数理解析研究所講究録
第 1580 巻 2008 年 136-149

Optimal Strategy with Uncertain Trade Execution [1]

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1 Introduction

In the real market, assets are not perfectly liquid. An investor sometimes fails to execute his trade. The risk associated with uncertain trade execution is called the execution risk. The execution risk is an important financial risk.

This paper studies a hedging problem in a discrete time model with the execution risk. The investor cannot always trade the assets and trade times are random. Usually the investor cannot hedges the contingent claim perfectly. The investor should measure the hedging error by a quadratic criterion. My problem is the mean-variance hedging problem in a random trade time model. The mean-variance hedging problem is studied both in a discrete time framework by Schäl [17], Schweizer [18] and Černý[3] and in a continuous time framework by Duffie and Richardson [5], Gourieroux, Laurent and Pham [6], Pham, Rheinländer and Schweizer [15], Pham [14], Schweizer [20], Arai [1]. The random trade time model is studied by Rogers and Zane [16] and Matsumoto [11, 12] in a continuous time framework.

First we study an optimal strategy with fixed initial condition. Secondly we find an optimal initial condition. The following results are shown.

1. An optimal strategy exists under some proper conditions.
2. The optimal strategy is expressed as the recursive formula, which can be calculated computationally.
3. In the fixed initial condition case, the hedging error case can be decomposed into two parts. One part is related to the contingent claim and the other part is related to the initial condition.
4. The optimal initial condition is given, using some signed measures.
5. The optimal initial cost is not suitable for the price of the contingent claim, when there is the execution risk.

This paper is organized as follows. Section 2 gives the model and the problem. And we explains the main results in this paper. We give sketches of proofs in Section 3. Finally in Section 4, we give some numerical examples.

[1]This paper is an abbreviated and revised version of Matsumoto [13]. Because we aim for concise presentation, we only sketch the proofs. Please see Matsumoto [13] for the details.

2 Model and Main Results

Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_k; k = 0, 1, \dots, T\})$ be a filtered probability space satisfying the usual condition. Here $T \in \boldsymbol{N}$ is a fixed time horizon. We assume that there is the saving account and one illiquid risky asset in the market. The trade of the illiquid asset does not always succeed and we define the trade indicator by

$$\nu(k) = \begin{cases} 1, & \text{if the trade succeeds at } k, \\ 0, & \text{if the trade fails at } k \end{cases}$$

for $k = 0, 1, \dots, T-1$ and set $\nu(T) = 1$ for formulation convenience. $\nu(k)$ is assumed to be $\mathcal{F}_k$-measurable.

For $n = 1, 2, \dots, T+1$, we define the n-th trade time of assets, τ_n by

$$\begin{aligned} \tau_1 &= \inf\{k \geq 0 : \nu(k) = 1\}, \\ \tau_n &= \inf\{k > \tau_{n-1} : \nu(k) = 1\} \wedge T \end{aligned}$$

where the infimum over the empty set is defined to be ∞. τ_n is an increasing sequence of $\mathcal{F}_k$-stopping times. For $k = 0, 1, \dots, T-1$, we define $\tau_{k,1}$ by

$$\tau_{k,1} = \inf\{\tau_n > k : n \geq 1\}.$$

Throughout this paper the saving account is the numeraire. We denote the discounted price process of the risky asset by $X(k)$. We consider a contingent claim whose discounted payoff at the expiration date T is H. Suppose that $X(k)$ is an adapted square-integrable process and H is an $\mathcal{F}_T$-measurable square-integrable random variable. We denote the investor's information by $\mathcal{G}_k$ which is defined by

$$\mathcal{G}_k = \sigma\{X(l); l \leq k\} \vee \sigma\{\nu(l); l < k\} \subset \mathcal{F}_k.$$

We assume that $X(0)$ is constant and then $\mathcal{G}_0$ is trivial. Note that τ_n for $n \geq 1$ is not a $\mathcal{G}_k$-stopping time but $\{\tau_n \geq k\}$ is $\mathcal{G}_k$-measurable.

We consider the investor who wants to hedge the contingent claim H using the risky asset X. The investor plans the number $\pi(k)$ of units of X at $k = 0, \dots, T-1$. We call π a trading strategy. The admissible strategy set $\mathcal{A}$ is defined by

$$\mathcal{A} = \{\pi : \pi \text{ is } \mathcal{G}_k\text{-adapted, } \pi(k) X_{k,1} \nu(k) \text{ is square-integrable for all } 0 \leq k < T\}$$

where

$$X_{k,1} = X(\tau_{k,1}) - X(k).$$

For $\pi \in \mathcal{A}$, we denote by $\theta^\pi(k)$ the realized number of units of X at $k = 0, \dots, T-1$. $\theta^\pi(k)$ satisfies

$$\begin{aligned} \theta^\pi(0) &= \pi(0)\nu(0) + \theta_0(1 - \nu(0)), \\ \theta^\pi(k) &= \pi(k)\nu(k) + \theta^\pi(k-1)(1 - \nu(k)), \quad k \geq 1 \end{aligned} \tag{2.1}$$

where $\theta_0 \in \boldsymbol{R}$ denotes the initial number of units invested in the risky asset.

For any $\theta_0 \in \boldsymbol{R}$ and $\pi \in \mathcal{A}$, the discounted gain process $G^{\pi}_{\theta_0}(k)$ is given by

$$G^{\pi}_{\theta_0}(k) = \sum_{0 \le j \le k-1} \theta^{\pi}(j)(X(j+1) - X(j)), \quad k = 0, 1, \ldots, T.$$

Especially when $k = T$, the gain is decomposed into the initial condition-dependent gain and the strategy-dependent gain as

$$G^{\pi}_{\theta_0}(T) = \theta_0 X_{0,1}(1 - \nu(0)) + G^{\pi}_{0}(T) \tag{2.2}$$

where

$$G^{\pi}_{0}(T) = \sum_{0 \le j \le T-1} \pi(j) X_{j,1} \nu(j). \tag{2.3}$$

The hedging portfolio consists of the initial cost and the gain process. I denote the initial cost by $c \in \boldsymbol{R}$ and then the value of the hedging portfolio is given by $c + G^{\pi}_{\theta_0}(T)$. Our main problems is

$$\inf_{(c,\theta_0,\pi) \in \boldsymbol{R}^2 \times \mathcal{A}} E[(H - c - G^{\pi}_{\theta_0}(T))^2]. \tag{2.4}$$

To prepare for solving the main problem, we consider the following auxiliary problem,

$$\inf_{\pi \in \mathcal{A}} E[(H - c - G^{\pi}_{\theta_0}(T))^2] = \inf_{G \in G^{\mathcal{A}}_{0}(T)} E[(H - c - \theta_0 X_{0,1}(1 - \nu(0)) - G)^2] \tag{2.5}$$

where

$$G^{\mathcal{A}}_{0}(T) = \{G^{\pi}_{0}(T) : \pi \in \mathcal{A}\}.$$

To solve this problem in the discrete time framework, Schweizer [18] shows that the nondegeneracy condition plays an important role and we use a similar assumption.

Assumption 2.1 *There exists a positive constant $C_X \in (0,1)$ such that*

$$E_{\nu(k)}[X_{k,1}|\mathcal{G}_k]^2 \le C_X E_{\nu(k)}[X^2_{k,1}|\mathcal{G}_k], \quad P\text{-a.s. for } k = 0, \ldots, T-1. \tag{2.6}$$

where

$$E_{\nu(k)}[\cdot|\mathcal{G}_k] = \frac{E[\cdot\,\nu(k)|\mathcal{G}_k]}{E[\nu(k)|\mathcal{G}_k]}, \quad \text{if } E[\nu(k)|\mathcal{G}_k] > 0,$$
$$\frac{dP_{\nu(k)}}{dP} = \frac{\nu(k)}{E[\nu(k)]}, \quad \text{if } E[\nu(k)] > 0$$

and $E_{\nu(k)}[\cdot|\mathcal{G}_k] = 0$ if $E[\nu(k)|\mathcal{G}_k] = 0$.

Throughout this paper, we suppose that Assumption 2.1 holds. Roughly speaking, Assumption 2.1 means the risky asset fluctuates sufficiently between trade times. Most of the standard models satisfy this assumption. For example, a standard multinomial model satisfies this assumption.

We define three auxiliary processes by $\beta(T) = \rho(T) = 0$, $Z(T) = 1$ and

$$\beta(k) = \frac{E[X_{k,1}Z(k+1)\nu(k)|\mathcal{G}_k]}{E[X_{k,1}^2 Z(k+1)\nu(k)|\mathcal{G}_k]} = \frac{E_{\nu(k)}[X_{k,1}Z(k+1)|\mathcal{G}_k]}{E_{\nu(k)}[X_{k,1}^2 Z(k+1)|\mathcal{G}_k]}, \tag{2.7}$$

$$\rho(k) = \frac{E[HX_{k,1}Z(k+1)\nu(k)|\mathcal{G}_k]}{E[X_{k,1}^2 Z(k+1)\nu(k)|\mathcal{G}_k]} = \frac{E_{\nu(k)}[HX_{k,1}Z(k+1)|\mathcal{G}_k]}{E_{\nu(k)}[X_{k,1}^2 Z(k+1)|\mathcal{G}_k]}, \tag{2.8}$$

$$Z(k) = \prod_{k \le \tau_l < T} (1 - \beta(\tau_l)X_{\tau_l,1}) \tag{2.9}$$

for $k = 0, \dots, T-1$. We use the conventions that a product over an empty set is 1 and 0/0=0.

Theorem 2.1 *There exists an optimal strategy $\pi^* \in \mathcal{A}$ which attains the optimal value of the auxiliary problem (2.5). For $k = 0, \dots, T-1$, π^* satisfies*

$$\pi^*(k) = \rho(k) - \beta(k)(c + G_{\theta_0}^{\pi^*}(k)). \tag{2.10}$$

Further if $\nu(k) = 1$, the hedging error is given by

$$H - c - G_{\theta_0}^{\pi^*}(T) = H - \sum_{k \le \tau_j < T} \rho(\tau_j)X_{\tau_j,1}Z(\tau_j + 1) - (c + G_{\theta_0}^{\pi^*}(k))Z(k) \tag{2.11}$$

for $k = 0, \dots, T$.

By Theorem 2.1, the expected square hedging error with $(c, \theta_0) = (0, 0)$ is

$$C_H = E[\left(H - \sum_{\tau_1 \le \tau_j < T} \rho(\tau_j)X_{\tau_j,1}Z(\tau_j + 1)\right)^2]. \tag{2.12}$$

Next we go back to the main problem (2.4) and we will find an optimal initial condition which minimize the expected square hedging error. Though π^* depends on (c, θ_0), we use the same notation π^* if there is no risk of confusion. We define a new measure P_Z by

$$\frac{dP_Z}{dP} = \frac{Z(0)}{E[Z(0)]}, \quad \text{if } E[Z(0)] > 0.$$

This is a signed measure. Under P_Z, we denote the expectation, the variance and the covariance by $E_Z[\cdot]$, $Var_Z[\cdot]$ and $Cov_Z[\cdot, \cdot]$, respectively, that are defined in the usual manner. Let

$$c_{\theta_0} = E_Z[H - \theta_0 (X(\tau_1) - X(0))]. \tag{2.13}$$

Theorem 2.2 *Suppose that $E[Z(0)] > 0$. Then $Var_Z[X(\tau_1)]$ is non-negative. If $Var_Z[X(\tau_1)] = 0$, $(c_{\theta_0}, \theta_0, \pi^*)$ is an optimal solution for all $\theta_0 \in \boldsymbol{R}$ and the least expected square hedging error is given by*

$$C_H - E_Z[H]^2 E[Z(0)]. \tag{2.14}$$

If $Var_Z[X(\tau_1)] > 0$, *an optimal solution* (c^*, θ_0^*, π^*) *is given by*

$$\theta_0^* = \frac{Cov_Z[H, X(\tau_1)]}{Var_Z[X(\tau_1)]}, \tag{2.15}$$

$$c^* = c_{\theta_0^*}, \tag{2.16}$$

$$\pi^*(k) = \rho(k) - \beta(k)(c^* + G_{\theta_0^*}^{\pi^*}(k)), \quad k = 0, \dots, T-1 \tag{2.17}$$

and the least expected square hedging error is given by

$$C_H - \left(E_Z[H]^2 + \frac{Cov_Z[H, X(\tau_1)]^2}{Var_Z[X(\tau_1)]} \right) E[Z(0)]. \tag{2.18}$$

Remark 2.1 *Most of the models with uncertainty of trade execution satisfy* $E[Z(0)] > 0$ *and* $Var_Z[X(\tau_1)] > 0$. *In this case, the optimal initial cost can be represented as*

$$c^* = E_Q[H]$$

where $E_Q[\cdot]$ *is the expectation under the signed measure* Q *defined by*

$$\frac{dQ}{dP_Z} = 1 - \frac{E_Z[X(\tau_1) - X(0)](X(\tau_1) - X(0)) - E_Z[X(\tau_1) - X(0)]^2}{Var_Z[X(\tau_1)]}.$$

Remark 2.2 *In our setting the investor plans the trading strategy without knowing the success of trade in advance. But the optimal solution has the same form even if the investor gets more information. For example, the investor's information* $\mathcal{G}_k$ *can be extended to*

$$\mathcal{G}_k = \sigma\{X(l); l \le k\} \vee \sigma\{\nu(l); l \le k\}, \quad k = 0, \dots, T-1.$$

In this case, the investor plans the trading strategy immediately after he knows the success of trade. Since $\nu(k)$ *is* $\mathcal{G}_k$*-adapted, the auxiliary processes* β *and* ρ *for* $k = 0, \dots, T-1$ *are*

$$\beta(k) = \frac{E[X_{k,1}Z(k+1)|\mathcal{G}_k]}{E[X_{k,1}^2 Z(k+1)|\mathcal{G}_k]},$$

$$\rho(k) = \frac{E[HX_{k,1}Z(k+1)|\mathcal{G}_k]}{E[X_{k,1}^2 Z(k+1)|\mathcal{G}_k]}$$

when $\nu(k) = 1$. *And we do not have to consider these variables when* $\nu(k) = 0$ *because it is no use planning* $\pi^*(k)$ *when the investor knows the failure of the trade at time* k. *Using these variables, the optimal solution is given by Theorem 2.2.*

3 Proofs (Sketch)

The increments of X can be decomposed as

$$X_{k,1} = A_{k,1} + M_{k,1}, \quad k = 0, 1, \dots, T-1 \tag{3.1}$$

where

$$A_{k,1} = E_{\nu(k)}[X(\tau_{k,1}) - X(k)|\mathcal{G}_k],$$

$$M_{k,1} = X(\tau_{k,1}) - X(k) - E_{\nu(k)}[X(\tau_{k,1}) - X(k)|\mathcal{G}_k].$$

Directly from Assumption 2.1 we can get the following lemma.

Lemma 3.1 *We have*

$$(1 - C_X)E_{\nu(k)}[X_{k,1}^2|\mathcal{G}_k] \leq E_{\nu(k)}[M_{k,1}^2|\mathcal{G}_k] \leq E_{\nu(k)}[X_{k,1}^2|\mathcal{G}_k], \quad P\text{-a.s.} \tag{3.2}$$

for all $k = 0, \ldots, T-1$.

Proposition 3.1 $G_0^{\mathcal{A}}(T)$ *is a closed subspace in* $\mathcal{L}^2(P)$.

Sketch of Proof of Proposition 3.1. It is clear that $G_0^{\mathcal{A}}(T)$ is a subspace in $\mathcal{L}^2(P)$. We will check that $G_0^{\mathcal{A}}(T)$ is closed. We consider a Cauchy sequence $\{G_n\}$ in $\mathcal{L}^2(P)$ in $G_0^{\mathcal{A}}(T)$. Let G_∞ be the limit of G_n in $\mathcal{L}^2(P)$ and let $\tilde{G}_\infty = G_\infty + \theta_0 X_{0,1}(1 - \nu(0))$. It suffices to show that

$$G_\infty \in G_0^{\mathcal{A}}(T).$$

By (3.1), we have

$$E[(G_{\theta_0}^{\pi}(T))^2] \geq E[\pi(T-1)^2 E[M_{T-1,1}^2 \nu(T-1)|\mathcal{G}_{T-1}]] \tag{3.3}$$

Because G_n is in $G_0^{\mathcal{A}}(T)$, there exists a sequence $\{\pi_n\}$ in $\mathcal{A}$ satisfying $G_0^{\pi_n}(T) = G_n$. Set

$$a_n(T-1) = \pi_n(T-1)\sqrt{E[M_{T-1,1}^2 \nu(T-1)|\mathcal{G}_{T-1}]}.$$

Since

$$G_n - G_m = G_0^{\pi_n}(T) - G_0^{\pi_m}(T) = G_{\theta_0}^{\pi_n}(T) - G_{\theta_0}^{\pi_m}(T) = G_{\theta_0}^{\pi_n - \pi_m}(T),$$

we have by (3.3)

$$E[|a_n(T-1) - a_m(T-1)|^2] \leq E[|G_n - G_m|^2].$$

From Lemma 3.1 and $\pi_n \in \mathcal{A}$, we have

$$E[|a_n(T-1)|^2] \leq E[\pi_n(T-1)^2 E[X_{T-1,1}^2 \nu(T-1)|\mathcal{G}_{T-1}]] \leq E[\pi_n(T-1)^2 X_{T-1,1}^2] < \infty.$$

Therefore $a_n(T-1)$ is a Cauchy sequence of $\mathcal{G}_{T-1}$-measurable random variable in $\mathcal{L}^2(P)$ and then $a_n(T-1)$ is convergent in $\mathcal{L}^2(P)$. We denote the limit of $a_n(T-1)$ by $a_\infty(T-1)$. Since $a_n(T-1)$ is an $\mathcal{G}_{T-1}$-measurable, $a_\infty(T-1)$ is also a $\mathcal{G}_{T-1}$-measurable random variable. Let

$$\pi_\infty(T-1) = \begin{cases} \dfrac{a_\infty(T-1)}{\sqrt{E[M_{T-1,1}^2 \nu(T-1)|\mathcal{G}_{T-1}]}}, & \text{if } E[M_{T-1,1}^2 \nu(T-1)|\mathcal{G}_{T-1}] \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Since we have from Lemma 3.1

$$E[|(\pi_n(T-1) - \pi_\infty(T-1))X_{T-1,1}\nu(T-1)|^2] \leq \frac{E[|a_n(T-1) - a_\infty(T-1)|^2]}{1 - C_X},$$

$\pi_n(T-1)X_{T-1,1}\nu(T-1)$ converges to $\pi_\infty(T-1)X_{T-1,1}\nu(T-1)$ in $\mathcal{L}^2(P)$ as $n \to \infty$.

Arguing in the same way, we can find an $\mathcal{G}_k$-adapted process $\pi_\infty = \{\pi_\infty(k); k = 0, \ldots, T-1\} \in \mathcal{A}$ satisfying

$$\lim_{n\to\infty} \pi_n(0) X_{0,1}\nu(0) = \pi_\infty(0) X_{0,1}\nu(0), \quad \text{in } \mathcal{L}^2(P),$$
$$\theta_0 X_{0,1}(1-\nu(0)) = \tilde{G}_\infty - \sum_{i=0}^{T-1} \pi_\infty(i) X_{i,1}\nu(i).$$

Therefore we have

$$G_\infty = \tilde{G}_\infty - \theta_0 X_{0,1}(1-\nu(0)) = \sum_{i=0}^{T-1} \pi_\infty(i) X_{i,1}\nu(i) = G_0^{\pi_\infty}(T) \in G_0^{\mathcal{A}}(T).$$

The result follows. □

Remark 3.1 *Proposition 3.1 is important because the closedness of $G_0^{\mathcal{A}}(T)$ ensures the existence of a solution of the basic problem (2.5) by the Hilbert space projection theorem. Proposition 3.1 is the same as Theorem 2.1 in Schweizer [18] if $\nu(k) = 1$ a.s. for all $0 \le k \le T-1$.*

We can prove that β, ρ and Z satisfy the following lemmas.

Lemma 3.2 *For all $k = 0, \ldots, T-1$, $\beta(k)$ and $\rho(k)$ are well-defined and satisfy the following properties.*

$$0 \le E[X_{k,1}^2 Z(k+1)^2 \nu(k)] = E[X_{k,1}^2 Z(k+1)\nu(k)] < \infty, \tag{3.4}$$
$$0 \le E[Z(k)^2 \nu(k)|\mathcal{G}_k] = E[Z(k)\nu(k)|\mathcal{G}_k] \le E[\nu(k)|\mathcal{G}_k], \quad P\text{-a.s.}, \tag{3.5}$$
$$E[X_{k,1} Z(k)\nu(k)|\mathcal{G}_k] = 0, \quad P\text{-a.s.}, \tag{3.6}$$
$$0 \le E[Z(k)^2|\mathcal{G}_k] = E[Z(k)|\mathcal{G}_k] \le 1, \quad P\text{-a.s.}. \tag{3.7}$$

Lemma 3.3 *For $k = 0, \ldots, T-1$, we have*

$$E[\rho(k) X_{k,1} Z(k+1)\nu(k)|\mathcal{G}_k] = E[\beta(k) H X_{k,1} Z(k+1)\nu(k)|\mathcal{G}_k] \tag{3.8}$$

Further we have

$$\begin{aligned} &E[\mathbf{1}_{\{l \le \tau_n \le m\}} \rho(\tau_n) X_{\tau_n,1} Z(\tau_n+1)\nu(k)|\mathcal{G}_k] \\ &= E[\mathbf{1}_{\{l \le \tau_n \le m\}} \beta(\tau_n) H X_{\tau_n,1} Z(\tau_n+1)\nu(k)|\mathcal{G}_k] \end{aligned} \tag{3.9}$$

for all $k \le l \le m < T$ and $n = 1, \ldots, T$.

Lemma 3.4 *Let Y be a $\mathcal{G}_k$-adapted process. Then we have*

$$E[Y(\tau_1) Z(0)] = E[Y(\tau_1) Z(0)^2]. \tag{3.10}$$

In particular, if Y is non-negative, $E[Y(\tau_1)Z(0)]$ is non-negative.

Sketch of Proof of Theorem 2.1. From Proposition 3.1, $H - c - \theta_0 X_{0,1}(1-\nu(0))$ can be projected on $G_0^{\mathcal{A}}(T)$. Therefore π^* exists. Further π^* satisfies for all $k = 0, \dots, T-1$

$$E[(H - c - G_{\theta_0}^{\pi^*}(T))X_{k,1}\nu(k)|\mathcal{G}_k] = 0, \quad P\text{-}a.s.. \tag{3.11}$$

We can prove the results for $k = T-1$. We use the backward induction. Suppose that (2.10) and (2.11) hold for $k+1, \dots, T-1$. From (3.11) and the inductive assumption (2.11), we have

$$I_0 - \sum_{i=k+1}^{T}\sum_{j=1}^{T} I_{ij} = 0$$

where

$$\begin{aligned} I_0 &= E[\big(H - (c + G_{\theta_0}^{\pi^*}(\tau_{k,1}))Z(\tau_{k,1})\big)\, X_{k,1}\nu(k)|\mathcal{G}_k], \\ I_{ij} &= E[\mathbf{1}_{\{\tau_{k,1}=i\}}\mathbf{1}_{\{i \le \tau_j \le T-1\}}(X(i) - X(k))\nu(k)\rho(\tau_j)X_{\tau_j,1}Z(\tau_j+1)|\mathcal{G}_k], \quad k+1 \le i. \end{aligned}$$

Recall that $\mathbf{1}_{\{\tau_{k,1}=i\}}$ can be decomposed into $\mathcal{G}_i$-measurable part and the other part as $\mathbf{1}_{\{\tau_{k,1}=i\}} = \mathbf{1}_{\{\tau_{k,1}\ge i\}}\nu(i)$. From Lemma 3.3, we have

$$\sum_{i=k+1}^{T}\sum_{j=1}^{T} I_{ij} = E[\left(\sum_{k<\tau_j<T} \beta(\tau_j)X_{\tau_j,1}Z(\tau_j+1)\right) HX_{k,1}\nu(k)|\mathcal{G}_k]$$

since $\{\tau_j : \tau_{k,1} \le \tau_j \le T-1\} = \{\tau_j : k < \tau_j < T\}$. If $\nu(k) = 1$, we have

$$G_{\theta_0}^{\pi^*}(\tau_{k,1}) = \pi^*(k)X_{k,1} + G_{\theta_0}^{\pi^*}(k), \tag{3.12}$$

and then we get

$$\begin{aligned} I_0 &= E[HX_{k,1}\nu(k)|\mathcal{G}_k] - (c + G_{\theta_0}^{\pi^*}(k))E[Z(k+1)X_{k,1}\nu(k)|\mathcal{G}_k] \\ &\quad -\pi^*(k)E[X_{k,1}^2 Z(k+1)\nu(k)|\mathcal{G}_k]. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} I_0 - \sum_{i=k+1}^{T}\sum_{j=1}^{T} I_{ij} &= E[Z(k+1)HX_{k,1}\nu(k)|\mathcal{G}_k] - (c + G_{\theta_0}^{\pi^*}(k))E[Z(k+1)X_{k,1}\nu(k)|\mathcal{G}_k] \\ &\quad -\pi^*(k)E[X_{k,1}^2 Z(k+1)\nu(k)|\mathcal{G}_k]. \end{aligned}$$

We get (2.10). By the definition of $\tau_{k,1}$, $\tau_{k,1} \ge k+1$ and $\nu(\tau_{k,1}) = 1$. From the inductive assumption (2.11) and (3.12), if $\nu(k) = 1$, we get

$$\begin{aligned} H - c - G_{\theta_0}^{\pi^*}(T) &= H - \left(\sum_{\tau_{k,1}\le\tau_j<T} \rho(\tau_j)X_{\tau_j,1}Z(\tau_j+1) + \rho(k)X_{k,1}Z(\tau_{k,1})\right) \\ &\quad - (c + G_{\theta_0}^{\pi^*}(k))(1 - \beta(k)X_{k,1})\, Z(\tau_{k,1}). \end{aligned}$$

The result follows.

□

Next we will find an optimal initial condition and we prove Theorem 2.2. By Theorem 2.1, we have known how the hedging error is represented and then we can get the following result.

Lemma 3.5 *Let*

$$F_p(c,\theta_0) \quad = \quad E[(H-c-G^{\pi^*}_{\theta_0}(T))^p]$$

for $p=1,2$ *and for all* $(c,\theta_0)\in \boldsymbol{R}^2$. *Then* $F_p(c,\theta_0)$ *satisfies*

$$\begin{aligned} F_1(c,\theta_0) &= E[\left(H-c-G^{\pi^*}_{\theta_0}(\tau_1)\right) Z(0)], && (3.13)\\ F_2(c,\theta_0) &= E[(c+G^{\pi^*}_{\theta_0}(\tau_1))^2 Z(0)] - 2E[H(c+G^{\pi^*}_{\theta_0}(\tau_1))Z(0)] + C_H. && (3.14)\end{aligned}$$

Note that $G^{\pi^*}_{\theta_0}(\tau_1)=\theta_0(X(\tau_1)-X(0))$ *does not depend on* π^*.

Sketch of Proof. First we consider $F_1(c,\theta_0)$. Using (2.11) with $k=\tau_1$, we have

$$F_1(c,\theta_0) \quad = \quad E[H-\sum_{\tau_1\le\tau_j<T}\rho(\tau_j)X_{\tau_j,1}Z(\tau_j+1)-(c+G^{\pi^*}_{\theta_0}(\tau_1))Z(\tau_1)].$$

From Lemma 3.3 we have

$$E[\sum_{\tau_1\le\tau_j<T}\rho(\tau_j)X_{\tau_j,1}Z(\tau_j+1)] \quad = \quad E[H\sum_{\tau_1\le\tau_j<T}\beta(\tau_j)X_{\tau_j,1}Z(\tau_j+1)].$$

Therefore we get

$$\begin{aligned} F_1(c,\theta_0) &= E[H\left(1-\sum_{\tau_1\le\tau_j<T}\beta(\tau_j)X_{\tau_j,1}Z(\tau_j+1)\right)-\left(c+G^{\pi^*}_{\theta_0}(\tau_1)\right)Z(\tau_1)]\\ &= E[\left(H-c-G^{\pi^*}_{\theta_0}(\tau_1)\right)Z(\tau_1)]\end{aligned}$$

since

$$E[H\left(1-\sum_{\tau_1\le\tau_j<T}\beta(\tau_j)X_{\tau_j,1}Z(\tau_j+1)\right)] \quad = \quad E[H\prod_{\tau_1\le\tau_l<T}(1-\beta(\tau_l)X_{\tau_l,1})].$$

From $Z(\tau_1)=Z(0)$, (3.13) follows.

Next we consider $F_2(c,\theta_0)$. Using (2.11) with $k=\tau_1$, we have

$$\begin{aligned} F_2(c,\theta_0) &= C_H+E[\left(c+G^{\pi^*}_{\theta_0}(\tau_1)\right)^2 Z(\tau_1)^2]\\ &\quad -2E[\left(H-\sum_{\tau_1\le\tau_j<T}\rho(\tau_j)X_{\tau_j,1}Z(\tau_j+1)\right)\left(c+G^{\pi^*}_{\theta_0}(\tau_1)\right)Z(\tau_1)].\end{aligned}$$

Note that C_H is defined by (2.12). The second term is

$$E[\left(c+G^{\pi^*}_{\theta_0}(\tau_1)\right)^2 Z(\tau_1)^2]=E[\left(c+G^{\pi^*}_{\theta_0}(\tau_1)\right)^2 Z(0)^2]=E[\left(c+G^{\pi^*}_{\theta_0}(\tau_1)\right)^2 Z(0)]$$

from Lemma 3.4. Therefore it suffices to show that

$$E[\mathbf{1}_{\{\tau_j<T\}}\rho(\tau_j)X_{\tau_j,1}Z(\tau_j+1)\left(c+G^{\pi^*}_{\theta_0}(\tau_1)\right)Z(\tau_1)] \quad = \quad 0$$

for $j = 1, \ldots, T$. For $j = 1$ we have

$$\begin{aligned} & E[\mathbf{1}_{\{\tau_1 < T\}} \rho(\tau_1) X_{\tau_1,1} Z(\tau_1 + 1) \left(c + G_{\theta_0}^{\pi^*}(\tau_1)\right) Z(\tau_1)] \\ = & \sum_{i=0}^{T-1} E[\mathbf{1}_{\{\tau_1 \geq i\}} \rho(i) \left(c + G_{\theta_0}^{\pi^*}(i)\right) E[\nu(i) X_{i,1} Z(\tau_{i,1}) Z(i) | \mathcal{G}_i]]. \end{aligned}$$

From (3.5), we have

$$E[\nu(i) X_{i,1} Z(\tau_{i,1}) Z(i) | \mathcal{G}_i] = E[\nu(i) X_{i,1} (1 - \beta(i) X_{i,1}) Z(\tau_{i,1})^2 | \mathcal{G}_i] = E[\nu(i) X_{i,1} Z(i) | \mathcal{G}_i] = 0.$$

The last equality follows by (3.6). Similarly for $j \geq 2$, we have

$$E[\mathbf{1}_{\{\tau_j < T\}} \rho(\tau_j) X_{\tau_j,1} Z(\tau_j + 1) \left(c + G_{\theta_0}^{\pi^*}(\tau_1)\right) Z(\tau_1)] = 0.$$

The result follows. □

Using Lemma 3.5, we can get the following propositions.

Proposition 3.2 *$E[Z(0)]$ is non-negative. If $E[Z(0)] = 0$, $F_1(c, \theta_0) = 0$ and $F_2(c, \theta_0)$ is constant.*

Proposition 3.3 *Fix $\theta_0 \in R$. Suppose that $E[Z(0)] > 0$. Then $F_2(c, \theta_0)$ is minimized at*

$$c = c_{\theta_0}.$$

Recall that c_{θ_0} is defined by (2.13). Further we have

$$F_1(c_{\theta_0}, \theta_0) = 0.$$

Sketch of Proof of Theorem 2.2. From Lemma 3.4, we have

$$E[(X(\tau_1) - E_Z[X(\tau_1)])^2 Z(0)] = E[(X(\tau_1) - E_Z[X(\tau_1)])^2 Z(0)^2] \geq 0$$

and the equality holds if and only if

$$X(\tau_1) Z(0) = E_Z[X(\tau_1)] Z(0), \quad P\text{-}a.s.. \tag{3.15}$$

Since $E[Z(0)] > 0$, $Var_Z[X(\tau_1)]$ is nonnegative.

By the definition of c_{θ_0}, we have

$$F_2(c_{\theta_0}, \theta_0) = E[Z(0)] f(\theta_0) + C_H$$

where

$$f(\theta_0) = Var_Z[X(\tau_1)] \theta_0^2 - 2Cov_Z[H, X(\tau_1)] \theta_0 - E_Z[H]^2.$$

If $Var_Z[X(\tau_1)] = 0$, we have from (3.15)

$$Cov_Z[H, X(\tau_1)] = 0$$

and then we get

$$F_2(c_{\theta_0}, \theta_0) = C_H - E_Z[H]^2 E[Z(0)]$$

which does not depend on θ_0. If $Var_Z[X(\tau_1)]$ is positive, $f(\theta_0)$ is a quadratic function and then $F_2(c_{\theta_0}, \theta_0)$ has an absolute minimum value (2.18) at $\theta_0 = \theta_0^*$. From Theorem 2.1 and Proposition 3.3, the result follows. □

4 Numerical Example

In this section, we give some numerical examples in a multi-period model.

In the multi-period binomial model, the tree of states braches into 4 at each period as well as the tree in Figure 1. Suppose that $\{(\nu(k), X(k+1)/X(k)); k = 0, 1, \ldots, T-1\}$ is independent and identically distributed and its probability distribution is given by

$$\begin{aligned}
p_1 &= P[\left\{\left(\nu(k), \frac{X(k+1)}{X(k)}\right) = (1, u)\right\} | \mathcal{G}_k], \\
p_2 &= P[\left\{\left(\nu(k), \frac{X(k+1)}{X(k)}\right) = (1, d)\right\} | \mathcal{G}_k], \\
p_3 &= P[\left\{\left(\nu(k), \frac{X(k+1)}{X(k)}\right) = (0, u)\right\} | \mathcal{G}_k], \\
p_4 &= P[\left\{\left(\nu(k), \frac{X(k+1)}{X(k)}\right) = (0, d)\right\} | \mathcal{G}_k], \\
\lambda &= p_1 + p_2
\end{aligned}$$

where p_i $(i = 1, \ldots, 4)$ and λ are positive constants.

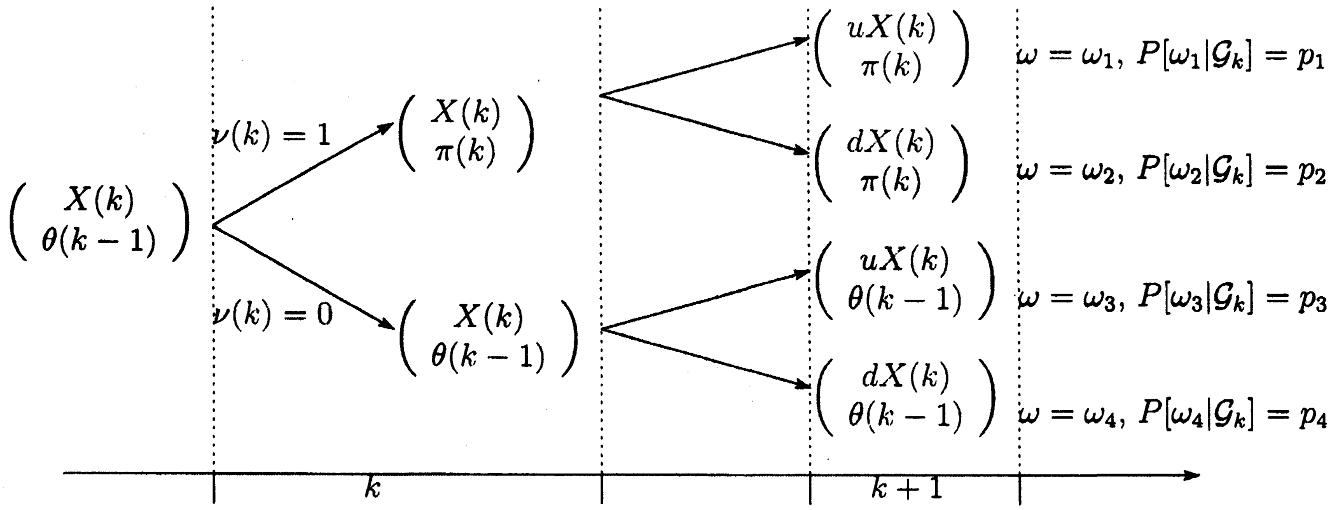

Figure 1: Binomial model with random trade times.

Since λ means the success probability of trade, λ represents the liquidity of the asset. Set

$$H = \max\{X(T) - K, 0\}.$$

The contingent claim is a European call option. Fix

$$T = 10, \quad X(0) = 100, \quad K = 120, \quad u = 1.2, \quad d = \frac{1}{u}.$$

We change λ from 0% to 100% and calculate the optimal solutions under the following two conditions.

1. X and ν are positively-correlated:

$$(p_1, p_2, p_3, p_4) = (0.9\lambda, 0.1\lambda, 0.75(1-\lambda), 0.25(1-\lambda)).$$

2. X and ν are negatively-correlated:

$$(p_1, p_2, p_3, p_4) = (0.6\lambda, 0.4\lambda, 0.75(1-\lambda), 0.25(1-\lambda)).$$

Note that there is no need for calculating $\pi^*(0)$ for $\lambda = 0$ and θ_0^* for $\lambda = 1$.

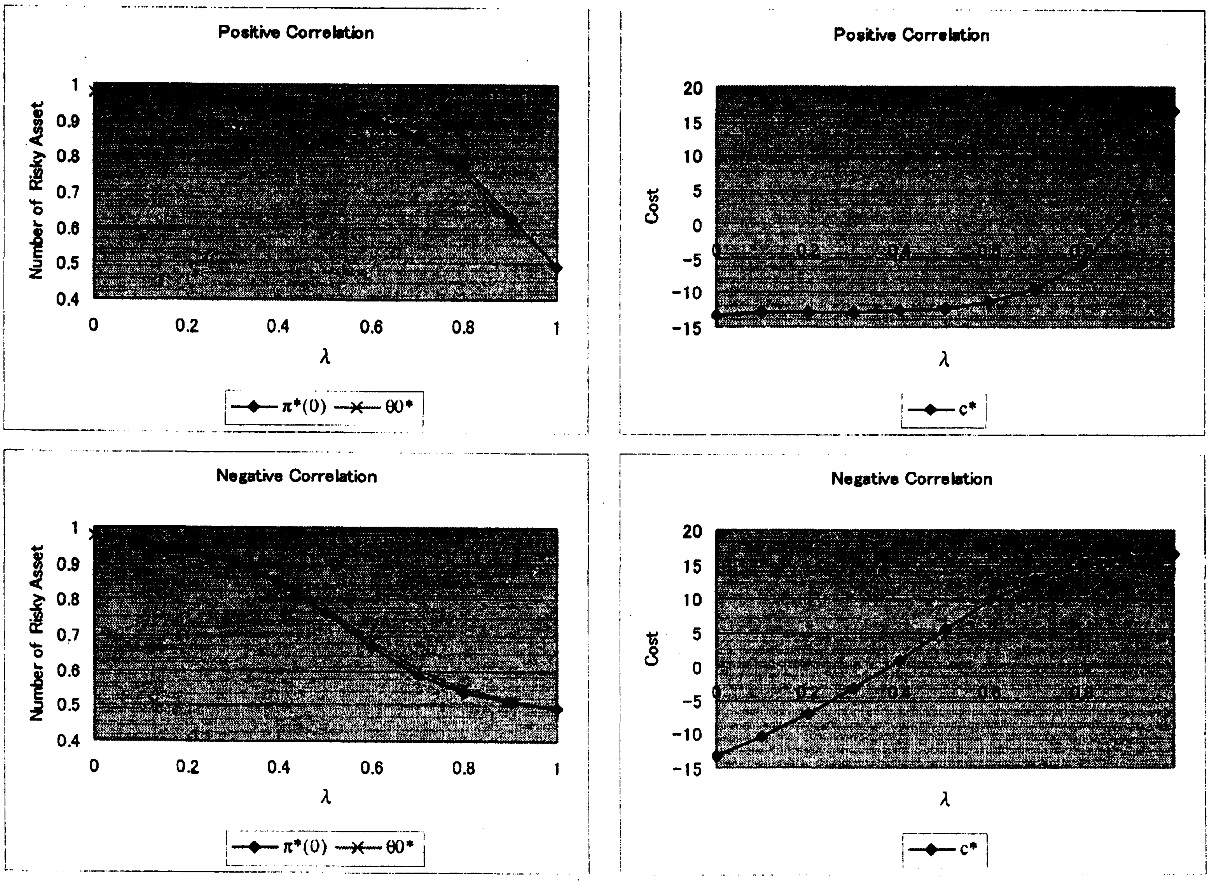

Figure 2: Optimal solution against λ (success probability of trade)

In Figure 2, the optimal solution $(\pi^*(0), \theta_0^*)$ and c^* are plotted against λ. The results are summarized as follows.

- When $\lambda = 1$, $\pi^*(0)$ and c^* correspond to the delta hedging strategy and the arbitrage price, respectively.

- When $\lambda = 0$, θ_0^* and c^* correspond to the mean-variance hedging strategy and the optimal initial cost for a one-period multinomial model. Note that c^* is negative.

- In the positively-correlated case, $\pi^*(0)$ is greater than θ_0^*. In the negatively-correlated case, $\pi^*(0)$ is less than θ_0^*.

- As λ goes to 1, c^* increases to the arbitrage price. In the positively-correlated case, c^* increases slowly. In the negatively-correlated case, c^* increases fast.
- c^* can be negative when λ is small, that is, the asset is illiquid. This example means that c^* should not be called the price.

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