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N- Fractional Calculus of Some Multi- Powers Functions

Katsuyuki Nishimoto

Institute for Applied Mathematics, Descartes Press Co.
2- 13- 10 Kaguike, Koriyama, 963 - 8833, JAPAN
Fax : + 81 - 24 - 922 - 7596

Abstract

In a previous article of the author, N- fractional calculus

$$\left(((z-b)^{\beta}-c)^{\alpha}\right)_{\gamma}\ ,\quad ((z-b)^{\beta}-c\neq 0)$$

are discussed. In this article that of more extended forms

$$\left((((z-b)^{\beta}-c)^{\alpha}-d)^{\delta}\right)_{\gamma}\ ,\quad (((z-b)^{\beta}-c)^{\alpha}-d\neq 0)$$

are discussed. Moreover their special cases

$$\left((((z-b)^{\beta}-c)^{\alpha}-d)^{\delta}\right)_{n}\ ,\quad (n\in Z_0^{+},((z-b)^{\beta}-c)^{\alpha}-d\neq 0)$$

are presented .

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D=\{D_-,D_+\}$, $C=\{C_-,C_+\}$,

C_- be a curve along the cut joining two points z and $-\infty+i\,\mathrm{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty+i\,\mathrm{Im}(z)$,

D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ .

(Here D contains the points over the curve C).

Moreover, let $f=f(z)$ be a regular function in $D (z\in D)$,

$$f_\nu=(f)_\nu={}_C(f)_\nu=\frac{\Gamma(\nu+1)}{2\pi i}\int_C\frac{f(\zeta)}{(\zeta-z)^{\nu+1}}d\zeta\qquad(\nu\notin Z^-),\tag{1}$$

$$(f)_{-m}=\lim_{\nu\to-m}(f)_\nu\qquad(m\in Z^+),\tag{2}$$

where $-\pi\le\arg(\zeta-z)\le\pi$ for C_- , $0\le\arg(\zeta-z)\le2\pi$ for C_+ ,

$\zeta\neq z$, $z\in C$, $\nu\in R$, Γ ; Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu>0$, and integrals of order $-\nu$ for $\nu<0$), with respect to z , of the function f , if $|(f)_\nu|<\infty$.

(I I) On the fractional calculus operator N^{ν} [3]

Theorem A. *Let fractional calculus operator (Nishimoto's Operator) N^{ν} be*

$$N^{\nu}=\left(\frac{\Gamma(\nu+1)}{2\pi i}\int_{C}\frac{d\zeta}{(\zeta-z)^{\nu+1}}\right)\qquad(\nu\notin Z^{-}),\quad [\,Refer\ to\,(1)\,] \tag{3}$$

with

$$N^{-m}=\lim_{\nu\to -m}N^{\nu}\qquad(m\in Z^{+}), \tag{4}$$

and define the binary operation $\circ$ *as*

$$N^{\beta}\circ N^{\alpha}f=N^{\beta}N^{\alpha}f=N^{\beta}(N^{\alpha}f)\qquad(\alpha,\beta\in R), \tag{5}$$

then the set

$$\left\{N^{\nu}\right\}=\left\{N^{\nu}\mid \nu\in R\right\} \tag{6}$$

is an Abelian product group (having continuous index ν *) which has the inverse transform operator* $(N^{\nu})^{-1}=N^{-\nu}$ *to the fractional calculus operator* N^{ν} *, for the function* f *such that* $f\in F=\left\{f;\ 0\neq|f_{\nu}|<\infty,\nu\in R\right\}$, *where* $f=f(z)$ *and* $z\in C$.
(*vis.* $-\infty<\nu<\infty$).

(For our convenience, we call $N^{\beta}\circ N^{\alpha}$ as product of N^{β} and N^{α}.)

Theorem B. " F.O.G. $\{N^{\nu}\}$ " *is an " Action product group which has continuous index* ν *" for the set of F* . (*F.O.G.* ; Fractional calculus operator group) [3]

(III) **Lemma.** We have [1]

(i) $$((z-c)^{b})_{\alpha}=e^{-i\pi\alpha}\frac{\Gamma(\alpha-b)}{\Gamma(-b)}(z-c)^{b-\alpha}\qquad\left(\left|\frac{\Gamma(\alpha-b)}{\Gamma(-b)}\right|<\infty\right), \tag{7}$$

(ii) $$(\log(z-c))_{\alpha}=-e^{-i\pi\alpha}\Gamma(\alpha)(z-c)^{-\alpha}\qquad\left(|\Gamma(\alpha)|<\infty\right), \tag{8}$$

(iii) $$((z-c)^{-\alpha})_{-\alpha}=-e^{i\pi\alpha}\frac{1}{\Gamma(\alpha)}\log(z-c)\qquad\left(|\Gamma(\alpha)|<\infty\right), \tag{9}$$

where $z-c\neq 0$ for (i) and $z-c\neq 0,1$ for (ii), (iii) ,

(iv) $$(u\cdot v)_{\alpha}:=\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)}u_{\alpha-k}v_{k}\qquad\left(\begin{matrix}u=u(z),\\ v=v(z)\end{matrix}\right). \tag{10}$$

§ 1. Preliminary

The Teorem below is reported by the author already (cf. J. Frac. Calc. Vol. 29, May (2006),pp.35 - 44.) . [12]

Theorem D. *We have*

(i) $$\left(((z-b)^{\beta}-c)^{\alpha}\right)_{\gamma}=e^{-i\pi\gamma}(z-b)^{\alpha\beta-\gamma}$$

$$\times \sum_{k=0}^{\infty} \frac{[-\alpha]_k \, \Gamma(\beta k - \alpha\beta + \gamma)}{k! \, \Gamma(\beta k - \alpha\beta)} \left(\frac{c}{(z-b)^{\beta}} \right)^{k} \qquad (1)$$

$$\left(\left| \frac{\Gamma(\beta k - \alpha\beta + \gamma)}{\Gamma(\beta k - \alpha\beta)} \right| < \infty \right)$$

and

(ii) $$\left(((z-b)^{\beta} - c)^{\alpha} \right)_n = (-1)^n (z-b)^{\alpha\beta - n}$$

$$\times \sum_{k=0}^{\infty} \frac{[-\alpha]_k \, [\beta k - \alpha\beta]_n}{k!} \left(\frac{c}{(z-b)^{\beta}} \right)^{k} \qquad (n \in \mathbf{Z}_0^{+}) \qquad (2)$$

where

$$\left| \frac{c}{(z-b)^{\beta}} \right| < 1 \, ,$$

and

$$[\lambda]_k = \lambda(\lambda+1)\cdots(\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda) \ \textit{with} \ [\lambda]_0 = 1 \, ,$$

(Notation of Pochhammer).

§ 2. N- Fractional Calculus of Functions $\left(((z-b)^{\beta} - c)^{\alpha} - d \right)^{\delta}$

Theorem 1. *We have*

(i) $$\left((((z-b)^{\beta} - c)^{\alpha} - d)^{\delta} \right)_{\gamma} = e^{-i\pi\gamma} (z-b)^{\alpha\beta\delta - \gamma}$$

$$\times \sum_{m,k=0}^{\infty} \frac{[-\delta]_m [-\alpha(\delta-m)]_k \, \Gamma(\beta k - \alpha\beta(\delta-m) + \gamma)}{m! \cdot k! \, \Gamma(\beta k - \alpha\beta(\delta-m))} \left(\frac{c}{(z-b)^{\beta}} \right)^{k} \left(\frac{d}{(z-b)^{\alpha\beta}} \right)^{m} \qquad (1)$$

$$\left(\left| \frac{\Gamma(\beta k - \alpha\beta(\delta-m) + \gamma)}{\Gamma(\beta k - \alpha\beta(\delta-m))} \right| < \infty \right)$$

and

(ii) $$\left(((z-b)^{\beta} - c)^{\alpha} - d)^{\delta} \right)_n = (-1)^n (z-b)^{\alpha\beta\delta - n}$$

$$\times \sum_{m,k=0}^{\infty} \frac{[-\delta]_m [-\alpha(\delta-m)]_k \, [\beta k - \alpha\beta(\delta-m)]_n}{m! \cdot k!} \left(\frac{c}{(z-b)^{\beta}} \right)^{k} \left(\frac{d}{(z-b)^{\alpha\beta}} \right)^{m} \, , \qquad (2)$$

where

$$((z-b)^{\beta}-c)^{\alpha}-d\neq 0, \quad \left|\frac{c}{(z-b)^{\beta}}\right|<1, \quad \left|\frac{d}{(z-b)^{\alpha\beta}}\right|<1, \quad n\in \mathbf{Z}_0^+.$$

Proof of (i). We have

$$(((z-b)^{\beta}-c)^{\alpha}-d)^{\delta}=X^{\delta}(1-\tfrac{d}{X})^{\delta} \qquad (X=((z-b)^{\beta}-c)^{\alpha}) \tag{3}$$

$$=X^{\delta}\sum_{m=0}^{\infty}\frac{[-\delta]_m}{m!}\left(\frac{d}{X}\right)^m \tag{4}$$

$$=\sum_{m=0}^{\infty}\frac{[-\delta]_m(d^m)}{m!}X^{\delta-m} \tag{5}$$

$$=\sum_{m=0}^{\infty}\frac{[-\delta]_m(d^m)}{m!}\left((z-b)^{\beta}-c\right)^{\alpha(\delta-m)}, \tag{6}$$

hence, operating N- fractional calculus operator N^{γ} to the both sides of (6), we obtain

$$\left((((z-b)^{\beta}-c)^{\alpha}-d)^{\delta}\right)_{\gamma}$$

$$=\sum_{m=0}^{\infty}\frac{[-\delta]_m(d^m)}{m!}\left(((z-b)^{\beta}-c)^{\alpha(\delta-m)}\right)_{\gamma} \tag{7}$$

$$=\sum_{m=0}^{\infty}\frac{[-\delta]_m(d^m)}{m!}\Big[e^{-i\pi\gamma}(z-b)^{\alpha\beta(\delta-m)-\gamma}$$

$$\times\sum_{k=0}^{\infty}\frac{[-\alpha(\delta-m)]_k\,\Gamma(\beta k-\alpha\beta(\delta-m)+\gamma)}{k!\,\Gamma(\beta k-\alpha\beta(\delta-m))}\left(\frac{c}{(z-b)^{\beta}}\right)^k, \tag{8}$$

applying Theorem D. (i), under the conditions stated before.

We have then (1) from (8) clearly.

Proof of (i i). Set $\gamma=n$ in (1).

Note 1. We use the notations $\sum_{m,k=0}^{\infty}\cdots=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\cdots$, for our convenience.

Note.2. When $d=0$ and $\delta=1$, Theorem 1 is reduced to Theorem D., clearly.

Corollary 1. *We have*

(i) $$\left(((z^{\beta}-c)^{\alpha}-d)^{\delta}\right)_{\gamma}=e^{-i\pi\gamma}z^{\alpha\beta\delta-\gamma}$$

$$\times\sum_{m,k=0}^{\infty}\frac{[-\delta]_m[-\alpha(\delta-m)]_k\Gamma(\beta k-\alpha\beta(\delta-m)+\gamma)}{m!\cdot k!\,\Gamma(\beta k-\alpha\beta(\delta-m))}\left(\frac{c}{z^{\beta}}\right)^{k}\left(\frac{d}{z^{\alpha\beta}}\right)^{m} \tag{9}$$

$$\left(\left|\frac{\Gamma(\beta k-\alpha\beta(\delta-m)+\gamma)}{\Gamma(\beta k-\alpha\beta(\delta-m))}\right|<\infty\right)$$

and

(ii) $$\left(((z^{\beta}-c)^{\alpha}-d)^{\delta}\right)_{n}=(-1)^{n}z^{\alpha\beta\delta-n}$$

$$\times\sum_{m,k=0}^{\infty}\frac{[-\delta]_m[-\alpha(\delta-m)]_k[\beta k-\alpha\beta(\delta-m)]_n}{m!\cdot k!}\left(\frac{c}{z^{\beta}}\right)^{k}\left(\frac{d}{z^{\alpha\beta}}\right)^{m}, \tag{10}$$

where

$$(z^{\beta}-c)^{\alpha}-d\neq 0,\quad \left|\frac{c}{z^{\beta}}\right|<1,\quad \left|\frac{d}{z^{\alpha\beta}}\right|<1,\quad n\in Z_0^{+}.$$

Proof. Set $b=0$ in Theorem 1.

§ 3. Some Special Cases

[I] When $\beta=\alpha=1$, we obtain

$$\left((((z-b-c-d)^{\delta}\right)_{\gamma}=e^{-i\pi\gamma}(z-b)^{\delta-\gamma}$$

$$\times\sum_{m,k=0}^{\infty}\frac{[-\delta]_m[m-\delta]_k\Gamma(k+m-\delta+\gamma)}{m!\cdot k!\,\Gamma(k+m-\delta)}\left(\frac{c}{z-b}\right)^{k}\left(\frac{d}{z-b}\right)^{m} \tag{1}$$

$$\left(\left|\frac{c}{z-b}\right|,\ \left|\frac{d}{z-b}\right|<1,\ \left|\frac{\Gamma(k+m-\delta+\gamma)}{\Gamma(k+m-\delta)}\right|<\infty\right)$$

from Theorem 1. (i).

Now we have the identities

$$\Gamma(m-\delta)=\Gamma(-\delta)[-\delta]_m\ , \tag{2}$$

$$\Gamma(k+m-\delta)=\Gamma(m-\delta)[m-\delta]_k\ , \tag{3}$$

$$\Gamma(k+m-\delta+\gamma)=\Gamma(m-\delta+\gamma)[m-\delta+\gamma]_k\ , \tag{4}$$

then applying (2) ~ (4) into the RHS of (1), we obtain

$$\text{RHS of } (1) = e^{-i\pi\gamma}(z-b)^{\delta-\gamma}$$

$$\times \sum_{m=0}^{\infty} \frac{\Gamma(m-\delta+\gamma)}{m!\,\Gamma(-\delta)}\left(\frac{d}{z-b}\right)^m \sum_{k=0}^{\infty} \frac{[m-\delta+\gamma]_k}{k!}\left(\frac{c}{z-b}\right)^k \tag{5}$$

$$= e^{-i\pi\gamma}(z-b)^{\delta-\gamma} \sum_{m=0}^{\infty} \frac{\Gamma(m-\delta+\gamma)}{m!\,\Gamma(-\delta)}\left(\frac{d}{z-b}\right)^m \left(1-\frac{c}{z-b}\right)^{\delta-\gamma-m} \tag{6}$$

$$= e^{-i\pi\gamma}(z-b-c)^{\delta-\gamma} \frac{\Gamma(-\delta+\gamma)}{\Gamma(-\delta)} \sum_{m=0}^{\infty} \frac{[-\delta+\gamma]_m}{m!}\left(\frac{d}{z-b-c}\right)^m \tag{7}$$

$$= e^{-i\pi\gamma}\frac{\Gamma(-\delta+\gamma)}{\Gamma(-\delta)}(z-b-c-d)^{\delta-\gamma}, \qquad \left(\left|\frac{\Gamma(-\delta+\gamma)}{\Gamma(-\delta)}\right| < \infty\right) \tag{8}$$

using the relationship

$$\sum_{k=0}^{\infty} \frac{[\lambda]_k}{k!} z^k = (1-z)^{-\lambda}. \tag{9}$$

The result (8) is same as the one obtained by Lemma (i).

[II] When $d = 0$ and $\delta = 1$, we obtain

$$\left(((z-b)^\beta - c)^\alpha\right)_\gamma = e^{-i\pi\gamma}(z-b)^{\alpha\beta-\gamma}$$

$$\times \sum_{m,k=0}^{\infty} \frac{[-1]_m[-\alpha(1-m)]_k\,\Gamma(\beta k-\alpha\beta(1-m)+\gamma)}{m!\cdot k!\,\Gamma(\beta k-\alpha\beta(1-m))}\left(\frac{c}{(z-b)^\beta}\right)^k\left(\frac{0}{(z-b)^{\alpha\beta}}\right)^m \tag{10}$$

$$= e^{-i\pi\gamma}(z-b)^{\alpha\beta-\gamma}\sum_{k=0}^{\infty} \frac{[-\alpha]_k\,\Gamma(\beta k-\alpha\beta+\gamma)}{k!\,\Gamma(\beta k-\alpha\beta)}\left(\frac{c}{(z-b)^\beta}\right)^k \tag{11}$$

$$\left(\left|\frac{c}{(z-b)^\beta}\right| < 1, \quad \left|\frac{\Gamma(\beta k-\alpha\beta+\gamma)}{\Gamma(\beta k-\alpha\beta)}\right| < \infty\right)$$

from Theorem 1. (i).

That is, in this case we have Theorem D. (i) it self, clearly.

[III] When $n = 0$, we have

$$\left(((z-b)^\beta - c)^\alpha - d)^\delta\right)_0 = (z-b)^{\alpha\beta\delta}$$

$$\times \sum_{m,k=0}^{\infty} \frac{[-\delta]_m[-\alpha(\delta-m)]_k}{m!\cdot k!}\left(\frac{c}{(z-b)^\beta}\right)^k\left(\frac{d}{(z-b)^{\alpha\beta}}\right)^m, \tag{12}$$

$$((z-b)^{\beta}-c)^{\alpha}-d \neq 0, \quad \left|\frac{c}{(z-b)^{\beta}}\right|<1, \quad \left|\frac{d}{(z-b)^{\alpha\beta}}\right|<1, .$$

from Theorem 1. (i i).

Indeed we obtain

RHS of (12) $= (z-b)^{\alpha\beta\delta}$

$$\times \sum_{m=0}^{\infty} \frac{[-\delta]_m}{m!} \left(\frac{d}{(z-b)^{\alpha\beta}}\right)^m \sum_{k=0}^{\infty} \frac{[-\alpha(\delta-m)]_k}{k!} \left(\frac{c}{(z-b)^{\beta}}\right)^k \tag{13}$$

$$= (z-b)^{\alpha\beta\delta} \sum_{m=0}^{\infty} \frac{[-\delta]_m}{m!} \left(\frac{d}{(z-b)^{\alpha\beta}}\right)^m \left(1-\frac{c}{(z-b)^{\beta}}\right)^{\alpha\delta-\alpha m} \tag{14}$$

$$= ((z-b)^{\beta}-c)^{\alpha\delta} \sum_{m=0}^{\infty} \frac{[-\delta]_m}{m!} \left(\frac{d}{((z-b)^{\beta}-c)^{\alpha}}\right)^m \tag{15}$$

$$= ((z-b)^{\beta}-c)^{\alpha\delta} \left(1-\frac{d}{((z-b)^{\beta}-c)^{\alpha}}\right)^{\delta} \tag{16}$$

$$= (((z-b)^{\beta}-c)^{\alpha}-d)^{\delta} \quad , \tag{17}$$

clearly.

[IV] When $n=1$, we have

$$\left(((z-b)^{\beta}-c)^{\alpha}-d)^{\delta}\right)_1 = -(z-b)^{\alpha\beta\delta-1}$$

$$\times \sum_{m,k=0}^{\infty} \frac{[-\delta]_m[-\alpha(\delta-m)]_k \cdot (\beta k-\alpha\beta\delta+\alpha\beta m)}{m!\cdot k!} \left(\frac{c}{(z-b)^{\beta}}\right)^k \left(\frac{d}{(z-b)^{\alpha\beta}}\right)^m , \tag{18}$$

from Theorem 1. (i i).

Then letting

$$R := \left(\frac{u}{(z-b)^{\beta}}\right)^{\alpha} \qquad (u=(z-b)^{\beta}-c) \tag{19}$$

and

$$S := \left(1-\frac{d}{u^{\alpha}}\right)^{\delta} \tag{20}$$

we have

$$\sum_{m,k=0}^{\infty} \frac{[-\delta]_m[-\alpha(\delta-m)]_k \cdot \beta k}{m!\cdot k!}\left(\frac{c}{(z-b)^{\beta}}\right)^k \left(\frac{d}{(z-b)^{\alpha\beta}}\right)^m$$

$$= \frac{-c\alpha\beta\delta}{u} R^{\delta} S - \frac{cd\alpha\beta\delta}{u(u^{\alpha}-d)} R^{\delta} S \quad , \tag{21}$$

$$-\alpha\beta\delta \sum_{m,k=0}^{\infty} \frac{[-\delta]_m[-\alpha(\delta-m)]_k}{m!\cdot k!}\left(\frac{c}{(z-b)^{\beta}}\right)^k \left(\frac{d}{(z-b)^{\alpha\beta}}\right)^m$$

$$= -\alpha\beta\delta R^{\delta} S \quad , \tag{22}$$

and

$$\alpha\beta \sum_{m,k=0}^{\infty} \frac{[-\delta]_m \cdot m[-\alpha(\delta-m)]_k}{m!\cdot k!}\left(\frac{c}{(z-b)^{\beta}}\right)^k \left(\frac{d}{(z-b)^{\alpha\beta}}\right)^m$$

$$= -\frac{d\alpha\beta\delta}{(u^{\alpha}-d)} R^{\delta} S \quad , \tag{23}$$

Therefore, applying (23), (22) and (21) into (18), we obtain

$$\left(((z-b)^{\beta}-c)^{\alpha}-d)^{\delta}\right)_1 = (z-b)^{\alpha\beta\delta-1} R^{\delta} S$$

$$\times\left[\frac{c\alpha\beta\delta}{u} + \frac{cd\alpha\beta\delta}{u(u^{\alpha}-d)} + \alpha\beta\delta + \frac{d\alpha\beta\delta}{u^{\alpha}-d}\right] \tag{24}$$

$$= \alpha\beta\delta(z-b)^{\alpha\beta\delta-1}\left(\frac{c}{u}+1\right)\left(\frac{d}{u^{\alpha}-d}+1\right) \tag{25}$$

$$= \alpha\beta\delta(z-b)^{-1}(u^{\alpha}-d)^{\delta}\left(\frac{(z-b)^{\beta}}{u}\right)\left(\frac{u^{\alpha}}{u^{\alpha}-d}\right) \tag{26}$$

$$= \alpha\beta\delta(u^{\alpha}-d)^{\delta-1}(u^{\alpha-1})(z-b)^{\beta-1} \tag{27}$$

$$= \alpha\beta\delta(((z-b)^{\beta}-c)^{\alpha}-d)^{\delta-1}((z-b)^{\beta}-c)^{\alpha-1}(z-b)^{\beta-1} \tag{28}$$

clearly.

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Katsuyuki Nishimoto
Institute for Applied Mathematics
Descartes Press Co.
2 - 13 - 10 Kaguike, Koriyama
963 - 8833 Japan