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CERTAIN SUBCLASSES OF MULTIVALENT FUNCTIONS

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ABSTRACT. The object of the present paper is to drive some properties of certain class $K_{n,p}(A,B)$ of multivalent analytic functions in the open unit disk E.

1. Introduction

Let A_p be the class of functions of the form

$$f(z)=z^p+\sum_{k=1}^{\infty}a_{p+k}z^{p+k} \tag{1.1}$$

which are analytic in the open unit disk $E=\{z\in\mathbb{C}:|z|<1\}$. A function $f\in A_p$ is said to be p-valently starlike functions of order α of it satisfies the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha \quad (0\leq\alpha<p, z\in E).$$

We denote by $S_p^*(\alpha)$.

On the other hand, a function $f\in A_p$ is sais to be p-valently close-to-convex functions of order α if it satisfies the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{g(z)}\right\}>\alpha \quad (0\leq\alpha<p, z\in E),$$

for some starlike function $g(z)$. We denote by $C_p(\alpha)$.

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For $f \in A_p$ given by (1.1), the generalized Bernardi integral operator F_c is defined by

$$\begin{aligned} F_c(z) &= \frac{c+p}{z^c}\int_0^z f(t)t^{c-1}dt \\ &= z^p + \sum_{k=1}^{\infty}\frac{c+p}{c+p+k}a_{p+k}z^{p+k} \quad (c+p>0, z\in E). \end{aligned} \tag{1.2}$$

For an analytic function g, defined in E by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k}z^{p+k}$$

and Flett [3] defined the multiplier transform I^η for a real number η by

$$I^\eta g(z) = \sum_{k=0}^{\infty}(p+k+1)^{-\eta}b_{p+k}z^{p+k} \quad (z\in E).$$

Clearly, the function $I^\eta g$ is analytic in E and

$$I^\eta(I^\mu g(z)) = I^{\eta+\mu}g(z)$$

for all real number η and μ.

For any integer n, J. Patel and P. Sahoo [5] also defined the operator D^n, for an analytic function f given by (1.1), by

$$\begin{aligned} D^n f(z) &= z^p + \sum_{k=1}^{\infty}\left(\frac{p+k+1}{1+p}\right)^{-n} a_{p+k}z^{p+k} \\ &= f(z) * z^{p-1}\left[z + \sum_{k=1}^{\infty}\left(\frac{k+1+p}{1+p}\right)^{-n} z^{k+1}\right] \quad (z\in E) \end{aligned} \tag{1.3}$$

where $*$ stands for the Hadamard product or convolution.

It follows from (1.3) that

$$z(D^n f(z))' = (p+1)D^{n-1}f(z) - D^n f(z). \tag{1.4}$$

We also have

$$D^0 f(z) = f(z) \quad \text{and} \quad D^{-1}f(z) = \frac{zf'(z)+f(z)}{p+1}.$$

If f and g are analytic functions in E, then we say that f is subordinate to g written $f \prec g$ or $f(z) \prec g(z)$, if there is a function w analytic in E, with $w(0) = 0$, $|w(z)| < 1$ for $z \in E$, such that $f(z) = g(w(z))$, for $z \in E$. If g is univalent then $f \prec g$ if and only if $f(0) = g(0)$ and $f(E) \subset g(E)$.

Making use of the operator notation D^n, we introduce a subclass of A_p as follows:

Definition 1.1. For any integer n and $-1 \leq B < A \leq 1$, a function $f \in A_p$ is said to be in the class $K_{n,p}(A, B)$ if

$$\frac{z(D^n f(z))'}{z^p} \prec \frac{p(1 + Az)}{1 + Bz} \tag{1.5}$$

where $\prec$ denotes subordination.

For convenience, we write

$$K_{n,p}\left(1 - \frac{2\alpha}{p}, -1\right) = K_{n,p}(\alpha),$$

where $K_{n,p}(\alpha)$ denote the class of function $f \in A_p$ satisfying the inequality

$$\text{Re}\left\{\frac{z(D^n f(z))'}{z^p}\right\} > \alpha \quad (0 \leq \alpha < p, \ z \in E).$$

We also note that $K_{0,p}(\alpha) \equiv C_p(\alpha)$ is the class of p-valently close-to-convex functions of order α.

In this present paper, we derive some properties of certain class $K_{n,p}(A, B)$ by using the differential subordination.

2. Preliminaries and Main Results

In our present investigation of the general class $K_{n,p}(A, B)$, we shall require the following lemmas.

Lemma 1 [4]. *If the function $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic in E, $h(z)$ is convex in E with $h(0) = 1$, and γ is complex number such that Re $\gamma > 0$. Then the Briot-Bouquet differential subordination*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z)$$

implies

$$p(z) \prec q(z) = \frac{\gamma}{z^\gamma} \int_0^z t^{\gamma-1} h(t) dt \prec h(z) \quad (z \in E)$$

and $q(z)$ is the best dominant.

For complex number a, b and $c \neq 0, -1, -2, \cdots$, the hypergeometric series

$$_2F_1(a, b; c; z) = 1 + \frac{ab}{c} z + \frac{a(a+1)b(b+1)}{2!c(c+1)} z^2 + \cdots \tag{2.1}$$

represents an analytic function in E. It is well known by [1] that

Lemma 2. *Let a, b and c be real $c \neq 0, -1, -2, \cdots$ and $c > b > 0$. Then*

$$\begin{aligned} &\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} {}_2F_1(a, b; c; z), \\ &{}_2F_1(a, b; c; z) = (1-z)^{-a} {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right) \end{aligned} \tag{2.2}$$

and

$$_2F_1(a, b; c; z) = {}_2F_1(b, a; c; z). \tag{2.3}$$

Lemma 3 [6]. *Let $\phi(z)$ be convex and $g(z)$ is starlike in E. Then for F analytic in E with $F(0) = 1$, $\frac{\phi * Fg}{\phi * g}(E)$ is contained in the convex hull of $F(E)$.*

Lemma 4 [2]. *Let $\phi(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$ and $\phi(z) \prec \frac{1+Az}{1+Bz}$. Then*

$$|c_k| \leq (A - B).$$

Theorem 1. *Let n be any integer and $-1 \le B < A \le 1$. If $f \in K_{n,p}(A,B)$, then*

$$\frac{z(D^{n+1}f(z))'}{z^p} \prec q(z) \prec \frac{p(1+Az)}{1+Bz} \quad (z \in E), \tag{2.4}$$

where

$$q(z) = \begin{cases} {}_2F_1(1,p+1;p+2;-Bz) \\ \quad +\frac{p+1}{p+2}Az\,{}_2F_1(1,p+2;p+3;-Bz), & B \neq 0 \\ 1+\frac{p+1}{p+2}Az, & B = 0 \end{cases} \tag{2.5}$$

and $q(z)$ is the best dominant of (2.4). Furthermore, $f \in K_{n+1,p}(\rho(p,A,B))$, where

$$\rho(p,A,B) = \begin{cases} p\,{}_2F_1(1,p+1;p+2;B) \\ \quad -\frac{p(p+1)}{p+2}A\,{}_2F_1(1,p+2;p+3;B), & B \neq 0 \\ 1-\frac{p+1}{p+2}A, & B = 0. \end{cases} \tag{2.6}$$

Proof. Let

$$p(z) = \frac{z(D^{n+1}f(z))'}{pz^p} \tag{2.7}$$

where $p(z)$ is analytic function with $p(0) = 1$.
Using the identity (1.4) in (2.7) and differentiating the resulting equation, we get

$$\frac{z(D^n f(z))'}{pz^p} = p(z) + \frac{zp'(z)}{p+1} \prec \frac{1+Az}{1+Bz} (\equiv h(z)). \tag{2.8}$$

Thus, by using Lemma 1 (for $\gamma = p+1$), we deduce that

$$\begin{aligned} p(z) &\prec (p+1)z^{-(p+1)} \int_0^z \frac{t^p(1+At)}{1+Bt}dt (\equiv q(z)) \\ &= (p+1)\int_0^1 \frac{s^p(1+Asz)}{1+Bsz}ds \\ &= (p+1)\int_0^1 \frac{s^p}{1+Bsz}ds + (p+1)Az\int_0^1 \frac{s^{p+1}}{1+Bsz}ds. \end{aligned} \tag{2.9}$$

By using (2.2) in (2.9), we obtain

$$p(z) \prec q(z) = \begin{cases} {}_2F_1(1, p+1; p+2; -Bz) \\ \quad + \dfrac{p+1}{p+2} Az {}_2F_1(1, p+2; p+3; -Bz), & B \neq 0 \\ 1 + \dfrac{p+1}{p+2} Az, & B = 0. \end{cases}$$

Thus, this proves (2.5).

Now, we show that

$$\text{Re } q(z) \geq q(-r) \quad (|z| = r < 1). \tag{2.10}$$

Since $-1 \leq B < A \leq 1$, the function $(1 + Az)/(1 + Bz)$ is convex(univalent) in E and

$$\text{Re } \left(\frac{1 + Az}{1 + Bz} \right) \geq \frac{1 - Ar}{1 - Br} > 0 \quad (|z| = r < 1).$$

Setting

$$g(s.z) = \frac{1 + Asz}{1 + Bsz} \quad (0 \leq s \leq 1, \ z \in E)$$

and $d\mu(s) = (p+1)s^p ds$, which is a positive measure on $[0, 1]$, we obtain from (2.9) that

$$q(z) = \int_0^1 g(s, z) d\mu(s) \quad (z \in E).$$

Therefore, we have

$$\text{Re } q(z) = \int_0^1 \text{Re } g(s, z) d\mu(s) \geq \int_0^1 \frac{1 - Asr}{1 - Bsr} d\mu(s)$$

which proves the inequality (2.10).

Now, using (2.10) in (2.9) and letting $r \to 1^-$, we obtain

$$\text{Re } \left\{ \frac{z(D^{n+1} f(z))'}{z^p} \right\} > \rho(p, A, B),$$

where

$$\rho(p,A,B)=\begin{cases} p\,{}_2F_1(1,p+1;p+2;B) \\ \quad -\dfrac{p(p+1)}{p+2}A\,{}_2F_1(1,p+2;p+3;B), & B\neq 0 \\ p-\dfrac{p(p+1)}{p+2}A, & B=0. \end{cases}$$

This proves the assertion of Theorem 1. The result is best possible because of the best dominent property of $q(z)$.

Putting $A=1-\dfrac{2\alpha}{p}$ and $B=-1$ in Theorem 1, we have the following:

Corollary 1. *For any integer n and $0\le\alpha<p$, we have*

$$K_{n,p}(\alpha)\subset K_{n+1,p}(\rho(p,\alpha)),$$

where

$$\rho(p,\alpha)=p\cdot{}_2F_1(1,p+1;p+2;-1)-\frac{p(p+1)}{p+2}(1-2\alpha)\,{}_2F_1(1,p+2;p+3;-1). \tag{2.11}$$

The result is best possible.

Taking $p=1$ in Corollary 1, we have the following:

Corollary 2. *For any integer n and $0\le\alpha<1$, we have*

$$K_n(\delta)\subset K_{n+1}(\delta(\alpha))$$

where

$$\delta(\alpha)=1+4(1-2\alpha)\sum_{k=1}^{\infty}\frac{1}{k+2}(-1)^k. \tag{2.12}$$

Theorem 2. *For any integer n and $0\le\alpha<p$, if $f(z)\in K_{n+1,p}(\alpha)$ then $f\in K_{n,p}(\alpha)$ for $|z|<R(p)$, where $R(p)=\dfrac{-1+\sqrt{1+(p+1)^2}}{p+1}$. The result is best possible.*

Proof. Since $f(z)\in K_{n+1,p}(\alpha)$, we have

$$\frac{z(D^{n+1}f(z))'}{z^p}=\alpha+(p-\alpha)w(z),\quad (0\le\alpha<p), \tag{2.13}$$

where $w(z) = 1 + w_1 z + w_2 z + \cdots$ is analytic and has a positive real part in E. Making use of the logarithmic differentiation and using identity (1.4) in (2.13), we get

$$\frac{z(D^n f(z))'}{z^p} - \alpha = (p-\alpha)\left[w(z) + \frac{zw'(z)}{p+1}\right]. \tag{2.14}$$

Now, using the well-known by [5],

$$\frac{|zw'(z)|}{\text{Re } w(z)} \le \frac{2r}{1-r^2} \quad \text{and} \quad \text{Re } w(z) \ge \frac{1-r}{1+r} \quad (|z| = r < 1),$$

in (2.14). We get

$$\begin{aligned} \text{Re}\left\{\frac{z(D^n f(z))'}{z^p} - \alpha\right\} &= (p-\alpha)\text{Re } w(z)\left\{1 + \frac{1}{p+1}\frac{\text{Re } zw'(z)}{\text{Re } w(z)}\right\} \\ &\ge (p-\alpha)\text{Re } w(z)\left\{1 - \frac{1}{p+1}\frac{|zw'(z)|}{\text{Re } w(z)}\right\} \\ &\ge (p-\alpha)\frac{1-r}{1+r}\left\{1 - \frac{1}{p+1}\frac{2r}{1-r^2}\right\}. \end{aligned}$$

It is easily seen that the right-hand side of the above expression is positive if $|z| < R(p) = \dfrac{-1+\sqrt{1+(p+1)^2}}{p+1}$. Hence $f \in K_{n,p}(\alpha)$ for $|z| < R(p)$.

To show that the bound $R(p)$ is best possible, we consider the function $f \in A_p$ defined by

$$\frac{z(D^{n+1} f(z))'}{z^p} = \alpha + (p-\alpha)\frac{1-z}{1+z} \quad (z \in E).$$

Noting that

$$\begin{aligned} \frac{z(D^n f(z))'}{z^p} - \alpha &= (p-\alpha)\cdot\frac{1-z}{1+z}\left\{1 + \frac{1}{p+1}\frac{-2z}{(p+1)(1-z^2)}\right\} \\ &= (p-\alpha)\cdot\frac{1-z}{1+z}\left\{\frac{(p+1)-(p+1)z^2-2z}{(p+1)-(p+1)z^2}\right\} \\ &= 0 \end{aligned}$$

for $z = \dfrac{-1+\sqrt{1+(p+1)^2}}{p+1}$, we complete the proof of Theorem 2.

Putting $n = -1$, $p = 1$ and $0 \le \alpha < 1$ in Theorem 2, we have the following:

Corollary 3. *If Re* $f'(z) > \alpha$, *then Re* $\{zf''(z) + 2f'(z)\} > \alpha$ *for* $|z| < \dfrac{-1+\sqrt{5}}{2}$.

Theorem 3. *(a) If* $f \in K_{n,p}(A,B)$, *then the function* F_c *defined by (1.2) belongs to* $K_{n,p}(A,B)$.

(b) $f \in K_{n,p}(A,B)$ *implies that* $F_c \in K_{n,p}(\eta(p,,c,A,B))$ *where*

$$\eta(p,c,A,B) = \begin{cases} p_2F_1(1,p+c;p+c+1;B) \\ \quad -\dfrac{p(p+c)}{p+c+1}A_2F_1(1,p+c+1;p+c+2;B), & B \neq 0 \\ p - \dfrac{p(p+c)}{p+c+1}A, & B = 0. \end{cases}$$

Proof. Let

$$\phi(z) = \frac{z(D^nF_c(z))'}{pz^p}, \tag{2.15}$$

where $\phi(z)$ is analytic function with $\phi(0) = 1$. Using the identity

$$z(D^nF_c(z))' = (p+c)D^nf(z) - cD^nF_c(z) \tag{2.16}$$

in (2.15) and differentiating the resulting equation, we get

$$\frac{z(D^nf(z))'}{pz^p} = \phi(z) + \frac{z\phi'(z)}{p+c}.$$

Since $f \in K_{n,p}(A,B)$,

$$\phi(z) + \frac{z\phi'(z)}{p+c} \prec \frac{1+Az}{1+Bz}.$$

By Lemma 1, we obtain $F_c(z) \in K_{n,p}(A,B)$. We deduce that

$$\phi(z) \prec q(z) \prec \frac{1+Az}{1+Bz} \tag{2.17}$$

where $q(z)$ is given (2.5) and $q(z)$ is best deminent of (2.17).

This proves the (a) part of theorem. Proceeding as in Theorem 3, the (b) part follows.

Putting $A = 1 - \dfrac{2\alpha}{p}$ and $B = -1$ in Theorem 2, we have the following:

Corollary 4. *If $f \in K_{n,p}(A,B)$ for $0 \le \alpha < p$, then $F_c \in K_{n,p}\mathcal{H}(p,c,\alpha)$ where*

$$\mathcal{H}(p,c,\alpha) = p \cdot {}_2F_1(1, p+c; p+c+1; -1) - \frac{p+c}{p+c+1}(p-2\alpha){}_2F_1(1, p+c; p+c+1; -1).$$

Setting $c = p = 1$ in Theorem 3, we get the following result.

Corollary 4. *If $f \in K_{n,p}(\alpha)$ for $0 \le \alpha < 1$, then the function*

$$G(z) = \frac{2}{z}\int_0^z f(t)dt$$

belongs to the class $K_n(\delta(\alpha))$, where $\delta(\alpha)$ is given by (2.12).

Theorem 4. *For any integer n and $0 \le \alpha < p$ and $c > -p$, if $F_c \in K_{n,p}(\alpha)$ then the function f defined by (1.1) belongs to $K_{n,p}(\alpha)$ for $|z| < R(p,c) = \dfrac{-1+\sqrt{1+(p+c)^2}}{p+c}$. The result is best possible.*

Proof. Since $F_c \in K_{n,p}(\alpha)$, we write

$$\frac{z(D^n F_c)'}{z^p} = \alpha + (p-\alpha)w(z), \tag{2.18}$$

where $w(z)$ is analytic, $w(0) = 1$ and $\operatorname{Re} w(z) > 0$ in E. Using (2.16) in (2.18) and differentiating be resulting equation, we obtain

$$\operatorname{Re}\left\{\frac{z(D^n f(z))'}{z^p} - \alpha\right\} = (p-\alpha)\operatorname{Re}\left\{w(z) + \frac{zw'(z)}{p+c}\right\}. \tag{2.19}$$

Now, by following the line of proof of Theorem 2, we get the assertion of Theorem 4.

Theorem 5. *Let $f \in K_{n,p}(A,B)$ and $\phi(z) \in A_p$ convex in E. Then*

$$(f * \phi(z))(z) \in K_{n,p}(A,B).$$

Proof. Since $f(z) \in K_{n,p}(A,B)$,

$$\frac{z(D^n f(z))'}{pz^p} \prec \frac{1+Az}{1+Bz}.$$

Now

$$\frac{z(D^n(f*\phi)(z))'}{pz^p * \phi(z)} = \frac{\phi(z) * z(D^n f)'}{\phi(z) * pz^p} = \frac{\phi(z) * \frac{z(D^n f(z))'}{pz^p} pz^p}{\phi(z) * pz^p}. \tag{2.20}$$

Then applying Lemma 3, we deduce that

$$\frac{\phi(z) * \frac{z(D^n f(z))'}{pz^p} pz^p}{\phi(z) * pz^p} \prec \frac{1+Az}{1+Bz}.$$

Hence $(f * \phi(z))(z) \in K_{n,p}(A,B)$.

Theorem 6. *Let a function $f(z)$ defined by (1.1) be in the class $K_{n,p}(A,B)$. Then*

$$|a_{p+k}| \leq \frac{p(A-B)(p+k+1)^n}{(1+p)^n(p+k)} \quad \text{for} \quad k = 1, 2, \cdots. \tag{2.21}$$

The result is sharp.

Proof. Since $f(z) \in K_{n,p}(A,B)$, we have

$$\frac{z(D^n f(z))'}{pz^p} \equiv \phi(z) \quad \text{and} \quad \phi(z) \prec \frac{1+Az}{1+Bz}.$$

Hence

$$z(D^n f(z))' = pz^p \phi(z) \quad \text{and} \quad \phi(z) = 1 + \sum_{k=1}^{\infty} c_k z^k. \tag{2.22}$$

From (2.22), we have

$$\begin{aligned} z(D^n f(z))' &= z\left(z^p + \sum_{k=1}^{\infty}\left(\frac{1+p}{p+k+1}\right)^n a_{p+k} z^{p+k}\right)' \\ &= pz^p + \sum_{k=1}^{\infty}\left(\frac{1+p}{p+k+1}\right)^n (p+k) a_{p+k} z^{p+k} \\ &= pz^p\left(1 + \sum_{k=1}^{\infty} c_k z^k\right). \end{aligned}$$

Therefore

$$\left(\frac{1+p}{p+k+1}\right)^n (p+k)a_{p+k} = pc_k. \tag{2.23}$$

By using Lemma 4 in (2.23),

$$\frac{\left(\frac{1+p}{p+k+1}\right)^n (p+k)|a_{p+k}|}{p} = |c_k| \leq A - B.$$

Hence

$$|a_{p+k}| \leq \frac{p(A-B)(p+k+1)^n}{(1+p)^n(p+k)}.$$

The equality sign in (2.21) holds for the function f given by

$$(D^n f(z))' = \frac{pz^{p-1} + p(A-B-1)z^p}{1-z}. \tag{2.24}$$

Hence

$$\frac{z(D^n f(z))'}{pz^p} = \frac{1+(A-B-1)z}{1-z} \prec \frac{1+Az}{1+Bz} \text{ for } k = 1, 2, \cdots.$$

The function $f(z)$ defined in (2.24) has the power series representation in E,

$$f(z) = z^p + \sum_{k=1}^{\infty} \frac{p(A-B)(p+k+1)^n}{(1+p)^n(p+k)} z^{p+k}.$$

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