say that $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$ if there exists an analytic function $w(z)$ satisfying $w(0)=0$, $|w(z)|<1 \quad (z\in\mathbb{U})$ and $f(z)=g(w(z)) \quad (z\in\mathbb{U})$. We denote this subordination by $f(z)\prec g(z)$. In particular, if $g(z)$ is univalent in $\mathbb{U}$, then $f(z)\prec g(z)$ is equivalent to $f(0)=g(0)$ and $f(\mathbb{U})\subset g(\mathbb{U})$.

We need the following lemma given by Miller and Mocanu [2] (see also [3, p. 132]).

**Lemma 1.1** *Let the function $q(z)$ be analytic and univalent in $\mathbb{U}$. Also let $\phi(\omega)$ and $\psi(\omega)$ be analytic in a domain $C$ containing $q(\mathbb{U})$, with*

$$\psi(\omega)\neq 0 \qquad (\omega\in q(\mathbb{U})\subset C).$$

*Set*

$$Q(z)=zq'(z)\psi(q(z)) \quad and \quad h(z)=\phi(q(z))+Q(z),$$

*and suppose that*

(*i*) $\qquad$ *$Q(z)$ is starlike and univalent in $\mathbb{U}$;*

*and*

(*ii*) $$\mathrm{Re}\left(\frac{zh'(z)}{Q(z)}\right)=\mathrm{Re}\left(\frac{\phi'(q(z))}{\psi(q(z))}+\frac{zQ'(z)}{Q(z)}\right)>0 \qquad (z\in\mathbb{U}).$$

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**2000 Mathematics Subject Classification** *: Primary 30C45.*
**Key Words and Phrases** *: Subordination, univalent, starlike.*

*If $p(z)$ is analytic in $\mathbb{U}$, with*

$$p(0) = q(0) \quad \textit{and} \quad p(\mathbb{U}) \subset \mathcal{C},$$

*and*

$$\phi(p(z)) + zp'(z)\psi(p(z)) \prec \phi(q(z)) + zq'(z)\psi(q(z)) =: h(z) \qquad (z \in \mathbb{U}),$$

*then*

$$p(z) \prec q(z) \qquad (z \in \mathbb{U})$$

*and $q(z)$ is the best dominant of this subordination.*

By making use of Lemma 1.1, Kuroki, Owa and Srivastava [1] deduced each of the following lemmas.

**Lemma 1.2** *Let the function $f(z) \in \mathcal{A}$ be so chosen that*

$$\frac{f(z)}{z} \neq 0 \qquad (z \in \mathbb{U}).$$

*Suppose also that the real parameters $\alpha$ ($\alpha \neq 0$) and $\beta$ ($-1 \leqq \beta \leqq 1$), as well as the complex parameters $A$ and $B$ constrained by*

$$|A| \leqq 1, \ |B| < 1, \ A \neq B, \ \textit{and} \ \operatorname{Re}(1 - A\overline{B}) \geqq |A - B|,$$

*are so prescribed that*

$$\frac{\beta(1-\alpha)}{\alpha} + \frac{(1+\beta)\{\operatorname{Re}(1-A\overline{B}) - |A-B|\}}{1-|B|^2} + \frac{1-\beta}{1+|A|} + \frac{1+\beta}{1+|B|} - 1 \geqq 0.$$

*If*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\beta}\left(1 + \alpha\frac{zf''(z)}{f'(z)}\right) \prec h(z) \qquad (z \in \mathbb{U}),$$

*where*

$$h(z) = \left(\frac{1+Az}{1+Bz}\right)^{\beta-1}\left\{(1-\alpha)\frac{1+Az}{1+Bz} + \frac{\alpha(1+Az)^2 + \alpha(A-B)z}{(1+Bz)^2}\right\},$$

*then*

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}).$$

**Lemma 1.3** *Let the function $f(z) \in \mathcal{A}$ be so chosen that*

$$\frac{f(z)}{z} \neq 0 \qquad (z \in \mathbb{U}).$$

*Suppose also that the real parameters $\alpha$ ($\alpha \neq 0$) and $\beta$ ($-1 \leqq \beta \leqq 1$), as well as the complex parameters $A$ and $B$ constrained by*

$$|A| \leqq 1, \ |B| = 1, \ A \neq B, \ \textit{and} \ 1 - A\overline{B} > 0,$$

*are so prescribed that*

$$\frac{\beta(1-\alpha)}{\alpha}+\frac{(1+\beta)(1-|A|^2)}{2(1-A\overline{B})}+\frac{(1-\beta)(1-|A|)}{2(1+|A|)} \geqq 0.$$

*If*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\beta}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right) \prec h(z) \qquad (z \in \mathbb{U}),$$

*where*

$$h(z)=\left(\frac{1+Az}{1+Bz}\right)^{\beta-1}\left\{(1-\alpha)\frac{1+Az}{1+Bz}+\frac{\alpha(1+Az)^2+\alpha(A-B)z}{(1+Bz)^2}\right\},$$

*then*

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}).$$

## 2 Main results

In this section, we begin by starting and proving one of our main results.

**Theorem 2.1** *Let the function $f(z) \in \mathcal{A}$ be so chosen that*

$$\frac{f(z)}{z} \neq 0 \qquad (z \in \mathbb{U}).$$

*Suppose also that the real parameters $\alpha$ $(\alpha \neq 0)$, $\beta$ $(-1 \leqq \beta \leqq 1)$, and*

$$\delta\left(1 \leqq \delta \leqq \frac{\pi}{2\left(\left|\sin^{-1}\frac{\operatorname{Im}(1-A\overline{B})}{|1-A\overline{B}|}\right|+\cos^{-1}\frac{\sqrt{(1-|A|^2)(1-|B|^2)}}{|1-A\overline{B}|}\right)}\right),$$

*as well as the complex parameters $A$ and $B$ constrained by*

$$|A| \leqq 1,\ |B|<1,\ A \neq B,\ \text{and}\ \operatorname{Re}(1-A\overline{B}) \geqq |A-B|,$$

*are so prescribed that*

$$1-|A||B|+\delta\beta|A|-\delta\beta|B| \geqq 0,$$

*and*

$$\frac{\beta(1-\alpha)}{\alpha}+(1+\beta)m_0+\frac{1-\delta\beta}{1+|A|}+\frac{1+\delta\beta}{1+|B|} \geqq 0,$$

*where*

$$m_0=\min_{w}\left\{|w|^{\delta}\cos\left(\delta\left(\left|\sin^{-1}\frac{\operatorname{Im}(1-A\overline{B})}{|1-A\overline{B}|}\right|+\cos^{-1}\frac{1-|A|^2+|w|^2(1-|B|^2)}{2|w||1-A\overline{B}|}\right)\right)\right\}:$$

$$\left(\frac{|1-A\overline{B}-|A-B||}{1-|B|^2} \leqq |w| \leqq \frac{\sqrt{(1-|A|^2)(1-|B|^2)}}{1-|B|^2}\right).$$

*If*

(1)
$$\left(\frac{zf'(z)}{f(z)}\right)^{\beta}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right) \prec h(z) \qquad (z \in \mathbb{U}),$$

*where*

$$h(z) = \left(\frac{1+Az}{1+Bz}\right)^{\delta\beta-1}\left\{(1-\alpha)\frac{1+Az}{1+Bz} + \frac{\alpha(1+Az)^{1+\delta}(1+Bz)^{1-\delta}+\alpha\delta(A-B)z}{(1+Bz)^2}\right\},$$

*then*

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+Az}{1+Bz}\right)^{\delta} \qquad (z \in \mathbb{U}).$$

*Proof.* If we define the function $p(z)$ by

$$p(z) = \frac{zf'(z)}{f(z)} \qquad (z \in \mathbb{U}),$$

then $p(z)$ is analytic in $\mathbb{U}$ and the condition (1) can be written as follows:

$$\{p(z)\}^{\beta}\{(1-\alpha)+\alpha p(z)\} + \alpha z p'(z)\{p(z)\}^{\beta-1} \prec h(z) \qquad (z \in \mathbb{U}).$$

We also set

$$q(z) = \left(\frac{1+Az}{1+Bz}\right)^{\delta}, \quad \phi(z) = z^{\beta}(1-\alpha+\alpha z), \quad \textit{and} \quad \psi(z) = \alpha z^{\beta-1}$$

for $z \in \mathbb{U}$. Then, clearly, the function $q(z)$ is analytic and univalent in $\mathbb{U}$.
Now, for $q_1(z) = \dfrac{1+Az}{1+Bz}$, it is clear that $q_1(z)$ is univalent in $\mathbb{U}$ and $q_1(\mathbb{U})$ is the open disk given by

$$\left|q_1 - \frac{1-A\overline{B}}{1-|B|^2}\right| < \frac{|A-B|}{1-|B|^2},$$

which shows that

$$\operatorname{Re}(q_1(z)) > \frac{\operatorname{Re}(1-A\overline{B})-|A-B|}{1-|B|^2} \geqq 0 \qquad (z \in \mathbb{U}).$$

Here, we define that

$$W := \left\{w : w \in \mathbb{C} \quad \textit{and} \quad \left|w - \frac{1-A\overline{B}}{1-|B|^2}\right| = \frac{|A-B|}{1-|B|^2}\right\}.$$

Then, for $w \in W$, $\arg(w)$ satisfy the following condition:

$$\sin^{-1}\frac{\operatorname{Im}(1-A\overline{B})}{|1-A\overline{B}|}-\cos^{-1}\frac{\sqrt{(1-|A|^2)(1-|B|^2)}}{|1-A\overline{B}|}\leqq \arg(w)$$

$$\leqq \sin^{-1}\frac{\operatorname{Im}(1-A\overline{B})}{|1-A\overline{B}|}+\cos^{-1}\frac{\sqrt{(1-|A|^2)(1-|B|^2)}}{|1-A\overline{B}|}.$$

From this condition, for

$$1\leqq\delta\leqq\frac{\pi}{2\left(\left|\sin^{-1}\frac{\operatorname{Im}(1-A\overline{B})}{|1-A\overline{B}|}\right|+cos^{-1}\frac{\sqrt{(1-|A|^2)(1-|B|^2)}}{|1-A\overline{B}|}\right)},$$

we obtain

$$|\delta\arg(w)|\leqq\frac{\pi}{2}.$$

Thus we see that

$$\operatorname{Re}(w^\delta)=|w|^\delta\cos(\delta\arg(w))\geqq 0.$$

Furthermore, we define that

$$W_1:=\left\{w_1 : w_1\in W \quad and \quad \arg(w_1)\geqq\arg\left(\frac{1-A\overline{B}}{1-|B|^2}\right)\right\},$$

and

$$W_2:=\left\{w_2 : w_2\in W \quad and \quad \arg(w_2)<\arg\left(\frac{1-A\overline{B}}{1-|B|^2}\right)\right\}.$$

Then, $\arg(w_1)$ and $\arg(w_2)$ can write to as follows:

$$\begin{cases}\arg(w_1)=\sin^{-1}\frac{\operatorname{Im}(1-A\overline{B})}{|1-A\overline{B}|}+\cos^{-1}\frac{1-|A|^2+|w_1|^2(1-|B|^2)}{2|w_1||1-A\overline{B}|},\\[2ex] \arg(w_2)=\sin^{-1}\frac{\operatorname{Im}(1-A\overline{B})}{|1-A\overline{B}|}-\cos^{-1}\frac{1-|A|^2+|w_2|^2(1-|B|^2)}{2|w_2||1-A\overline{B}|}.\end{cases}$$

Here, the following two things can be said about the minimum value of $\operatorname{Re}(w^\delta)$ $(w\in W)$.

(*i*) When $\operatorname{Im}(1-A\overline{B})\geqq 0$, for $w_1\in W_1$,

$$\min_{w_1}\{\operatorname{Re}(w_1^\delta)\} : \left(\frac{|1-A\overline{B}-|A-B||}{1-|B|^2}\leqq|w_1|\leqq\frac{\sqrt{(1-|A|^2)(1-|B|^2)}}{1-|B|^2}\right)$$

is minimum value of $\operatorname{Re}(w^\delta)$ $(w\in W)$.
Then, we obtain

$$\operatorname{Re}(w_1^\delta)=|w_1|^\delta\cos\left(\delta\left(\sin^{-1}\frac{\operatorname{Im}(1-A\overline{B})}{|1-A\overline{B}|}+\cos^{-1}\frac{1-|A|^2+|w_1|^2(1-|B|^2)}{2|w_1||1-A\overline{B}|}\right)\right).$$

(*ii*) When $\mathrm{Im}(1-A\overline{B})<0$, for $w_2\in W_2$,

$$\min_{w_2}\{\mathrm{Re}(w_2^\delta)\}:\left(\frac{|1-A\overline{B}-|A-B||}{1-|B|^2}\leqq |w_2|\leqq \frac{\sqrt{(1-|A|^2)(1-|B|^2)}}{1-|B|^2}\right)$$

is minimum value of $\mathrm{Re}(w^\delta)$ $(w\in W)$.
Then, we can write

$$\mathrm{Re}(w_2^\delta)=|w_2|^\delta\cos\left(\delta\left(\sin^{-1}\frac{\mathrm{Im}(1-A\overline{B})}{|1-A\overline{B}|}-\cos^{-1}\frac{1-|A|^2+|w_2|^2(1-|B|^2)}{2|w_2||1-A\overline{B}|}\right)\right)$$
$$=|w_2|^\delta\cos\left(\delta\left(\left|\sin^{-1}\frac{\mathrm{Im}(1-A\overline{B})}{|1-A\overline{B}|}\right|+\cos^{-1}\frac{1-|A|^2+|w_2|^2(1-|B|^2)}{2|w_2||1-A\overline{B}|}\right)\right).$$

From the above-mentioned, for $w\in W$, we see that

$$m_0:=\min_{w}\left\{|w|^\delta\cos\left(\delta\left(\left|\sin^{-1}\frac{\mathrm{Im}(1-A\overline{B})}{|1-A\overline{B}|}\right|+\cos^{-1}\frac{1-|A|^2+|w|^2(1-|B|^2)}{2|w||1-A\overline{B}|}\right)\right)\right\}:$$
$$\left(\frac{|1-A\overline{B}-|A-B||}{1-|B|^2}\leqq |w|\leqq \frac{\sqrt{(1-|A|^2)(1-|B|^2)}}{1-|B|^2}\right)$$

is minimum value of $\mathrm{Re}(w^\delta)$ $(w\in W)$.

Thus, we obtain

$$\mathrm{Re}(q(z))>m_0\geqq 0\qquad (z\in\mathbb{U}).$$

Therefore, $\phi$ and $\psi$ are analytic in a domain $\mathcal{C}$ containing $q(\mathbb{U})$, with

$$\psi(\omega)\neq 0\qquad (\omega\in q(\mathbb{U})\subset\mathcal{C}).$$

The function $Q(z)$ given by

$$Q(z)=zq'(z)\psi(q(z))=\frac{\alpha(A-B)z(1+Az)^{\beta-1}}{(1+Bz)^{\beta+1}}$$

is univalent and starlike in $\mathbb{U}$, because

$$\mathrm{Re}\left(\frac{zQ'(z)}{Q(z)}\right)=(1-\delta\beta)\mathrm{Re}\left(\frac{1}{1+Az}\right)+(1+\delta\beta)\mathrm{Re}\left(\frac{1}{1+Bz}\right)-1$$
$$>\frac{1-\delta\beta}{1+|A|}+\frac{1+\delta\beta}{1+|B|}-1=\frac{1-|A||B|+\delta\beta|A|-\delta\beta|B|}{(1+|A|)(1+|B|)}\geqq 0\qquad (z\in\mathbb{U}).$$

Furthermore, we have

$$h(z)=\phi(q(z))+Q(z)$$
$$=\left(\frac{1+Az}{1+Bz}\right)^{\delta\beta}\left\{1-\alpha+\alpha\left(\frac{1+Az}{1+Bz}\right)^{\delta}\right\}+\frac{\alpha\delta(A-B)z(1+Az)^{\delta\beta-1}}{(1+Bz)^{\delta\beta+1}},$$

and

$$\mathrm{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \frac{\beta(1-\alpha)}{\alpha} + (1+\beta)\mathrm{Re}(q(z)) + \mathrm{Re}\left(\frac{zQ'(z)}{Q(z)}\right)$$
$$> \frac{\beta(1-\alpha)}{\alpha} + (1+\beta)m_0 + \frac{1-\delta\beta}{1+|A|} + \frac{1+\delta\beta}{1+|B|} - 1 \geqq 0 \qquad (z \in \mathbb{U}).$$

Since all conditions of Lemma 1.1 are satisfied, we conclude that

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+Az}{1+Bz}\right)^{\delta} \qquad (z \in \mathbb{U}).$$

This completes the proof of Theorem 2.1. □

**Remark 2.1** Letting $\delta = 1$ in Theorem 2.1, we obtain Lemma 1.2 proven by Kuroki, Owa and Srivastava [1, Theorem 2].

Also, taking $A, B \in \mathbb{R}$ $(-1 < B < A \leqq 1$ *or* $-1 \leqq A < B < 1)$ in Theorem 2.1, we get the following Corollary 2.1 and Corollary 2.2.

**Corollary 2.1** *Let the function $f(z) \in \mathcal{A}$ be so chosen that*

$$\frac{f(z)}{z} \neq 0 \qquad (z \in \mathbb{U}).$$

*Suppose also that the parameters*

$$\alpha\ (\alpha \neq 0),\ \beta\ (-1 \leqq \beta \leqq 1),\ A,\ B\ (-1 < B < A \leqq 1),$$

*and*

$$\delta\left(1 \leqq \delta \leqq \frac{\pi}{2\cos^{-1}\dfrac{\sqrt{(1-A^2)(1-B^2)}}{1-AB}}\right)$$

*are so prescribed that*

$$1 - |A||B| + \delta\beta|A| - \delta\beta|B| \geqq 0,$$

*and*

$$\frac{\beta(1-\alpha)}{\alpha} + (1+\beta)m_0 + \frac{1-\delta\beta}{1+|A|} + \frac{1+\delta\beta}{1+|B|} \geqq 0,$$

*where*

$$m_0 = \min_{w}\left\{|w|^{\delta}\cos\left(\delta\cos^{-1}\frac{1-A^2+|w|^2(1-B^2)}{2|w|(1-AB)}\right)\right\} : \left(\frac{1-A}{1-B} \leqq |w| \leqq \sqrt{\frac{1-A^2}{1-B^2}}\right).$$

*If*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\beta}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right) \prec h(z) \qquad (z \in \mathbb{U}),$$

*where*

$$h(z)=\left(\frac{1+Az}{1+Bz}\right)^{\delta\beta-1}\left\{(1-\alpha)\frac{1+Az}{1+Bz}+\frac{\alpha(1+Az)^{1+\delta}(1+Bz)^{1-\delta}+\alpha\delta(A-B)z}{(1+Bz)^2}\right\},$$

*then*

$$\frac{zf'(z)}{f(z)}\prec\left(\frac{1+Az}{1+Bz}\right)^{\delta}\qquad(z\in\mathbb{U}).$$

**Corollary 2.2** *Let the function $f(z)\in\mathcal{A}$ be so chosen that*

$$\frac{f(z)}{z}\neq 0\qquad(z\in\mathbb{U}).$$

*Suppose also that the parameters*

$$\alpha\ (\alpha\neq 0),\ \beta\ (-1\leqq\beta\leqq 1),\ A,\ B\ (-1\leqq A<B<1),$$

*and*

$$\delta\left(1\leqq\delta\leqq\frac{\pi}{2\cos^{-1}\dfrac{\sqrt{(1-A^2)(1-B^2)}}{1-AB}}\right)$$

*are so prescribed that*

$$1-|A||B|+\delta\beta|A|-\delta\beta|B|\geqq 0,$$

*and*

$$\frac{\beta(1-\alpha)}{\alpha}+(1+\beta)m_0+\frac{1-\delta\beta}{1+|A|}+\frac{1+\delta\beta}{1+|B|}\geqq 0,$$

*where*

$$m_0=\min_{w}\left\{|w|^{\delta}\cos\left(\delta\cos^{-1}\frac{1-A^2+|w|^2(1-B^2)}{2|w|(1-AB)}\right)\right\}:\left(\frac{1+A}{1+B}\leqq|w|\leqq\sqrt{\frac{1-A^2}{1-B^2}}\right).$$

*If*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\beta}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right)\prec h(z)\qquad(z\in\mathbb{U}),$$

*where*

$$h(z)=\left(\frac{1+Az}{1+Bz}\right)^{\delta\beta-1}\left\{(1-\alpha)\frac{1+Az}{1+Bz}+\frac{\alpha(1+Az)^{1+\delta}(1+Bz)^{1-\delta}+\alpha\delta(A-B)z}{(1+Bz)^2}\right\},$$

*then*

$$\frac{zf'(z)}{f(z)}\prec\left(\frac{1+Az}{1+Bz}\right)^{\delta}\qquad(z\in\mathbb{U}).$$

Next, we derive our second main result contained in Theorem 2.2 below.

**Theorem 2.2** *Let the function $f(z) \in \mathcal{A}$ be so chosen that*

$$\frac{f(z)}{z} \neq 0 \qquad (z \in \mathbb{U}).$$

*Suppose also that real parameters $\alpha$ $(\alpha \neq 0)$, $\beta$ $(-1 \leqq \beta \leqq 1)$, and*

$$\varepsilon \left(0 \leqq \varepsilon \leqq \frac{1-|A|^2}{2(1-A\overline{B})}\right),$$

*as well as the complex parameters $A$ and $B$ constrained by*

$$|A| \leqq 1, \ |B| = 1, \ A \neq B, \ \text{and} \ 1 - A\overline{B} > 0,$$

*are so prescribed that*

$$\frac{\beta(1-\alpha)}{\alpha} + \frac{(1+\beta)\{1-|A|^2-2\varepsilon(1-A\overline{B})\}}{2(1-\varepsilon)(1-A\overline{B})} + \frac{(1-\beta)\{1-\varepsilon-|A-\varepsilon B|\}}{2\{1-\varepsilon+|A-\varepsilon B|\}} \geqq 0.$$

*If*

(2)
$$\left(\frac{zf'(z)}{f(z)}\right)^{\beta}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right) \prec h(z) \qquad (z \in \mathbb{U}),$$

*where*

$$h(z) = \left\{\frac{1-\varepsilon+(A-\varepsilon B)z}{(1-\varepsilon)(1+Bz)}\right\}^{\beta-1}\left\{(1-\alpha)\frac{1-\varepsilon+(A-\varepsilon B)z}{(1-\varepsilon)(1+Bz)} + \frac{\alpha\{1-\varepsilon+(A-\varepsilon B)z\}^2+\alpha(1-\varepsilon)(A-B)z}{(1-\varepsilon)^2(1+Bz)^2}\right\},$$

*then*

$$\frac{zf'(z)}{f(z)} \prec \frac{1-\varepsilon+(A-\varepsilon B)z}{(1-\varepsilon)(1+Bz)} \left(= \frac{\frac{1+Az}{1+Bz}-\varepsilon}{1-\varepsilon}\right) \qquad (z \in \mathbb{U}).$$

*Proof.* Let us define the function $p(z)$, $q(z)$, $\phi(z)$, and $\psi(z)$ by

$$p(z) = \frac{zf'(z)}{f(z)}, \ q(z) = \frac{1-\varepsilon+(A-\varepsilon B)z}{(1-\varepsilon)(1+Bz)}, \ \phi(z) = z^{\beta}(1-\alpha+\alpha z), \quad \text{and} \quad \psi(z) = \alpha z^{\beta-1}$$

for $z \in \mathbb{U}$. Then, clearly, the function $q(z)$ is analytic and univalent in $\mathbb{U}$.
Now, for $q_1(z) = \dfrac{1+Az}{1+Bz}$, it is clear that $q_1(z)$ is univalent in $\mathbb{U}$ and $q_1(\mathbb{U})$ is the right half plane satisfying

$$\mathrm{Re}(q_1(z)) > \frac{1-|A|^2}{2(1-A\overline{B})} \geqq 0.$$

Thus we see that

$$\mathrm{Re}(q(z)) = \mathrm{Re}\left(\frac{q_1(z)-\varepsilon}{1-\varepsilon}\right) = \frac{\mathrm{Re}(q_1(z))-\varepsilon}{1-\varepsilon}$$

$$> \frac{\frac{1-|A|^2}{2(1-A\overline{B})}-\varepsilon}{1-\varepsilon} = \frac{1-|A|^2-2\varepsilon(1-A\overline{B})}{2(1-\varepsilon)(1-A\overline{B})} \geqq 0 \qquad (z\in\mathbb{U}).$$

Also, the functions $\phi$ and $\psi$ satisfy the conditions required by Lemma 1.1. The function $Q(z)$ given by

$$Q(z) = zq'(z)\psi(q(z)) = \frac{\alpha(A-B)z\{1-\varepsilon+(A-\varepsilon B)z\}^{\beta-1}}{(1-\varepsilon)^{\beta}(1+Bz)^{\beta+1}}$$

is univalent and starlike in $\mathbb{U}$, because

$$\mathrm{Re}\left(\frac{zQ'(z)}{Q(z)}\right) = (1-\beta)(1-\varepsilon)\mathrm{Re}\left(\frac{1}{1-\varepsilon+(A-\varepsilon B)z}\right) + (1+\beta)\mathrm{Re}\left(\frac{1}{1+Bz}\right) - 1$$

$$> (1-\beta)(1-\varepsilon)\frac{1}{1-\varepsilon+|A-\varepsilon B|} + \frac{1}{2}(1+\beta) - 1$$

$$= \frac{(1-\beta)\{1-|A|^2-2\varepsilon(1-A\overline{B})\}}{2\{1-\varepsilon+|A-\varepsilon B|\}^2} \geqq 0 \qquad (z\in\mathbb{U}).$$

Furthermore, we have

$$h(z) = \phi(q(z)) + Q(z)$$

$$= \left\{\frac{(1-\varepsilon)+(A-\varepsilon B)z}{(1-\varepsilon)(1+Bz)}\right\}^{\beta}\left\{1-\alpha+\alpha\frac{(1-\varepsilon)+(A-\varepsilon B)z}{(1-\varepsilon)(1+Bz)}\right\}$$

$$+\frac{\alpha(A-B)z\{(1-\varepsilon)+(A-\varepsilon B)z\}^{\beta-1}}{(1-\varepsilon)^{\beta}(1+Bz)^{\beta+1}},$$

and

$$\mathrm{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \frac{\beta(1-\alpha)}{\alpha} + (1+\beta)\mathrm{Re}(q(z)) + \mathrm{Re}\left(\frac{zQ'(z)}{Q(z)}\right)$$

$$> \frac{\beta(1-\alpha)}{\alpha} + \frac{(1+\beta)\{1-|A|^2-2\varepsilon(1-A\overline{B})\}}{2(1-\varepsilon)(1-A\overline{B})}$$

$$+\frac{(1-\beta)\{(1-\varepsilon)-|A-\varepsilon B|\}}{2\{(1-\varepsilon)+|A-\varepsilon B|\}} \geqq 0 \qquad (z\in\mathbb{U}).$$

Similarly, the other conditions of Lemma 1.1 are also seen to be satisfied. Therefore, we conclude that

$$\frac{zf'(z)}{f(z)} \prec \frac{1-\varepsilon+(A-\varepsilon B)z}{(1-\varepsilon)(1+Bz)} \qquad (z\in\mathbb{U}),$$

which completes the proof of Theorem 2.2. □

**Remark 2.2** Setting $\varepsilon = 0$ in Theorem 2.2, we obtain Lemma 1.3 proven by Kuroki, Owa and Srivastava [1, Theorem 1].

Also, making $A, B \in \mathbb{R}$ ($B = -1$; $-1 < A \leqq 1$ *or* $B = 1$; $-1 \leqq A < 1$) in Theorem 2.2, we find the following Corollary 2.3 and Corollary 2.4.

**Corollary 2.3** *Let the function $f(z) \in \mathcal{A}$ be so chosen that*

$$\frac{f(z)}{z} \neq 0 \qquad (z \in \mathbb{U}).$$

*Suppose also that the parameters*

$$\alpha\ (\alpha \neq 0),\ \beta\ (-1 \leqq \beta \leqq 1),\ A\ (-1 < A \leqq 1),$$

*and*

$$\varepsilon\left(0 \leqq \varepsilon \leqq \frac{1}{2}(1-A)\right)$$

*are so prescribed that*

$$\frac{\beta(1-\alpha)}{\alpha} + \frac{(1+\beta)(1-A-2\varepsilon)}{2(1-\varepsilon)} + \frac{(1-\beta)\{1-\varepsilon-|A+\varepsilon|\}}{2\{1-\varepsilon+|A+\varepsilon|\}} \geqq 0.$$

*If*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\beta}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right) \prec h(z) \qquad (z \in \mathbb{U}),$$

*where*

$$h(z) = \left\{\frac{1-\varepsilon+(A+\varepsilon)z}{(1-\varepsilon)(1-z)}\right\}^{\beta-1}\left\{(1-\alpha)\frac{1-\varepsilon+(A+\varepsilon)z}{(1-\varepsilon)(1-z)} + \frac{\alpha\{1-\varepsilon+(A+\varepsilon)z\}^2+\alpha(1-\varepsilon)(A+1)z}{(1-\varepsilon)^2(1-z)^2}\right\},$$

*then*

$$\frac{zf'(z)}{f(z)} \prec \frac{1-\varepsilon+(A+\varepsilon)z}{(1-\varepsilon)(1-z)} \qquad (z \in \mathbb{U}).$$

**Corollary 2.4** *Let the function $f(z) \in \mathcal{A}$ be so chosen that*

$$\frac{f(z)}{z} \neq 0 \qquad (z \in \mathbb{U}).$$

*Suppose also that the parameters*

$$\alpha\ (\alpha \neq 0),\ \beta\ (-1 \leqq \beta \leqq 1),\ A\ (-1 \leqq A < 1),$$

*and*

$$\varepsilon\left(0 \leqq \varepsilon \leqq \frac{1}{2}(1+A)\right)$$

*are so prescribed that*

$$\frac{\beta(1-\alpha)}{\alpha}+\frac{(1+\beta)(1+A-2\varepsilon)}{2(1-\varepsilon)}+\frac{(1-\beta)\{1-\varepsilon-|A-\varepsilon|\}}{2\{1-\varepsilon+|A-\varepsilon|\}} \geqq 0.$$

*If*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\beta}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right) \prec h(z) \qquad (z \in \mathbb{U}),$$

*where*

$$h(z)=\left\{\frac{1-\varepsilon+(A-\varepsilon)z}{(1-\varepsilon)(1+z)}\right\}^{\beta-1}\left\{(1-\alpha)\frac{1-\varepsilon+(A-\varepsilon)z}{(1-\varepsilon)(1+z)} + \frac{\alpha\{1-\varepsilon+(A-\varepsilon)z\}^{2}+\alpha(1-\varepsilon)(A-1)z}{(1-\varepsilon)^{2}(1+z)^{2}}\right\},$$

*then*

$$\frac{zf'(z)}{f(z)} \prec \frac{1-\varepsilon+(A-\varepsilon)z}{(1-\varepsilon)(1+z)} \qquad (z \in \mathbb{U}).$$

# References


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Kazuo Kuroki
*Department of Mathematics*
*Kinki University*
*Higashi-Osaka, Osaka 577-8502*
*Japan*
*e-mail : freedom@sakai.zaq.ne.jp*

Shigeyoshi Owa
*Department of Mathematics*
*Kinki University*
*Higashi-Osaka, Osaka 577-8502*
*Japan*
*e-mail : owa@math.kindai.ac.jp*