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N- Fractional Calculus of Some Functions Which Include A Root Sign

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Abstract

In a previous article of the author, N- fractional calculus of some composite algebraic functions are derived. Applying this fresh results, N- fractional calculus of functions

$$\frac{1}{\sqrt[m]{(z-b)^m-c}} \quad \text{and} \quad \sqrt[m]{(z-b)^m-c} \qquad (m\in Z^+)$$

are reported in this paper. That is , we have the below, for example.

$$\text{(i)} \qquad \left(\frac{1}{\sqrt[m]{(z-b)^m-c}}\right)_\gamma = e^{-i\pi\gamma}(z-b)^{-1-\gamma}$$

$$\times\sum_{k=0}^{\infty}\frac{[1/m]_k\Gamma(mk+1+\gamma)}{k!\;\Gamma(mk+1)}\left(\frac{c}{(z-b)^m}\right)^k \qquad (|\Gamma(mk+1+\gamma)|<\infty)$$

and

$$\text{(ii)} \qquad \left(\sqrt[m]{(z-b)^m-c}\right)_\gamma = e^{-i\pi\gamma}(z-b)^{1-\gamma}$$

$$\times\sum_{k=0}^{\infty}\frac{[-1/m]_k\Gamma(mk-1+\gamma)}{k!\;\Gamma(mk-1)}\left(\frac{c}{(z-b)^m}\right)^k \qquad (|\Gamma(mk-1+\gamma)|<\infty)$$

where

$$|c/(z-b)^m|<1, \quad m\in Z^+,$$

and

$$[\lambda]_k = \lambda(\lambda+1)\cdots(\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda) \text{ with } [\lambda]_0 = 1,$$

(Notation of Pochhammer).

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$,

C_- be a curve along the cut joining two points z and $-\infty + i\,\mathrm{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty + i\,\mathrm{Im}(z)$,

D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ .

(Here D contains the points over the curve C).

Moreover, let $f = f(z)$ be a regular function in $D(z \in D)$,

$$f_\nu = (f)_\nu = {}_C(f)_\nu = \frac{\Gamma(\nu+1)}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{\nu+1}}\,d\zeta \qquad (\nu \notin \mathbb{Z}^-), \tag{1}$$

$$(f)_{-m} = \lim_{\nu \to -m} (f)_\nu \qquad (m \in \mathbb{Z}^+), \tag{2}$$

where $\quad -\pi \le \arg(\zeta - z) \le \pi$ for C_- , $\quad 0 \le \arg(\zeta - z) \le 2\pi$ for C_+ ,

$\zeta \ne z$, $z \in \boldsymbol{C}$, $\nu \in \boldsymbol{R}$, Γ ; Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z , of the function f , if $|(f)_\nu| < \infty$.

(I I) On the fractional calculus operator N^ν [3]

Theorem A. *Let fractional calculus operator (Nishimoto's Operator) N^ν be*

$$N^\nu = \left(\frac{\Gamma(\nu+1)}{2\pi i}\int_C \frac{d\zeta}{(\zeta - z)^{\nu+1}}\right) \qquad (\nu \notin \mathbb{Z}^-), \quad [\textit{Refer to} (1)] \tag{3}$$

with

$$N^{-m} = \lim_{\nu \to -m} N^\nu \qquad (m \in \mathbb{Z}^+), \tag{4}$$

and define the binary operation $\circ$ *as*

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\beta (N^\alpha f) \qquad (\alpha, \beta \in \boldsymbol{R}), \tag{5}$$

then the set

$$\{N^\nu\} = \{N^\nu \mid \nu \in \boldsymbol{R}\} \tag{6}$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator N^ν , for the function f such that $f \in F = \{f ;\, 0 \ne |f_\nu| < \infty, \nu \in \boldsymbol{R}\}$, where $f = f(z)$ and $z \in \boldsymbol{C}$.
(vis. $-\infty < \nu < \infty$).

(For our convenience, we call $N^\beta \circ N^\alpha$ as product of N^β and N^α.)

Theorem B. " F.O.G. $\{N^\nu\}$ " *is an " Action product group which has continuous index ν " for the set of F . (F.O.G.* ; Fractional calculus operator group) [3]

Theorem C. *Let*

$$S := \{\pm N^{\nu}\} \cup \{0\} = \{N^{\nu}\} \cup \{-N^{\nu}\} \cup \{0\} \quad (\nu \in \boldsymbol{R}) . \tag{7}$$

Then the set S is a commutative ring for the function $f \in F$, *when the identity*

$$N^{\alpha} + N^{\beta} = N^{\gamma} \quad (N^{\alpha}, N^{\beta}, N^{\gamma} \in S) . \tag{8}$$

(III) Lemma. We have [1]

(i) $$((z-c)^{b})_{\alpha} = e^{-i\pi\alpha} \frac{\Gamma(\alpha-b)}{\Gamma(-b)} (z-c)^{b-\alpha} \quad \left(\left| \frac{\Gamma(\alpha-b)}{\Gamma(-b)} \right| < \infty \right),$$

(ii) $$(\log(z-c))_{\alpha} = - e^{-i\pi\alpha} \Gamma(\alpha) (z-c)^{-\alpha} \quad \left(|\Gamma(\alpha)| < \infty \right),$$

(iii) $$((z-c)^{-\alpha})_{-\alpha} = - e^{i\pi\alpha} \frac{1}{\Gamma(\alpha)} \log(z-c) \quad \left(|\Gamma(\alpha)| < \infty \right),$$

where $z-c \neq 0$ for (i) and $z-c \neq 0, 1$ for (ii), (iii) ,

(iv) $$(u \cdot v)_{\alpha} := \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k!\, \Gamma(\alpha+1-k)} u_{\alpha-k} v_{k} \quad \begin{pmatrix} u = u(z), \\ v = v(z) \end{pmatrix}.$$

§ 1. Preliminary

The Teorem below is reported by the author already (cf. J. Frac. Calc. Vol. 29, May (2006),pp.35 - 44.) . [12]

Theorem D. *We have*

(i) $$\left(((z-b)^{\beta} - c)^{\alpha} \right)_{\gamma} = e^{-i\pi\gamma} (z-b)^{\alpha\beta-\gamma}$$

$$\times \sum_{k=0}^{\infty} \frac{[-\alpha]_k \Gamma(\beta k - \alpha\beta + \gamma)}{k!\, \Gamma(\beta k - \alpha\beta)} \left(\frac{c}{(z-b)^{\beta}} \right)^{k} \tag{1}$$

$$\left(\left| \frac{\Gamma(\beta k - \alpha\beta + \gamma)}{\Gamma(\beta k - \alpha\beta)} \right| < \infty \right)$$

and

(ii) $$\left(((z-b)^{\beta} - c)^{\alpha} \right)_{n} = (-1)^{n} (z-b)^{\alpha\beta-n}$$

$$\times \sum_{k=0}^{\infty} \frac{[-\alpha]_k [\beta k - \alpha\beta]_n}{k!} \left(\frac{c}{(z-b)^{\beta}} \right)^{k} \quad (n \in \boldsymbol{Z}_0^{+}) \tag{2}$$

where

$$\left| \frac{c}{(z-b)^{\beta}} \right| < 1 ,$$

and

$$[\lambda]_k = \lambda(\lambda+1)\cdots(\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda) \ \textit{with} \ [\lambda]_0 = 1 \ ,$$

(Notation of Pochhammer).

§ 2. N- Fractional Calculus of Functions $\dfrac{1}{\sqrt[m]{(z-b)^m - c}}$ and $\sqrt[m]{(z-b)^m - c}$, $(m \in \mathbf{Z}^+)$

Applying Theorem D in § 1. Preliminary we obtain the following theorems.

Theorem 1. *We have*

$$\text{(i)} \qquad \left(\frac{1}{\sqrt[m]{(z-b)^m - c}}\right)_\gamma = e^{-i\pi\gamma}(z-b)^{-1-\gamma}$$

$$\times \sum_{k=0}^{\infty} \frac{[1/m]_k \Gamma(mk+1+\gamma)}{k!\ \Gamma(mk+1)} \left(\frac{c}{(z-b)^m}\right)^k \qquad (|\Gamma(mk+1+\gamma)| < \infty) \tag{1}$$

and

$$\text{(ii)} \qquad \left(\frac{1}{\sqrt[m]{(z-b)^m - c}}\right)_n = (-1)^n (z-b)^{-1-n}$$

$$\times \sum_{k=0}^{\infty} \frac{[1/m]_k [mk+1]_n}{k!} \left(\frac{c}{(z-b)^m}\right)^k \qquad (n \in Z_0^+) \tag{2}$$

where

$$|c/(z-b)^m| < 1, \quad m \in \mathbf{Z}^+.$$

Proof of (i). We have

$$\left(\frac{1}{\sqrt[m]{(z-b)^m - c}}\right)_\gamma = \left(((z-b)^m - c)^{-1/m}\right)_\gamma , \tag{3}$$

therefore, setting $\beta = m$ and $\alpha = -1/m$ in Theorem D. (i), we obtain (1) clearly, under the conditions.

Proof of (i i). Set $\gamma = n$ in (1), we have then (2) , since

$$\frac{\Gamma(mk+1+n)}{\Gamma(mk+1)} = [mk+1]_n \ . \tag{4}$$

Corollary 1. *We have*

(i) $$\left(\frac{1}{\sqrt{(z-b)^2-c}}\right)_\gamma = e^{-i\pi\gamma}(z-b)^{-1-\gamma}$$

$$\times\sum_{k=0}^{\infty}\frac{[1/2]_k\,\Gamma(2k+1+\gamma)}{k!\,\Gamma(2k+1)}\left(\frac{c}{(z-b)^2}\right)^k \quad (\,|\,\Gamma(2k+1+\gamma)\,|<\infty\,) \tag{5}$$

and

(ii) $$\left(\frac{1}{\sqrt{(z-b)^2-c}}\right)_n = (-1)^n(z-b)^{-1-n}$$

$$\times\sum_{k=0}^{\infty}\frac{[1/2]_k[2k+1]_n}{k!}\left(\frac{c}{(z-b)^2}\right)^k \quad (\,n\in\mathbf{Z}_0^+\,) \tag{6}$$

where

$$|\,c/(z-b)^2\,|<1\ .$$

Proof. Set $m=2$ in Theorem 1.

Corollary 2. *We have*

(i) $$\left(\frac{1}{\sqrt{z^2-2bz+p}}\right)_\gamma = e^{-i\pi\gamma}(z-b)^{-1-\gamma}$$

$$\times\sum_{k=0}^{\infty}\frac{[1/2]_k\,\Gamma(2k+1+\gamma)}{k!\,\Gamma(2k+1)}\left(\frac{b^2-p}{(z-b)^2}\right)^k \quad (\,|\,\Gamma(2k+1+\gamma)\,|<\infty\,) \tag{7}$$

and

(ii) $$\left(\frac{1}{\sqrt{z^2-2bz+p}}\right)_n = (-1)^n(z-b)^{-1-n}$$

$$\times\sum_{k=0}^{\infty}\frac{[1/2]_k[2k+1]_n}{k!}\left(\frac{b^2-p}{(z-b)^2}\right)^k \quad (\,n\in\mathbf{Z}_0^+\,) \tag{8}$$

where

$$|\,(b^2-p)/(z-b)^2\,|<1\ .$$

Proof of (i). We have

$$z^2 - 2bz + p = (z-b)^2 - c \quad (c = b^2 - p), \tag{9}$$

hence

$$\left(\frac{1}{\sqrt{z^2 - 2bz + p}}\right)_\gamma = \left(\frac{1}{\sqrt{(z-b)^2 - c}}\right)_\gamma . \tag{10}$$

Therefore, we obtain (7) from (10) and (5), under the conditions.

Proof of (i i). Set $\gamma = n$ in (7), we have then (8) clearly.

Theorem 2. *We have*

(i) $$\left(\sqrt[m]{(z-b)^m - c}\right)_\gamma = e^{-i\pi\gamma}(z-b)^{1-\gamma}$$

$$\times \sum_{k=0}^{\infty} \frac{[-1/m]_k \Gamma(mk-1+\gamma)}{k!\ \Gamma(mk-1)} \left(\frac{c}{(z-b)^m}\right)^k \quad \left(\left|\frac{\Gamma(mk-1+\gamma)}{\Gamma(mk-1)}\right| < \infty\right) \tag{11}$$

and

(i i) $$\left(\sqrt[m]{(z-b)^m - c}\right)_n = (-1)^n (z-b)^{1-n}$$

$$\times \sum_{k=0}^{\infty} \frac{[-1/m]_k [mk-1]_n}{k!} \left(\frac{c}{(z-b)^m}\right)^k \quad (n \in \mathbf{Z}_0^+) \tag{12}$$

where

$$|c/(z-b)^m| < 1, \quad m \in \mathbf{Z}^+.$$

Proof of (i). We have

$$\left(\sqrt[m]{(z-b)^m - c}\right)_\gamma = \left(((z-b)^m - c)^{1/m}\right)_\gamma , \tag{13}$$

therefore, setting $\beta = m$ and $\alpha = 1/m$ in Theorem D. (i), we obtain (11) clearly, under the conditions.

Proof of (i i). Set $\gamma = n$ in (11).

Corollary 3. *We have*

(i) $$\left(\sqrt{(z-b)^2 - c}\right)_\gamma = e^{-i\pi\gamma}(z-b)^{1-\gamma}$$

$$\times \sum_{k=0}^{\infty} \frac{[-1/2]_k \Gamma(2k-1+\gamma)}{k!\ \Gamma(2k-1)} \left(\frac{c}{(z-b)^2}\right)^k \quad \left(\left|\frac{\Gamma(2k-1+\gamma)}{\Gamma(2k-1)}\right| < \infty\right) \tag{14}$$

and

(ii) $$\left(\sqrt{(z-b)^2-c}\right)_n = (-1)^n (z-b)^{1-n}$$

$$\times \sum_{k=0}^{\infty} \frac{[-1/2]_k [2k-1]_n}{k!} \left(\frac{c}{(z-b)^2}\right)^k \quad \left(n \in \mathbf{Z}_0^+\right) \tag{15}$$

where

$$|c/(z-b)^2| < 1 .$$

Proof . Set $m = 2$ in Theorem 2.

Corollary 4. *We have*

(i) $$\left(\sqrt{z^2-2bz+p}\right)_\gamma = e^{-i\pi\gamma} (z-b)^{1-\gamma}$$

$$\times \sum_{k=0}^{\infty} \frac{[-1/2]_k \Gamma(2k-1+\gamma)}{k!\ \Gamma(2k-1)} \left(\frac{b^2-p}{(z-b)^2}\right)^k \quad \left(\left|\frac{\Gamma(2k-1+\gamma)}{\Gamma(2k-1)}\right| < \infty\right) \tag{16}$$

and

(ii) $$\left(\sqrt{z^2-2bz+p}\right)_n = (-1)^n (z-b)^{1-n}$$

$$\times \sum_{k=0}^{\infty} \frac{[-1/2]_k [2k-1]_n}{k!} \left(\frac{b^2-p}{(z-b)^2}\right)^k \quad \left(n \in \mathbf{Z}_0^+\right) \tag{17}$$

where

$$|(b^2-p)/(z-b)^2| < 1 .$$

Proof of (i). We have (9), hence

$$\left(\sqrt{z^2-2bz+p}\right)_\gamma = \left(\sqrt{(z-b)^2-c}\right)_\gamma . \tag{18}$$

Therefore, we obtain (16) from (18) and (14), under the conditions.

Proof of (i i). Set $\gamma = n$ in (16).

§ 3. Special Cases of Corollaries 1. (i i) and 3. (i i)

[I] Special case of Corollary 1. (i i) (for $n = 0, 1, 2$)

1.) When $n = 0$ we have

$$\left(\frac{1}{\sqrt{(z-b)^2-c}}\right)_0 = (z-b)^{-1} \sum_{k=0}^{\infty} \frac{[1/2]_k}{k!}\left(\frac{c}{(z-b)^2}\right)^k \tag{1}$$

$$= \frac{1}{z-b}\left(1-\frac{c}{(z-b)^2}\right)^{-1/2} \tag{2}$$

$$= \left(\frac{1}{(z-b)^2-c}\right)^{1/2} \tag{3}$$

from § 2. (6).

2.) When $n = 1$ we have

$$\left(\frac{1}{\sqrt{(z-b)^2-c}}\right)_1 = -(z-b)^{-2} \sum_{k=0}^{\infty} \frac{[1/2]_k[2k+1]_1}{k!}\left(\frac{c}{(z-b)^2}\right)^k \tag{4}$$

$$= -(z-b)^{-2} \sum_{k=0}^{\infty} \frac{[1/2]_k(2k+1)}{k!} T^k \qquad \left(T = \frac{c}{(z-b)^2}\right) \tag{5}$$

$$= -2(z-b)^{-2} \sum_{k=0}^{\infty} \frac{[1/2]_k k}{k!} T^k - (z-b)^{-2} \sum_{k=0}^{\infty} \frac{[1/2]_k}{k!} T^k \tag{6}$$

$$= -2(z-b)^{-2} \cdot \frac{T}{2}(1-T)^{-3/2} - (z-b)^{-2}(1-T)^{-1/2} \tag{7}$$

$$= -(z-b)^{-2}(1-T)^{-3/2} \tag{8}$$

$$= -(z-b)\left((z-b)^2-c\right)^{-3/2} \tag{9}$$

from § 2. (6). (see Appendix I)

3.) When $n = 2$ we have

$$\left(\frac{1}{\sqrt{(z-b)^2-c}}\right)_2 = (z-b)^{-3} \sum_{k=0}^{\infty} \frac{[1/2]_k[2k+1]_2}{k!}\left(\frac{c}{(z-b)^2}\right)^k \tag{10}$$

$$= (z-b)^{-3} \sum_{k=0}^{\infty} \frac{[1/2]_k(2k+1)[2k+2]_1}{k!}\left(\frac{c}{(z-b)^2}\right)^k \tag{11}$$

$$= (z-b)^{-3}\left\{2\sum_{k=0}^{\infty}\frac{[1/2]_k k[2k+2]_1}{k!}T^k + \sum_{k=0}^{\infty}\frac{[1/2]_k[2k+2]_1}{k!}T^k\right\} \tag{12}$$

$$= (z-b)^{-3}\left\{\frac{4T-T^2}{(1-T)^{5/2}} + \frac{2-T}{(1-T)^{3/2}}\right\} \tag{13}$$

$$= (z-b)^{-3}\frac{1}{(1-T)^{3/2}}\left(\frac{T+2}{1-T}\right) \tag{14}$$

$$= \left((z-b)^2-c\right)^{-5/2}\left(2(z-b)^2+c\right), \tag{15}$$

from § 2. (6). (see Appendix II and III).

The results (9) and (15) coincide with the ones obtained by the classical calculations

$$\frac{d}{dz}\left(\frac{1}{\sqrt{(z-b)^2-c}}\right) \quad \text{and} \quad \frac{d^2}{dz^2}\left(\frac{1}{\sqrt{(z-b)^2-c}}\right), \tag{16}$$

respectively. And so on.

Therefore we can see that the presentation of Corollary 1. (i i) holds true for $n \in Z_0^+$.

[I I] Special case of Corollary 3. (i i) (for $n = 0, 1, 2$)

1.) When $n = 0$ we have

$$\left(\sqrt{(z-b)^2-c}\right)_0 = (z-b)\sum_{k=0}^{\infty}\frac{[-1/2]_k}{k!}\left(\frac{c}{(z-b)^2}\right)^k \tag{17}$$

$$= (z-b)\left(1-\frac{c}{(z-b)^2}\right)^{1/2} \tag{18}$$

$$= \left((z-b)^2-c\right)^{1/2} \tag{19}$$

from § 2. (15).

2.) When $n = 1$ we have

$$\left(\sqrt{(z-b)^2-c}\right)_1 = -\sum_{k=0}^{\infty}\frac{[-1/2]_k[2k-1]_1}{k!}\left(\frac{c}{(z-b)^2}\right)^k \tag{20}$$

$$= -\sum_{k=0}^{\infty} \frac{[-1/2]_k(2k-1)}{k!} T^k \qquad \left(T = \frac{c}{(z-b)^2}\right) \tag{21}$$

$$= -2\sum_{k=0}^{\infty} \frac{[-1/2]_k k}{k!} T^k + \sum_{k=0}^{\infty} \frac{[-1/2]_k}{k!} T^k \tag{22}$$

$$= T\,(1-T)^{-1/2} + (1-T)^{1/2} \tag{23}$$

$$= (1-T)^{-1/2} \tag{24}$$

$$= (z-b)\,((z-b)^2 - c)^{-1/2}\ , \tag{25}$$

from § 2. (6). (see Appendix IV.)

3.) When $n = 2$ we have

$$\left(\sqrt{(z-b)^2 - c}\right)_2 = (z-b)^{-1} \sum_{k=0}^{\infty} \frac{[-1/2]_k[2k-1]_2}{k!} \left(\frac{c}{(z-b)^2}\right)^k \tag{26}$$

$$= (z-b)^{-1} \sum_{k=0}^{\infty} \frac{[-1/2]_k(2k-1)[2k]_1}{k!} T^k \tag{27}$$

$$= (z-b)^{-1} \left\{2\sum_{k=0}^{\infty} \frac{[-1/2]_k k[2k]_1}{k!} T^k - \sum_{k=0}^{\infty} \frac{[-1/2]_k[2k]_1}{k!} T^k\right\} \tag{28}$$

$$= (z-b)^{-1} \left\{T(T-2)(1-T)^{-3/2} + T(1-T)^{-1/2}\right\} \tag{29}$$

$$= -(z-b)^{-1}\, T(1-T)^{-3/2} \tag{30}$$

$$= -\frac{c}{\left((z-b)^2 - c\right)^{3/2}}\ , \tag{31}$$

from § 2. (6). (see Appendix V and VI).

The results (25) and (31) coincide with the ones obtained by the classical calculations

$$\frac{d}{dz}\left(\sqrt{(z-b)^2 - c}\right) \quad \text{and} \quad \frac{d^2}{dz^2}\left(\sqrt{(z-b)^2 - c}\right),$$

respectively, again. And so on.

Therefore we can see that the presentation of Corollary 3. (i i) holds true for $n \in Z_0^+$.

Appendix

I. We have

$$\sum_{k=0}^{\infty} \frac{[1/2]_k k}{k!} T^k = \sum_{k=1}^{\infty} \frac{[1/2]_k}{(k-1)!} T^k = T \sum_{k=0}^{\infty} \frac{[1/2]_{k+1}}{k!} T^k \qquad (1)$$

$$= \frac{1}{2} T \sum_{k=0}^{\infty} \frac{[3/2]_k}{k!} T^k = \frac{1}{2} T (1-T)^{-3/2} \qquad (2)$$

II.

$$\sum_{k=0}^{\infty} \frac{[1/2]_k k[2k+2]_1}{k!} T^k = \sum_{k=1}^{\infty} \frac{[1/2]_k [2k+2]_1}{(k-1)!} T^k \qquad (3)$$

$$= T \sum_{k=0}^{\infty} \frac{[1/2]_{k+1} [2k+4]_1}{k!} T^k = \frac{1}{2} T \sum_{k=0}^{\infty} \frac{[3/2]_k (2k+4)}{k!} T^k \qquad (4)$$

$$= T \sum_{k=0}^{\infty} \frac{[3/2]_k k}{k!} T^k + 2T \sum_{k=0}^{\infty} \frac{[3/2]_k}{k!} T^k \qquad (5)$$

$$= T^2 \sum_{k=0}^{\infty} \frac{[3/2]_{k+1}}{k!} T^k + 2T(1-T)^{-3/2} \qquad (6)$$

$$= \frac{3}{2} T^2 \sum_{k=0}^{\infty} \frac{[5/2]_k}{k!} T^k + 2T(1-T)^{-3/2} \qquad (7)$$

$$= \frac{3}{2} T^2 (1-T)^{-5/2} + 2T(1-T)^{-3/2} \qquad (8)$$

$$= \frac{4T - T^2}{2(1-T)^{5/2}} \qquad (9)$$

III.

$$\sum_{k=0}^{\infty} \frac{[1/2]_k [2k+2]_1}{k!} T^k = \sum_{k=1}^{\infty} \frac{[1/2]_k (2k+2)}{k!} T^k \qquad (10)$$

$$= 2 \sum_{k=0}^{\infty} \frac{[1/2]_k k}{k!} T^k + 2 \sum_{k=0}^{\infty} \frac{[1/2]_k}{k!} T^k \qquad (11)$$

$$= T(1-T)^{-3/2} + 2(1-T)^{-1/2} \qquad (12)$$

$$= \frac{2-T}{(1-T)^{3/2}} \qquad (13)$$

IV.

$$\sum_{k=0}^{\infty} \frac{[-1/2]_k k}{k!} T^k = \sum_{k=1}^{\infty} \frac{[-1/2]_k}{(k-1)!} T^k \tag{14}$$

$$= T \sum_{k=0}^{\infty} \frac{[-1/2]_{k+1}}{k!} T^k = -\frac{1}{2} T \sum_{k=0}^{\infty} \frac{[1/2]_k}{k!} T^k \tag{15}$$

$$= -\frac{1}{2} T (1-T)^{-1/2} . \tag{16}$$

V.

$$\sum_{k=0}^{\infty} \frac{[-1/2]_k k [2k]_1}{k!} T^k = \sum_{k=1}^{\infty} \frac{[-1/2]_k [2k]_1}{(k-1)!} T^k \tag{17}$$

$$= T \sum_{k=0}^{\infty} \frac{[-1/2]_{k+1} [2k+2]_1}{k!} T^k = -\frac{T}{2} \sum_{k=0}^{\infty} \frac{[1/2]_k (2k+2)}{k!} T^k \tag{18}$$

$$= -T \sum_{k=0}^{\infty} \frac{[1/2]_k k}{k!} T^k - T \sum_{k=0}^{\infty} \frac{[1/2]_k}{k!} T^k \tag{19}$$

$$= -\frac{1}{2} T^2 (1-T)^{-3/2} - T (1-T)^{-1/2} \tag{20}$$

$$= \frac{1}{2} T(T-2)(1-T)^{-3/2} . \tag{21}$$

VI.

$$\sum_{k=0}^{\infty} \frac{[-1/2]_k [2k]_1}{k!} T^k = \sum_{k=0}^{\infty} \frac{[-1/2]_k (2k)}{(k-1)!} T^k \tag{22}$$

$$= 2 \sum_{k=1}^{\infty} \frac{[-1/2]_k}{(k-1)!} T^k = 2T \sum_{k=0}^{\infty} \frac{[-1/2]_{k+1}}{k!} T^k \tag{23}$$

$$= -T \sum_{k=0}^{\infty} \frac{[1/2]_k}{k!} T^k \tag{24}$$

$$= -T (1-T)^{-1/2} . \tag{25}$$

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