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N-Fractional Calculus of Some Algebraic Functions

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Abstract

In this article, N-fractional calculus for the functions

$$\frac{1}{z^2+(a+b)z+ab} \quad \text{and} \quad \frac{z+\frac{a+b}{2}}{z^2+(a+b)z+ab}$$

where $z^2+(a+b)z+ab \neq 0$ are discussed.

1 Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto, [1] Vol. 1)

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$, C_- be a curve along the cut joining two points z and $-\infty + iIm(z)$, C_+ be a curve along the cut joining two points z and $\infty + iIm(z)$, D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ (Here D contains the points over the curve C).

Moreover, let $f = f(z)$ be a regular function in $D(z \in D)$,

$$\begin{aligned} f_\nu &= (f)_\nu = {}_C(f)_\nu \\ &= \frac{\Gamma(\nu+1)}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z)^{\nu+1}} \quad (\nu \notin Z^-), \end{aligned} \tag{1}$$

$$(f)_{-m} = \lim_{\nu\to -m}(f)_\nu \quad (m \in Z^+), \tag{2}$$

where

$$-\pi \leq arg(\zeta - z) \leq \pi \ for\ C_-, \quad 0 \leq arg(\zeta - z) \leq 2\pi \ \ for\ C_+,$$

$$\zeta \neq z, \quad z \in C, \quad \nu \in R, \quad \Gamma;\ Gamma\ function,$$

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z , of the function f , if $|(f)_\nu| < \infty$.

(II) On the fractional calculus operator N^ν [3]

Theorem A. Let fractional calculus operator (Nishimoto's Operator) N^ν be

$$N^\nu = (\frac{\Gamma(\nu+1)}{2\pi i}\int_C \frac{d\zeta}{(\zeta - z)^{\nu+1}}) \quad (\nu \notin Z^-), \quad (Refer\ to[1]) \tag{3}$$

with

$$N^{-m} = \lim_{\nu \to -m} N^\nu \quad (m \in Z^+), \tag{4}$$

and define the binary operation $\circ$ as

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\beta(N^\alpha f) \quad (\alpha, \beta \in R), \tag{5}$$

then the set

$$\{N^\nu\} = \{N^\nu | \nu \in R\} \tag{6}$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator N^ν , for the function f such that $f \in F = \{f; 0 \neq |f_\nu| < \infty, \nu \in R\}$, where $f = f(z)$ and $z \in C$. (vis. $-\infty < \nu < \infty$).

(For our convenience, we call $N^\beta \circ N^\alpha$ as product of N^β and N^α .)

Theorem B. " F.O.G. $\{N^\nu\}$) " is an " Action product group which has continuous index ν " for the set of F . (F.O.G. ; Fractional calculus operator group)

Theorem C. Let

$$S := \{\pm N^\nu\} \cup \{0\} = \{N^\nu\} \cup \{-N^\nu\} \cup \{0\} \quad (\nu \in R). \tag{7}$$

Then the set S is a commutative ring for the function $f \in F$, when the identity

$$N^\alpha + N^\beta = N^\gamma \quad (N^\alpha, N^\beta, N^\gamma \in S) \tag{8}$$

holds. [5]

(III) **Lemma.** We have [1]

(i)

$$((z-c)^{\beta})_{\alpha} = e^{-i\pi\alpha}\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}(z-c)^{\beta-\alpha} \qquad (|\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}| < \infty)$$

(ii)

$$(log(z-c))_{\alpha} = -e^{-i\pi\alpha}\Gamma(\alpha)(z-c)^{-\alpha} \qquad (|\Gamma(\alpha)| < \infty)$$

(iii)

$$((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha}\frac{1}{\Gamma(\alpha)}\log(z-c), \quad (|\Gamma(\alpha)| < \infty)$$

where $z-c \neq 0$ in (i), and $z-c \neq 0,1$ in (ii) and (iii) ,

(iv)

$$(u \cdot v)_{\alpha} := \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha+1-k)} u_{\alpha-k} v_k. \quad (u = u(z), v = v(z))$$

2 Preliminary

The following theorem has been reported by K. Nishimoto [12].

Theorem D. We have

(i)

$$(((z-b)^{\beta}-c)^{\alpha})_{\gamma} = e^{-i\pi\gamma}(z-b)^{\alpha\beta-\gamma}\sum_{k=0}^{\infty}\frac{[-\alpha]_k\Gamma(\beta k-\alpha\beta+\gamma)}{k!\Gamma(\beta k-\alpha\beta)}(\frac{c}{(z-b)^{\beta}})^k \tag{1}$$

$$(|\frac{\Gamma(\beta k-\alpha\beta+\gamma)}{\Gamma(\beta k-\alpha\beta)}| < \infty),$$

and

(ii)

$$(((z-b)^{\beta}-c)^{\alpha})_n = (-1)^n(z-b)^{\alpha\beta-n}\sum_{k=0}^{\infty}\frac{[-\alpha]_k[\beta k-\alpha\beta]_n}{k!}(\frac{c}{(z-b)^{\beta}})^k \tag{2}$$

$$(n \in Z_0^+, \quad |\frac{c}{(z-b)^{\beta}}| < 1),$$

where

$$[\lambda]_k = \lambda(\lambda+1)\cdots(\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda) \;\; with \;\; [\lambda]_0 = 1,$$

(Pochhammer's Notation).

We apply this theorem to have some theorems for some algebraic functions.

3 Some Theorems and Identities

We descrive some results for the functions $\frac{1}{z^2+(a+b)z+ab}$ and $\frac{z+\frac{a+b}{2}}{z^2+(a+b)z+ab}$ by applying the theorem in §2.

Theorem 1. We have

(i)

$$\left(\frac{1}{z^2+(a+b)z+ab}\right)_\gamma = e^{-i\pi\gamma}\left(\frac{2z+a+b}{2}\right)^{-2-\gamma} \times \sum_{k=0}^{\infty} \frac{[1]_k \Gamma(2k+2+\gamma)}{k!\Gamma(2k+2)} \left(\left(\frac{a-b}{2z+a+b}\right)^2\right)^k \quad (1)$$

$$(|\Gamma(2k+2+\gamma)| < \infty)$$

and

(ii)

$$\left(\frac{1}{z^2+(a+b)z+ab}\right)_n = (-1)^n\left(\frac{2z+a+b}{2}\right)^{-2-n} \times \sum_{k=0}^{\infty} [2k+2]_n \left(\left(\frac{a-b}{2z+a+b}\right)^2\right)^k, \quad (n \in Z_0^+) \quad (2)$$

where $z^2+(a+b)z+ab \neq 0, \quad |(a-b)/(2z+a+b)| < 1.$

Proof of (i). Setting $p = -(a+b)/2,\ q = (\frac{a+b}{2})^2 - ab$, we have

$$z^2+(a+b)z+ab = (z-p)^2 - q, \quad (3)$$

hence

$$\left(\frac{1}{z^2+(a+b)z+ab}\right)_\gamma = (((z-p)^2-q)^{-1})_\gamma = e^{-i\pi\gamma}(z-p)^{-2-\gamma} \sum_{k=0}^{\infty} \frac{[1]_k \Gamma(2k+2+\gamma)}{k!\Gamma(2k+2)} \left(\frac{q}{(z-p)^2}\right)^k \quad (4)$$

by Theorem D, (i), and we can rewrite this equation immediately to (1).

Proof of (ii). We have the result by setting $\gamma = n$ in the equation (1).

Theorem 2. We have

(i)

$$\left(\frac{1}{z^2+(a+b)z+ab}\right)_\gamma = e^{-i\pi\gamma}\frac{\Gamma(\gamma+1)}{b-a}\left(\frac{1}{(z+a)^{1+\gamma}}-\frac{1}{(z+b)^{1+\gamma}}\right) \tag{5}$$

$$(|\Gamma(1+\gamma)|<\infty)$$

and

(ii)

$$\left(\frac{1}{z^2+(a+b)z+ab}\right)_n = (-1)^n\frac{n!}{b-a}\left(\frac{1}{(z+a)^{1+n}}-\frac{1}{(z+b)^{1+n}}\right) \tag{6}$$

$$(n\in Z_0^+)$$

$$where \quad b-a\neq 0,\quad z^2+(a+b)z+ab\neq 0.$$

Proof of (i). According to Lemma (i), we have

$$\left(\frac{1}{z^2+(a+b)z+ab}\right)_\gamma = \frac{1}{b-a}\left(\frac{1}{z+a}-\frac{1}{z+b}\right)_\gamma \tag{7}$$

$$=\frac{1}{b-a}\{((z+a)^{-1})_\gamma-((z+b)^{-1})_\gamma\} \tag{8}$$

$$=e^{-i\pi\gamma}\frac{\Gamma(\gamma+1)}{b-a}\{(z+a)^{-1-\gamma}-(z+b)^{-1-\gamma}\}, \tag{9}$$

This is the result.

Proof of (ii). We have the result by setting $\gamma=n$ in the equation (5).

{*Note* } When $a+b=0$, we obtain the following case from this theorem,

$$\left(\frac{1}{z^2-a^2}\right)_\gamma=\frac{1}{2a}\left(\frac{1}{z-a}+\frac{1}{z+a}\right)_\gamma \quad (z^2-a^2\neq 0).$$

Theorem 3. We have the identities

(i)

$$\left(\frac{2z+a+b}{2}\right)^{-2-\gamma}\sum_{k=0}^{\infty}\frac{[1]_k\Gamma(2k+2+\gamma)}{k!\Gamma(2k+2)}\left(\left(\frac{a-b}{2z+a+b}\right)^2\right)^k$$

$$=\frac{\Gamma(\gamma+1)}{b-a}\left(\frac{1}{(z+a)^{1+\gamma}}-\frac{1}{(z+b)^{1+\gamma}}\right) \tag{10}$$

$$(|\Gamma(2k+2+\gamma)|<\infty,\quad |\Gamma(\gamma+1)|<\infty)$$

and

(ii)

$$\left(\frac{2z+a+b}{2}\right)^{-2-n}\sum_{k=0}^{\infty}[2k+2]_n\left(\left(\frac{a-b}{2z+a+b}\right)^2\right)^k$$
$$=\frac{n!}{b-a}\left(\frac{1}{(z+a)^{1+n}}-\frac{1}{(z+b)^{1+n}}\right) \tag{11}$$
$$(n\in Z_0^+)$$

where $|\frac{a-b}{2z+a+b}|<1,\ \ b-a\neq 0,\ \ z^2+(a+b)z+ab\neq 0.$

Proof. It is valid from Theorem 1 and Theorem 2.

Theorem 4. We have

(i)

$$\left(\frac{z+\frac{a+b}{2}}{z^2+(a+b)z+ab}\right)_\gamma = e^{-i\pi\gamma}(\frac{2}{2z+a+b})^{1+\gamma}$$
$$\times\sum_{m=0}^{\infty}\frac{\Gamma(2m+1+\gamma)}{\Gamma(2m+1)}\left((\frac{a-b}{2z+a+b})^2\right)^m \tag{12}$$
$$(|\Gamma(2m+1+\gamma)|<\infty)$$

and

(ii)

$$\left(\frac{z+\frac{a+b}{2}}{z^2+(a+b)z+ab}\right)_n = (-1)^n(\frac{2}{2z+a+b})^{1+n}$$
$$\times\sum_{m=0}^{\infty}[2m+1]_n\left((\frac{a-b}{2z+a+b})^2\right)^m \tag{13}$$
$$(n\in Z_0^+)$$

where $z^2+(a+b)z+ab\neq 0,\quad |(a-b)/(2z+a+b)|<1.$

Proof of (i). Setting $p=-(a+b)/2,\ q=(\frac{a+b}{2})^2-ab$, we have

$$z^2+(a+b)z+ab=(z-p)^2-q, \tag{14}$$

hence we obtain

$$\left(\frac{z+\frac{a+b}{2}}{z^2+(a+b)z+ab}\right)_\gamma = \left(((z-p)^2-q)^{-1}\cdot(z-p)\right)_\gamma \tag{15}$$

$$= \sum_{k=0}^{1} \frac{\Gamma(\gamma+1)}{k!\Gamma(\gamma+1-k)}(((z-p)^2-q)^{-1})_{\gamma-k}(z-p)_k \tag{16}$$

by Lemma (iv). According to Theorem D, (i) and Lemma (i), we have

$$(((z-p)^2-q)^{-1})_{\gamma-k} = e^{-i\pi(\gamma-k)}(z-p)^{k-2-\gamma}$$
$$\times \sum_{m=0}^{\infty} \frac{[1]_m}{m!}\frac{\Gamma(2m+2+\gamma-k)}{\Gamma(2m+2)}\left(\frac{q}{(z-p)^2}\right)^m, \tag{17}$$
$$(|q/(z-p)^2|<1, \quad |\Gamma(2m+2+\gamma-k)|<\infty)$$

and

$$(z-p)_k = e^{-i\pi k}\frac{\Gamma(k-1)}{\Gamma(-1)}(z-p)^{1-k}. \tag{18}$$

So we have

$$\left(((z-p)^2-q)^{-1})\cdot(z-p)\right)_\gamma = e^{-i\pi\gamma}\{\sum_{m=0}^{\infty}\frac{\Gamma(2m+2+\gamma)}{\Gamma(2m+2)}(z-p)^{-1-\gamma}\left(\frac{q}{(z-p)^2}\right)^m$$
$$-\gamma\sum_{m=0}^{\infty}\frac{\Gamma(2m+1+\gamma)}{\Gamma(2m+2)}(z-p)^{-1-\gamma}\left(\frac{q}{(z-p)^2}\right)^m\} \tag{19}$$
$$= e^{-i\pi\gamma}(z-p)^{-1-\gamma}\sum_{m=0}^{\infty}\frac{(2m+1)\Gamma(2m+1+\gamma)}{\Gamma(2m+2)}\left(\frac{q}{(z-p)^2}\right)^m \tag{20}$$
$$= e^{-i\pi\gamma}(z+\frac{a+b}{2})^{-1-\gamma}\sum_{m=0}^{\infty}\frac{\Gamma(2m+1+\gamma)}{\Gamma(2m+1)}\left(\frac{(\frac{a+b}{2})^2-ab}{(z+\frac{a+b}{2})^2}\right)^m. \tag{21}$$

Therefore we have (12) from (21) clearly under the conditions stated before.

Proof of (ii). We have the result by setting $\gamma = n$ in the equation (12).

Corollary 1. We have

(i)

$$\left(\frac{z}{z^2-a^2}\right)_\gamma = e^{-i\pi\gamma}\left(\frac{1}{z}\right)^{1+\gamma}\sum_{m=0}^{\infty}\frac{\Gamma(2m+1+\gamma)}{\Gamma(2m+1)}\left(\left(\frac{a}{z}\right)^2\right)^m \tag{22}$$
$$(|\Gamma(2m+1+\gamma)|<\infty)$$

and

(ii)

$$\left(\frac{z}{z^2-a^2}\right)_n = (-1)^n\left(\frac{1}{z}\right)^{1+n}\sum_{m=0}^{\infty}[2m+1]_n\left(\left(\frac{a}{z}\right)^2\right)^m \tag{23}$$

$$(n \in Z_0^+),$$

$$\text{where}\quad z^2-a^2 \neq 0, \quad |\frac{a}{z}| < 1.$$

Proof. It is clear from Theorem 4, by setting $a+b=0$.

Theorem 5. We have

(i)

$$\left(\frac{z+\frac{a+b}{2}}{z^2+(a+b)z+ab}\right)_\gamma = e^{-i\pi\gamma}\frac{\Gamma(1+\gamma)}{2}\{\frac{1}{(z+a)^{1+\gamma}}+\frac{1}{(z+b)^{1+\gamma}}\} \tag{24}$$

$$(|\Gamma(1+\gamma)| < \infty)$$

and

(ii)

$$\left(\frac{z+\frac{a+b}{2}}{z^2+(a+b)z+ab}\right)_n = (-1)^n\frac{n!}{2}\{\frac{1}{(z+a)^{1+n}}+\frac{1}{(z+b)^{1+n}}\} \tag{25}$$

$$(n \in Z_0^+)$$

$$\text{where}\quad z^2+(a+b)z+ab \neq 0, \qquad z+\frac{a+b}{2} \neq 0.$$

Proof of (i). We have

$$\frac{z+\frac{a+b}{2}}{z^2+(a+b)z+ab} = \frac{1}{2}\left(\frac{1}{z+a}+\frac{1}{z+b}\right) \tag{26}$$

hence, based on Lemma (i) we have the followings under the conditions,

$$LHS\ of\ (24) = \frac{1}{2}\left(\frac{1}{z+a}+\frac{1}{z+b}\right)_\gamma \tag{27}$$

$$= \frac{1}{2}\left(\left(\frac{1}{z+a}\right)_\gamma+\left(\frac{1}{z+b}\right)_\gamma\right) \tag{28}$$

$$= \frac{1}{2}e^{-i\pi\gamma}\Gamma(\gamma+1)\left(\frac{1}{(z+a)^{1+\gamma}}+\frac{1}{(z+b)^{1+\gamma}}\right). \tag{29}$$

Proof of (ii). We have the result by setting $\gamma = n$ in (i).

Theorem 6. We have the identities

(i)

$$\left(\frac{2}{2z+a+b}\right)^{1+\gamma} \sum_{m=0}^{\infty} \frac{\Gamma(2m+1+\gamma)}{\Gamma(2m+1)} \left(\left(\frac{a-b}{2z+a+b}\right)^2\right)^m = \frac{\Gamma(\gamma+1)}{2} \left(\frac{1}{(z+a)^{1+\gamma}} + \frac{1}{(z+b)^{1+\gamma}}\right) \tag{30}$$

$$(|\Gamma(2m+1+\gamma)| < \infty)$$

and

(ii)

$$\left(\frac{2}{2z+a+b}\right)^{1+n} \sum_{m=0}^{\infty} [2m+1]_n \left(\left(\frac{a-b}{2z+a+b}\right)^2\right)^m = \frac{n!}{2} \left(\frac{1}{(z+a)^{1+n}} + \frac{1}{(z+b)^{1+n}}\right) \tag{31}$$

$$(n \in Z_0^+)$$

where $|\frac{a-b}{2z+a+b}| < 1, \quad z + (a+b)/2 \neq 0, \quad z^2 + (a+b)z + ab \neq 0.$

Proof. It is valid from Theorem 4 and Theorem 5.

4 Semi Derivatives and Integrals

In this section we give semi derivatives and integrals from Theorems 1 , 4 , 5 and Corollary.

(I) From Theorem 1, (i), we have

(i)

$$\left(\frac{1}{z^2+(a+b)z+ab}\right)_{\frac{1}{2}} = -i\left(\frac{2z+a+b}{2}\right)^{-\frac{5}{2}} \sum_{k=0}^{\infty} \frac{[1]_k \Gamma(2k+\frac{5}{2})}{k!\Gamma(2k+2)} \left(\left(\frac{a-b}{2z+a+b}\right)^2\right)^k, \tag{1}$$

(*semi derivative*)

(ii)

$$\left(\frac{1}{z^2+(a+b)z+ab}\right)_{-\frac{1}{2}} = i\left(\frac{2z+a+b}{2}\right)^{-\frac{3}{2}} \sum_{k=0}^{\infty} \frac{[1]_k\Gamma(2k+\frac{3}{2})}{k!\Gamma(2k+2)} \left(\left(\frac{a-b}{2z+a+b}\right)^2\right)^k \tag{2}$$

(*semi integral*)

where

$$z^2+(a+b)z+ab \neq 0 \quad \text{and} \quad |(a-b)/(2z+a+b)| < 1.$$

(II) From Theorem 4, (i), we have

(i)

$$\left(\frac{1+\frac{a+b}{2}}{z^2+(a+b)z+ab}\right)_{\frac{1}{2}} = -i\left(\frac{2}{2z+a+b}\right)^{\frac{3}{2}} \sum_{m=0}^{\infty} \frac{\Gamma(2m+\frac{3}{2})}{\Gamma(2m+1)} \left(\left(\frac{a-b}{2z+a+b}\right)^2\right)^m \tag{3}$$

(*semi derivative*)

and

(ii)

$$\left(\frac{1+\frac{a+b}{2}}{z^2+(a+b)z+ab}\right)_{-\frac{1}{2}} = i\left(\left(\frac{2}{2z+a+b}\right)^2\right)^m \sum_{m=0}^{\infty} \frac{\Gamma(2m+\frac{1}{2})}{\Gamma(2m+1)} \left(\left(\frac{a-b}{2z+a+b}\right)^2\right)^m \tag{4}$$

(*semi integral*)

where

$$z^2+(a+b)z+ab \neq 0 \quad \text{and} \quad |(a-b)/(2z+a+b)| < 1.$$

(III) From Corollary 1, (i), we have

(i)

$$\left(\frac{z}{z^2-a^2}\right)_{\frac{1}{2}} = -i\left(\frac{1}{z}\right)^{\frac{3}{2}} \sum_{m=0}^{\infty} \frac{\Gamma(2m+\frac{3}{2})}{\Gamma(2m+1)} \left(\frac{a}{z}\right)^{2m}, \tag{5}$$

(*semi derivative*)

(ii)

$$\left(\frac{z}{z^2-a^2}\right)_{-\frac{1}{2}} = i\left(\frac{1}{z}\right)^{\frac{1}{2}} \sum_{m=0}^{\infty} \frac{\Gamma(2m+\frac{1}{2})}{\Gamma(2m+1)} \left(\frac{a}{z}\right)^{2m} \tag{6}$$

(*semi integral*)

where

$$z^2 - a^2 \neq 0 \text{ and } |a/z| < 1.$$

(IV) From Theorem 5, (i), we have

(i)

$$\left(\frac{z+\frac{a+b}{2}}{z^2+(a+b)z+ab}\right)_{\frac{1}{2}} = -i\frac{\sqrt{\pi}}{4}\left\{\frac{1}{(z+a)^{(3/2)}} + \frac{1}{(z+b)^{(3/2)}}\right\} \tag{7}$$

(*semi derivative*)

and

(ii)

$$\left(\frac{z+\frac{a+b}{2}}{z^2+(a+b)z+ab}\right)_{-\frac{1}{2}} = i\frac{\sqrt{\pi}}{2}\left\{\frac{1}{\sqrt{z+a}} + \frac{1}{\sqrt{z+b}}\right\} \tag{8}$$

(*semi integral*)

where

$$z^2 + (a+b)z + ab \neq 0 \text{ and } |z + (a+b)/2| \neq 0.$$

5 Examples of Theorem 3, (ii)

We illustrate some examples of Theorem 3 in cases of which the order of derivative are integers $n = 0, 1, 2$.

(I) When $n = 0$, from Theorem 3 the left-hand side of (11) in §3 is derived as follows,

$$\left(\frac{2z+a+b}{2}\right)^{-2} \sum_{k=0}^{\infty} [2k+2]_0 \left(\left(\frac{a-b}{2z+a+b}\right)^2\right)^k$$

$$= \left(\frac{2}{2z+a+b}\right)^2 \sum_{k=0}^{\infty} \frac{[1]_k}{k!} \left(\left(\frac{a-b}{2z+a+b}\right)^2\right)^k \tag{1}$$

$$= \left(\frac{2}{2z+a+b}\right)^2 \left(1-\left(\frac{a-b}{2z+a+b}\right)^2\right)^{-1} \tag{2}$$

$$= \frac{1}{(z+a)(z+b)}. \tag{3}$$

And we derive the followings from the right-hand side of (11),

$$\frac{1}{b-a}\left(\frac{1}{z+a} - \frac{1}{z+b}\right) = \frac{1}{(z+a)(z+b)}. \tag{4}$$

Then (11) holds true for $n = 0$.

(II) When $n = 1$, the LHS of (11) is

$$\left(\frac{2z+a+b}{2}\right)^{-3} \sum_{k=0}^{\infty} [2k+2]_1 \left(\left(\frac{a-b}{2z+a+b}\right)^2\right)^k$$

$$= \left(\frac{2}{2z+a+b}\right)^3 \sum_{k=0}^{\infty} \frac{[1]_k}{k!}(2k+2) \left(\left(\frac{a-b}{2z+a+b}\right)^2\right)^k \tag{5}$$

$$= \left(\frac{2}{2z+a+b}\right)^3 \left(\frac{(a-b)^2(2z+a+b)^2}{8(z+a)^2(z+b)^2} + \frac{(2z+a+b)^2}{2(z+a)(z+b)}\right) \tag{6}$$

$$= \frac{2z+a+b}{(z+a)^2(z+b)^2}, \tag{7}$$

since

$$\sum_{k=0}^{\infty} \frac{[1]_k}{k!} k \left(\left(\frac{a-b}{2z+a+b}\right)^2\right)^k = \sum_{k=1}^{\infty} \frac{[1]_k}{(k-1)!} \left(\left(\frac{a-b}{2z+a+b}\right)^2\right)^k \tag{8}$$

$$= \left(\frac{a-b}{2z+a+b}\right)^2 \sum_{k=0}^{\infty} \frac{[2]_k}{k!} \left(\left(\frac{a-b}{2z+a+b}\right)^2\right)^k \tag{9}$$

$$= \left(\frac{a-b}{2z+a+b}\right)^2 \left(1-\left(\frac{a-b}{2z+a+b}\right)^2\right)^{-2} \tag{10}$$

$$= \frac{(a-b)^2(2z+a+b)^2}{4^2(z+a)^2(z+b)^2} \tag{11}$$

and

$$\sum_{k=0}^{\infty} \frac{[1]_k}{k!} \left(\left(\frac{a-b}{2z+a+b} \right)^2 \right)^k = \left(1 - \left(\frac{a-b}{2z+a+b} \right)^2 \right)^{-1} \tag{12}$$

$$= \frac{(2z+a+b)^2}{4(z+a)(z+b)}. \tag{13}$$

Furthermore from RHS of (11) in §3, we have

$$\frac{1}{b-a} \left(\frac{1}{(z+a)^2} - \frac{1}{z+b)^2} \right) = \frac{2z+a+b}{(z+a)^2(z+b)^2}. \tag{14}$$

Then for $n = 1$, section 3.(11) holds.

(III) When $n = 2$, setting

$$T = \left(\frac{a-b}{2z+a+b} \right)^2, \tag{15}$$

we have

$$LHS\ of\ section\ 3.(11) = \left(\frac{2}{2z+a+b} \right)^4 \sum_{k=0}^{\infty} (2k+2)[2k+3]_1 T^k \tag{16}$$

$$= \left(\frac{2}{2z+a+b} \right)^4 \{ 2 \sum_{k=0}^{\infty} k[2k+3]_1 T^k + 2 \sum_{k=0}^{\infty} [2k+3]_1 T^k \} \tag{17}$$

$$= \left(\frac{2}{2z+a+b} \right)^4 2 \left(\frac{5T-T^2}{(1-T)^3} + \frac{3-T}{(1-T)^2} \right) \tag{18}$$

$$= \left(\frac{2}{2z+a+b} \right)^4 \left(\frac{6+2T}{(1-T)^3} \right) \tag{19}$$

$$= \frac{6z^2 + 6z(a+b) + 2(a^2+ab+b^2)}{(z+a)^3(z+b)^3}. \tag{20}$$

On the other hand we obtain the following from the right side of (11),

$$RHS\ of\ section\ 3.(11) = \frac{2}{2z+a+b} \left(\frac{1}{(z+a)^3} - \frac{1}{(z+b)^3} \right) \tag{21}$$

$$= \frac{6z^2(b-a) + 6z(b^2-a^2) + 2(b^3-a^3)}{(b-a)(z+a)^3(z+b)^3} \tag{22}$$

$$= \frac{6z^2 + 6z(a+b) + 2(a^2+ab+b^2)}{(z+a)^3(z+b)^3}. \tag{23}$$

Therefore (11) in §3 holds for the case of $n = 2$.
And so on.

{ *Note* } Examples for Theorem 6 (ii) shall be shown in another paper.

6 Appendix

In this section, we give some remarks on the infinite series. We have used these results in the previous section.

We have

$$\begin{aligned}
\sum_{k=0}^{\infty} k[2k+3]_1 T^k &= \sum_{k=1}^{\infty} \frac{[1]_k}{(k-1)!}[2k+3]_1 T^k & (1)\\
&= T\sum_{k=0}^{\infty} \frac{[2]_k}{k!}(2(k+1)+3)T^k & (2)\\
&= 2T\sum_{k=0}^{\infty} \frac{[2]_k}{k!}kT^k + 5T\sum_{k=0}^{\infty} \frac{[2]_k}{k!}T^k & (3)\\
&= 2T^2\sum_{k=0}^{\infty} \frac{[2]_{k+1}}{k!}T^k + 5T(1-T)^{-2} & (4)\\
&= 2^2T^2\sum_{k=0}^{\infty} \frac{[3]_k}{k!}T^k + 5T(1-T)^{-2} & (5)\\
&= 4T^2(1-T)^{-3} + 5T(1-T)^{-2} & (6)\\
&= \frac{5T-T^2}{(1-T)^3}, & (7)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{\infty} [2k+3]_1 T^k &= \sum_{k=0}^{\infty} \frac{[1]_k}{k!}(2k+3)T^k & (8)\\
&= 2\sum_{k=0}^{\infty} \frac{[1]_k}{k!}kT^k + 3\sum_{k=0}^{\infty} \frac{[1]_k}{k!}T^k & (9)\\
&= 2T\sum_{k=0}^{\infty} \frac{[2]_k}{k!}T^k + 3(1-T)^{-1} & (10)\\
&= 2T(1-T)^{-2} + 3(1-T)^{-1} & (11)\\
&= \frac{3-T}{(1-T)^2}. & (12)
\end{aligned}$$

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