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Premium Calculation and Optimal Intertemporal Risk Diversification *

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Abstract

We present a general approach to the pricing of products in finance and insurance in the multi-period setting. It is a combination of the utility indifference pricing and optimal intertemporal risk allocation. We give a characterization of the optimal intertemporal risk allocation by a first order condition. Applying this result to the exponential utility function, we obtain an essentially new type of premium calculation method for a popular type of multi-period insurance contract. This method is simple and can be easily implemented numerically. We see that the results of numerical calculations are well coincident with the risk loading level determined by traditional and experimental practices. The results also suggest a possible implied utility approach to insurance pricing.

1 Introduction

The insurer of an insurance contract needs to ensure that the premium contains a conservative margin — the so called risk loading or safety loading — to put up the necessary risk capital. When determining this margin in a multi-period insurance contract, the insurer faces two types of risks to evaluate. The first one comes from unfavorable fluctuations in the level of investment funded by accumulated premiums. The second risk comes from the uncertainty of (life) time, i.e., the risk of the unfavorable event occurring at an inopportune time, e.g., before the funding target is reached. It is desirable to determine the margin that reflects both types of risks adequately. However, there seems to be no theoretically established solution to this challenging problem. The main difficulty is in the inseparable nature of the two types of risks themselves; the insurance contract guarantees a defined payment at an uncertain time of the insured event occurring by uncertain funding.

In this paper, toward a solution to the problem above, we present a fairly general approach to the multi-period pricing problem. It is a combination of the *utility indifference pricing* and *optimal intertemporal risk allocation*. Though both are quite general concepts, their combination leads us to an interesting new premium calculation method in a multi-period setting.

The general setting of the utility indifference pricing is as follows: we define the *indifference price* $H(Z)$ of a risk Z by

$$U(w + H(Z) - Z) = U(w), \tag{IP}$$

where $U(W)$ denotes the utility of a risk W and the constant w the initial wealth of the seller of Z. The price $H(Z)$ is the so-called *selling* indifference price: $H(Z)$ is the amount that leaves the seller of the risk Z indifferent between selling and being paid for Z, and neither selling nor being paid for Z. In mathematical finance, the indifference pricing approach has been becoming one of the major pricing methods in incomplete markets (see, e.g., Hodges and Neuberger [21], Rouge and El Karoui [25], Musiela and Zariphopoulou [23], Bielecki et al. [5],

*This paper is an abbreviated version of Fukuda et al. [17]. All proofs are omitted due to the page restriction.

and Møller and Steffensen [24]). The indifference pricing also fits the pricing of insurance well. For example, in the single period, we can show that many premium principles are obtained by this method. The expectation, variance and exponential principles are among them. Thus, the utility indifference pricing approach has the potential advantage of pricing products in finance and insurance coherently.

We write $\mathcal{A}(W)$ for the class of admissible intertemporal risk allocations $(Y_t)_{t\in\mathbb{T}}$ of W over the time axis $\mathbb{T} := \{1, 2, \ldots, T\}$ (see Definition 2.1 below): $(Y_t)_{t\in\mathbb{T}}$ is an essentially bounded adapted process satisfying

$$\sum\nolimits_{t\in\mathbb{T}} \tilde{Y}_t = W \quad \text{a.s.}, \tag{IRA}$$

where $\tilde{Y}_t$ denotes the discounted value of Y_t. In this paper, we adopt the following utility $U(\cdot)$ in (IP):

$$U(W) := \sup\left\{\sum\nolimits_{t\in\mathbb{T}} E[u_t(\tilde{Y}_t)] : (Y_t)_{t\in\mathbb{T}} \in \mathcal{A}(W)\right\}, \tag{U}$$

where $u_t(x)$ is a time-dependent utility function describing the intertemporal preferences of an economic agent such as an insurance company. This definition says that if an allocation $(X_t) \in \mathcal{A}(W)$ attains the supremum in (U), then the utility of W is based on the choice of (X_t). Thus, to precisely investigate $U(\cdot)$, whence $H(\cdot)$, we are led to the problem of finding $(X_t) \in \mathcal{A}(W)$ that attains the supremum in (U), which we call the *optimal intertemporal risk allocation* of W.

The optimal risk allocation problems date back to the classical work of Arrow [2] and Borch [6, 7], where Pareto optimality and equilibria in uncertain circumstances are studied extensively, motivated mainly by insurance and reinsurance. Since then, various types optimal risk allocation problems have been considered by Bühlmann [8], Gerber [18], and many others. See also Gerber and Pafumi [19], Duffie [14], Dana and Jeanblanc [11] and Dana and Scarsini [12]. Recently, many authors consider the problems based on preferences defined by coherent or convex risk measures introduced by Artzner et al. [3], Delbaen [14], and Föllmer and Schied [15] (see also [16]). See Heath and Ku [20], Barrieu and El Karoui [4], Burgert and Rüschendorf [9], Acciao [1], and Jouini et al. [22].

Unlike most of these references where the problems of optimal risk allocation among several economic agents are discussed, we consider a single agent in the multi-period setting who seeks to find the optimal intertemporal allocation of her/his risk. As the definition itself suggests, this optimality is closely related to Pareto optimality. It should be noticed that, however, classical Pareto optimality is concerned with allocations of risk among economic agents in a single period, while the Pareto optimality we consider in this paper is concerned with intertemporal allocations of the aggregate risk of a single agent, whence it may be called *time* Pareto optimality.

Our key finding about the optimal intertemporal risk allocation is that an allocation $(X_t) \in \mathcal{A}(W)$ is optimal if and only if the following first order condition is satisfied:

$$(u_t'(\tilde{X}_t))_{t\in\mathbb{T}} \text{ is a martingale}, \tag{FOC}$$

where $u_t'(x) := (du_t/dx)(x)$. It is perhaps interesting that this first order condition involves a martingale property. By applying this characterization to the exponential utility, we can derive an algorithm to compute the optimal intertemporal risk allocation and indifference price $H(\cdot)$. To illustrate the usefulness of this algorithm, we apply it to a popular type of multi-period insurance contract, whereby obtain an essentially new type of premium calculation method in the multi-period setting. This method is simple and can be easily implemented numerically. We see that the results of numerical calculations are well coincident with the risk loading level determined by traditional and experimental practices. The results also suggest a possible *implied utility approach* to insurance pricing.

In §2, we give some results on the optimal intertemporal risk allocation, including its characterization by (FOC) and its relationship to Pareto optimality. In §3, we apply the results in §3 to the exponential utility function and derive the optimal intertemporal risk allocation and the indifference price for it. Section 4 is devoted to the applications of the results in §3 to insurance pricing. We also give some numerical examples.

2 Optimal intertemporal risk allocation

We work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}\cup\{0\}}, P)$ with $\mathcal{F}_T = \mathcal{F}$, where $\mathbb{T} := \{1, 2, \ldots, T\}$. Throughout the paper, we write $L^\infty := L^\infty(\Omega, \mathcal{F}_T, P)$ for the space of all essentially bounded, real-valued $\mathcal{F}_T$-measurable random variables. Let $(r_t)_{t\in\mathbb{T}}$ be a spot rate process. We assume that the process (r_t) is bounded, nonnegative and predictable, i.e., r_t is bounded, nonnegative and $\mathcal{F}_{t-1}$-measurable for all $t = 1, \ldots, T$. Let B_t be the price of the riskless bond:

$$B_0 = 1, \qquad B_t = \prod_{k=1}^{t}(1 + r_k) \quad \text{for } t = 1, \ldots, T.$$

Throughout the paper, we use (B_t) as the numéraire, and for each price process (X_t), we denote by $(\tilde{X}_t)$ its discounted price process:

$$\tilde{X}_t := X_t/B_t, \qquad t \in \mathbb{T}.$$

2.1 Definition

We consider an economic agent such as an insurance company who wishes to allocate some aggregate risk W over the time axis $\mathbb{T}$. In the next definition, we define the collection of all such possible intertemporal allocations of W.

Definition 2.1. For $W \in L^\infty$, we write $\mathcal{A}(W)$ for the set of *admissible intertemporal allocations* $(Y_t)_{t\in\mathbb{T}}$ of W: (Y_t) is an $(\mathcal{F}_t)$-adapted process satisfying $Y_t \in L^\infty$ for $t \in \mathbb{T}$ and (IRA).

Example 2.2. We consider the aggregate risk W of a life insurance contract with duration T in which the insured receives c_t dollars at time t if she/he dies in the period $(t-1, t]$. Then, we have $W = \sum_{t\in\mathbb{T}} \tilde{Y}_t$ with $Y_t := c(t)1_{(t-1<\tau\le t)}$, where τ is a stopping time representing the lifetime of the insured. Notice that $(Y_t)_{t\in\mathbb{T}}$ itself is an admissible intertemporal allocation of W or $(Y_t) \in \mathcal{A}(W)$. If we define (X_t) by

$$X_t = \begin{cases} 0, & t = 1, \\ (1 + r_t)Y_{t-1}, & t = 2, \ldots, T-1, \\ Y_T + (1 + r_T)Y_{T-1}, & t = T, \end{cases}$$

then (X_t) is also in $\mathcal{A}(W)$. Insurance companies which have many contracts are able to regard W as the aggregate risk of (X_t), rather than that of (Y_t), without cost.

We assume that the intertemporal preferences of the agent is described by the time-dependent utility function $u_t(x)$. This means that a rational choice of the agent's allocation $(Y_t) \in \mathcal{A}(W)$ is based on the integrated expected utility $\sum_{t\in\mathbb{T}} E[u_t(\tilde{Y}_t)]$. Throughout this section, we assume that the utility function $u_t(x)$ satisfies the following condition:

$$\begin{cases} \text{for } t \in \mathbb{T},\ \mathbb{R} \ni x \mapsto u_t(x) \in \mathbb{R} \text{ is a strictly concave,} \\ C^1\text{-class function such that } u_t'(x) := (du_t/dx)(x) > 0 \text{ for } x \in \mathbb{R}. \end{cases} \tag{2.1}$$

We define the utility $U(W) \in \mathbb{R} \cup \{+\infty\}$ of the risk $W \in L^\infty$ by (U).

Definition 2.3. An intertemporal allocation $(X_t)_{t\in\mathbb{T}} \in \mathcal{A}(W)$ of the risk $W \in L^\infty$ is called *optimal* if it attains the supremum in (U).

Proposition 2.4. *The optimal intertemporal allocation $(X_t)_{t\in\mathbb{T}} \in \mathcal{A}(W)$ of $W \in L^\infty$ is unique if it exists.*

2.2 Indifference pricing

In this section, we assume that $U(W) < \infty$ for all $W \in L^\infty$. This condition holds, for example, if $u_t(x)$ is bounded from above. This also holds if the optimal intertemporal risk allocation exists for all $W \in L^\infty$. We thus have the utility functional $U : L^\infty \to \mathbb{R}$. We write $w \in \mathbb{R}$ for the initial wealth of the agent.

Proposition 2.5. *The functional U has the following properties for $W, Z \in L^\infty$.*

(a) *Strict Monotonicity: If $W \geq Z$ a.s. and $P(W > Z) > 0$, then $U(W) > U(Z)$.*

(b) *Concavity: If $a \in [0,1]$, then $U(aW + (1-a)Z) \geq aU(W) + (1-a)U(Z)$.*

From Proposition 2.5, we see that for $Z \in L^\infty$, the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) := U(w + x - Z)$ is concave (whence continuous) and strictly increasing. Moreover, since Z is bounded, we have $U(w + x - Z) < U(w)$ for x small enough and $U(w + x - Z) > U(w)$ for x large enough. We are thus led to the following definition.

Definition 2.6. We define the *indifference price* $H(Z) = H(Z; w) \in \mathbb{R}$ of $Z \in L^\infty$ by $U(w + H(Z) - Z) = U(w)$.

From Proposition 2.5, we immediately obtain the next proposition.

Proposition 2.7. *The indifference price functional $H : L^\infty \to \mathbb{R}$ has the following properties for $W, Z \in L^\infty$.*

(a) *Strict Monotonicity: If $W \geq Z$ a.s. and $P(W > Z) > 0$, then $H(W) > H(Z)$.*

(b) *Convexity: If $a \in [0,1]$, then $H(aW + (1-a)Z) \leq aH(W) + (1-a)H(Z)$.*

2.3 Characterization by the first order condition

It should be noticed that, in general, the optimal intertemporal risk allocation may not exist. However, to precisely investigate the utility $U(\cdot)$, whence the indifference price $H(\cdot)$, it seems indispensable to find and describe the optimal intertemporal risk allocation. In this section, we give a necessary and sufficient first order condition for $(X_t) \in \mathcal{A}(W)$ to be optimal. This characterization plays a key role in the next section. In the proof below, and throughout the paper, we write

$$E_t[Y] := E[Y|\mathcal{F}_t], \qquad Y \in L^1(\Omega, \mathcal{F}, P),\ t \in \mathbb{T}.$$

Here is the characterization of the optimal intertemporal risk allocation.

Theorem 2.8. *For $W \in L^\infty$ and $(X_t) \in \mathcal{A}(W)$, the following conditions are equivalent:*

(a) *The intertemporal allocation (X_t) of W is optimal.*

(b) *The first order condition* (FOP) *is satisfied.*

Remark 2.9. We clearly find similarity between the theorem above and Borch's theorem which characterizes (classical) Pareto optimality by a first order condition (see Borch [6]; see also Bühlmann [8] and Gerber and Pafumi [19]).

2.4 Pareto optima

In this section, we introduce Pareto optimality of intertemporal risk allocations. It is closely related with the optimality introduced in the previous section.

Definition 2.10. For $W \in L^\infty$, the allocation $(X_t)_{t\in\mathbb{T}} \in \mathcal{A}(W)$ is *Pareto optimal* if there does not exist $(Y_t)_{t\in\mathbb{T}} \in \mathcal{A}(W)$ satisfying

$$E[u_t(\tilde{Y}_t)] \geq E[u_t(\tilde{X}_t)] \quad \text{for all } t \in \mathbb{T}$$

and

$$E[u_{t_0}(\tilde{Y}_{t_0})] > E[u_{t_0}(\tilde{X}_{t_0})] \quad \text{for at least one } t_0 \in \mathbb{T}.$$

For $\lambda = (\lambda_1, \ldots, \lambda_T) \in \mathbb{R}^T_+ \setminus \{0\}$, we consider the problem

$$\text{maximize} \quad \sum_{t\in\mathbb{T}} \lambda_t E[u_t(\tilde{X}_t)] \quad \text{subject to} \quad (X_t)_{t\in\mathbb{T}} \in \mathcal{A}(W). \tag{P_λ}$$

Lemma 2.11. *Let $\lambda = (\lambda_1, \ldots, \lambda_T) \in \mathbb{R}^T_+ \setminus \{0\}$.*

(a) *If $(X_t) \in \mathcal{A}(W)$ is the solution to Problem P_λ, then $(\lambda_t u_t'(\tilde{X}_t))_{t\in\mathbb{T}}$ is a martingale.*

(b) *If Problem P_λ has a solution, then $\lambda \in (0,\infty)^T$.*

Proposition 2.12. *Let $\lambda \in (0,\infty)^T$. If a solution $(X_t)_{t\in\mathbb{T}} \in \mathcal{A}(W)$ to Problem P_λ exists, then it is unique.*

The proof is almost the same as that of Proposition 2.4, whence we omit it.
The next theorem is an analogue of the *second fundamental theorem of welfare economics.*

Theorem 2.13. *For $(X_t)_{t\in\mathbb{T}} \in \mathcal{A}(W)$, the following conditions are equivalent:*

(a) *X is Pareto optimal.*

(b) *There exists $\lambda \in (0,\infty)^T$ such that X is the solution to Problem P_λ.*

By Theorem 2.13, we see that the set of Pareto optimal intertemporal risk allocations in $\mathcal{A}(W)$ is parametrized by the $T-1$ parameters $(\lambda_2/\lambda_1, \ldots, \lambda_T/\lambda_1) \in (0,\infty)^{T-1}$. We also see that the Pareto optimal allocation $(X_t) \in \mathcal{A}(W)$ corresponding to Problem (P_λ) with $\lambda = (\lambda_1, \ldots, \lambda_T)$ is optimal with respect to the intertemporal preferences described by the utility function $v_t(x) := \lambda_t u_t(x)$.

3 Exponential utility

Let $(r_t)_{t=1}^T$ and $(B_t)_{t\in\mathbb{T}}$ be as in Section 2. In this section, we adopt the following time-dependent exponential utility function:

$$\begin{cases} u_t(x) = \dfrac{1}{\alpha_t}\left[1 - \exp\left(-\alpha_t x\right)\right], & t \in \mathbb{T},\ x \in \mathbb{R} \\ \text{with } \alpha := (\alpha_1, \ldots, \alpha_t) \in (0,\infty)^T. \end{cases} \tag{EU}$$

In what follows, we may also write $\alpha(t) = \alpha_t$. We have

$$u_t'(x) = \exp\left(-\alpha_t x\right), \qquad u_t(0) = 0, \qquad u_t'(0) = 1. \tag{3.1}$$

3.1 The optimal intertemporal risk allocation for exponential utility

In this section, we describe the optimal intertemporal risk allocation for the exponential utility function $u_t(x)$ in (EU). Thus, the problem that we consider here is

$$\text{maximize} \quad \sum_{t\in\mathbb{T}} E[u_t(\tilde{X}_t)] \quad \text{subject to} \quad (X_t)_{t\in\mathbb{T}} \in \mathcal{A}(W). \tag{EP}$$

To derive the optimal allocation $(X_t)_{t\in\mathbb{T}} \in \mathcal{A}(W)$ or the solution to (EP), we consider the transform

$$M_t = \exp(-\alpha_t \tilde{X}_t), \qquad t \in \mathbb{T}.$$

Then, by Theorem 2.8, Problem (EP) reduces to

Problem M. For $W \in L^\infty$ and $\alpha = (\alpha_1, \dots, \alpha_T) \in (0,\infty)^T$, derive a positive martingale $(M_t)_{t\in\mathbb{T}}$ satisfying

$$\prod_{t\in\mathbb{T}} M_t^{1/\alpha(t)} = \exp(-W) \qquad \text{a.s.} \tag{3.2}$$

For $W \in L^\infty$ and $\alpha = (\alpha_1, \dots, \alpha_T) \in (0,\infty)^T$, we define the adapted process $(L_t(\alpha, W))_{t\in\mathbb{T}}$ by the following backward iteration:

$$\begin{cases} L_T(\alpha, W) := \exp(-\alpha_t W), \\ L_{t-1}(\alpha, W) := E_{t-1}[L_t(\alpha, W)]^{\beta(t-1)/\beta(t)}, \quad t = 2, \dots, T, \end{cases} \tag{L1}$$

where $E_t[Y] := E[Y|\mathcal{F}_t]$ as before, and we define β_t, or $\beta(t)$, in $(0,\infty)$ by

$$\frac{1}{\beta_t} = \sum_{k=t}^{T} \frac{1}{\alpha_k}, \qquad t \in \mathbb{T}. \tag{β}$$

Notice that for all $t \in \mathbb{T}$, $L_t(\alpha, W)$ is bounded away from 0 and ∞. We also define the adapted process $(M_t(\alpha, W))_{t\in\mathbb{T}}$ by

$$\begin{cases} M_t(\alpha, W) = L_t(\alpha, W) \cdot \prod_{k=1}^{t-1} L_k(\alpha, W)^{-\beta(k+1)/\alpha(k)}, \quad t = 2, \dots, T, \\ M_1(\alpha, W) = L_1(\alpha, W). \end{cases} \tag{M}$$

Here is the solution to the martingale problem M above.

Theorem 3.1. *For $W \in L^\infty$ and $\alpha = (\alpha_1, \dots, \alpha_T) \in (0,\infty)^T$, the solution $(M_t)_{t\in\mathbb{T}}$ to Problem M is unique and given by $M_t = M_t(\alpha, W)$ for $t \in \mathbb{T}$.*

Theorem 3.2. *The optimal intertemporal allocation $(X_t)_{t\in\mathbb{T}} \in \mathcal{A}(W)$ of $W \in L^\infty$ for the exponential utility function $u_t(x)$ in* (EU) *is unique and given by*

$$\exp(-\alpha_t \tilde{X}_t) = M_t(\alpha, W), \qquad t \in \mathbb{T}. \tag{3.3}$$

Proof. The theorem follows from Theorems 2.8 and 3.1. □

We need the next proposition later.

Proposition 3.3. *Let $x \in \mathbb{R}$, $W \in L^\infty$ and $\alpha = (\alpha_1, \dots, \alpha_T) \in (0,\infty)^T$. Then, the following assertions hold:*

(a) $L_t(\alpha, x) = \exp(-\beta_t x)$ *for* $t \in \mathbb{T}$.

(b) $L_t(\alpha, x - Z) = \exp(-\beta_t x) L_t(\alpha, -Z)$ *for* $t \in \mathbb{T}$.

3.2 The indifference prices for the exponential utility

In this section, we derive the indifference prices for the exponential utility $u_t(x)$ in (EU). Let $U, H : L^\infty \to \mathbb{R}$ be the utility and indifference price functionals for $u_t(x)$, respectively. Recall β_t, $L_t(\alpha, Z)$ and $M_t(\alpha, Z)$ from Section 3.1.

For the exponential utility, the next theorem reduces the computation of the indifference price $H(Z)$ to that of $L_1(\alpha, -Z)$.

Theorem 3.4. *We assume* (EU). *Then, for $x \in \mathbb{R}$ and $Z \in L^\infty$, the following assertions hold:*

(a) $U(Z) = \frac{1}{\beta_1}\{1 - E[L_1(\alpha, Z)]\}$.

(b) $U(x - Z) = \frac{1}{\beta_1}\{1 - \exp(-\beta_1 x) \cdot E[L_1(\alpha, -Z)]\}$.

(c) $H(Z) = \frac{1}{\beta_1} \log E[L_1(\alpha, -Z)]$.

From Theorem 3.4 (c), we see that the indifference price $H(Z)$ does not depend on the level w of the initial wealth.

The next proposition describes the optimal intertemporal allocation of the selling position $w + H(Z) - Z$ for the exponential utility.

Proposition 3.5. *We assume* (EU). *For $x \in \mathbb{R}$ and $Z \in L^\infty$, let $(X_t) \in \mathcal{A}(x - Z)$ be the optimal intertemporal allocation of $x - Z$: $\sum_{t\in\mathbb{T}} E[u_t(\tilde{X}_t)] = U(x - Z)$. Then, $(X_t)_{t\in\mathbb{T}}$ is given by*

$$X_1 = \frac{B_1}{\alpha_1}\left[\beta_1 x - \log L_1(\alpha, -Z)\right],$$
$$X_t = \frac{B_t}{\alpha_t}\left[\beta_1 x - \log L_t(\alpha, -Z) + \sum_{k=1}^{t-1} \frac{\beta_{k+1}}{\alpha_k} \log L_k(\alpha, -Z)\right], \quad t = 2, \ldots, T.$$

4 Insurance pricing

In this section, we apply the optimal-intertemporal-risk-allocation approach to the computation of insurance premiums.

4.1 Life insurance contract

We consider a life insurance contract with duration T in which the insurer pays the insured c_t dollars at time $t \in \mathbb{T}$ if the insured dies in the interval $(t-1, t]$. Here c_t's are deterministic. The insured pays the insurer a one-time premium at time $t = 0$.

We denote by τ the future life time of the insured, i.e., she/he dies at time τ. We assume that τ is a random variable on $(\Omega, \mathcal{F}, P)$ satisfying $\tau(\omega) > 0$ for all $\omega \in \Omega$ and $P(\tau = t) = 0$ for all $t \in [0, \infty)$.

If the insured pays the insurer H dollars as one time premium at time $t = 0$, then the present value of the cashflow of the insurer is given by $H - Z$ with

$$Z = \sum_{t\in\mathbb{T}} \tilde{c}_t 1_{(t-1<\tau\le t)}$$

and

$$\tilde{c}_t := c_t/B_t, \qquad t \in \mathbb{T}.$$

In the classical pricing, the premium $H_0(Z)$ based on the *principle of equivalence* is often used: $H_0(Z)$ is defined by $E[H_0(Z) - Z] = 0$ or $H_0(Z) = E[Z]$. If the interest rates are deterministic, $H_0(Z)$ is given by

$$H_0(Z) = \sum_{t \in \mathbb{T}} \tilde{c}_t P(t-1 < \tau \leq t)$$

or, in the actuarial notation,

$$H_0(Z) = \sum_{t \in \mathbb{T}} \tilde{c}_t \cdot {}_{t-1|}q_x,$$

where x is the age of the insured at time $t = 0$. It should be noticed that this price lacks the *safety loading* if the real mortality table is used. Usually, insurance companies use modified mortality tables to ensure the necessary safety loading.

We define a discrete-time process $(D_t)_{t=0}^T$ by

$$D_t := 1_{(\tau \leq t)}, \qquad t = 0, 1, \ldots, T.$$

Then, $(D_t)_{t=0}^T$ is a $\{0,1\}$-valued nondecreasing process satisfying $D_0 = 0$. Notice that for $t \in \mathbb{T}$, $D_t = 0$ (resp., $D_t = 1$) if and only if the insurer is alive (resp., dead) at time t. We denote by $(\mathcal{H}_t)_{t=0}^T$ the filtration associated with the process $(D_t)_{t=0}^T$:

$$\mathcal{H}_t := \sigma(D_s : 0 \leq s \leq t), \qquad t = 0, 1, \ldots, T. \tag{4.1}$$

We consider the following conditional probabilities:

$$\begin{aligned} q_t &:= P(\tau \leq t+1 \mid \tau > t), \qquad t = 0, \ldots, T-1, \\ p_t &:= 1 - q_t = P(\tau > t+1 \mid \tau > t), \qquad t = 0, \ldots, T-1. \end{aligned}$$

In the actuarial notation, for $t = 0, \ldots, T-1$,

$$q_t = {}_1q_{x+t}, \qquad p_t = {}_1p_{x+t}.$$

We have the following equalities:

$$q_t + p_t = 1 \quad (t = 0, \ldots, T-1), \quad q_0 = P(\tau \leq 1), \quad p_0 = P(1 < \tau).$$

We use the following well-known result.

Lemma 4.1. *The following assertions hold:*

(a) $E[D_t | \mathcal{H}_{t-1}] = D_{t-1} + (1 - D_{t-1})q_{t-1}$ *for* $t \in \mathbb{T}$.

(b) $E[(1 - D_t) | \mathcal{H}_{t-1}] = (1 - D_{t-1})p_{t-1}$ *for* $t \in \mathbb{T}$.

4.2 Algorithm for the premium computation

The aim of this section is to derive an algorithm to compute the indifference premium of the life insurance contract. To this end, in addition to (EU), we assume the following conditions:

The interest rate process $(r_t)_{t=1}^T$ is deterministic. (R1)

The filtration $(\mathcal{F}_t)_{t=0}^T$ is given by $(\mathcal{H}_t)$ in (4.1): $\mathcal{F}_t = \mathcal{H}_t$ for $t = 0, \ldots, T$. (F1)

The condition (R1) implies that the riskless bond price process $(B_t)_{t \in \mathbb{T}}$ is also deterministic.

The σ-algebra $\mathcal{F}_T$ is generated by the following decomposition of Ω:

$$\Omega = (0 < \tau \leq 1) \cup (1 < \tau \leq 2) \cup \cdots \cup (T-1 < \tau \leq T) \cup (T < \tau).$$

Hence, if $Z \in L^\infty(\Omega, \mathcal{F}_T, P)$, then Z has the decomposition of the form

$$Z = \sum_{t=1}^{T} z_t 1_{(t-1<\tau\leq t)} + z_{T+1} 1_{(T<\tau)} \tag{Z}$$

with some real deterministic coefficients z_t, $t = 1, \ldots, T+1$. In the life insurance contract considered in the previous section, we have

$$z_t = \begin{cases} \tilde{c}_t, & t = 1, \ldots, T, \\ 0, & t = T+1. \end{cases}$$

Recall β_t from (β). For $Z \in L^\infty$ with representation (Z), we define the real deterministic sequence $(h_t)_{t=0}^T$ by the following backward iteration:

$$\begin{cases} h_t = e^{z(T+1)}, \\ h_{t-1} = \left[e^{\beta(t) z(t)} q_{t-1} + h_t^{\beta(t)} p_{t-1} \right]^{1/\beta(t)}, \quad t = 1 \ldots, T. \end{cases} \tag{h}$$

Recall the definition of the process $(L_t(\alpha, -Z))_{t=0}^T$ from Section 2.

Proposition 4.2. *We assume* (EU), (R1) *and* (F1). *Then, for $Z \in L^\infty$ with* (Z), *the process $(L_t(\alpha, -Z))_{t=0}^T$ is given by*

$$\begin{cases} L_1(\alpha, -Z) = e^{\beta(1) z(1)} D_1 + h_1^{\beta(1)} (1 - D_1), \\ L_t(\alpha, -Z) = \exp\left[\beta_t \sum_{s=1}^{t-1} \{z_s - z_{s+1}\} D_s \right] \\ \qquad\qquad \times \left[e^{\beta(t) z(t)} D_t + h_t^{\beta(t)} (1 - D_t) \right], \quad t = 2, \ldots, T. \end{cases} \tag{L2}$$

We are ready to give the algorithms to compute the indifference premium $H(Z)$ and corresponding optimal allocation of the selling position $w + H(Z) - Z$. We see that the computations are reduced to those of h_t, $t = 0, \ldots, T$, in (h).

Theorem 4.3. *We assume* (EU), (R1) *and* (F1). *Let $Z \in L^\infty$ with representation* (Z). *Then, the following assertions hold.*

(a) *The indifference price $H(Z)$ is given by $H(Z) = \log h_0$.*

(b) *Let $(X_t) \in \mathcal{A}(w + H(Z) - Z)$ be the optimal intertemporal allocation of $w + H(Z) - Z$: $\sum_{t \in \mathbb{T}} E[u_t(\tilde{X}_t)] = U(w + H(Z) - Z) = U(w)$. Then, $(X_t)_{t \in \mathbb{T}}$ is given by*

$$X_1 = \frac{B_1}{\alpha_1} \left[\beta_1 (w + H(Z)) - \beta_1 z_1 \cdot 1_{(0<\tau\leq 1)} - \beta_1 \log h_1 \cdot 1_{(1<\tau)} \right],$$

$$X_t = \frac{B_t}{\alpha_t} \left[\beta_1 (w + H(Z)) - \sum_{k=1}^{t} \beta_k z_k \cdot 1_{(k-1<\tau\leq k)} - \beta_t \log h_t \cdot 1_{(t<\tau)} \right.$$
$$\left. + \sum_{k=1}^{t-1} \frac{\beta_{k+1}}{\alpha_k} \beta_k \log h_k \cdot 1_{(k<\tau)} \right], \qquad t = 2, \ldots, T.$$

4.3 Dependence on the risk aversion coefficients

As in the previous section, we assume (EU), (R1) and (F1). The aim of this section is to investigate the dependence of the indifference price $H(Z)$ on the absolute risk aversion coefficient set $\alpha = (\alpha_1, \dots, \alpha_t) \in (0,\infty)^T$. To emphasize the dependence on α, we write $u_t(x;\alpha)$, $U_\alpha(Z)$, $H_\alpha(Z)$ and $h_t(\alpha)$ for the exponential utility function $u_t(x)$, utility $U(Z)$, indifference price $H(Z)$ and h_t in (h), respectively. In what follows, $\alpha \to 0+$ (resp., $\alpha \to \infty$) means that $\alpha_t \to +0$ (resp., $\alpha_t \to \infty$) for all $t \in \mathbb{T}$.

To study the asymptotic behavior of $H_\alpha(Z)$ as $\alpha \to 0+$, we need the next lemma.

Lemma 4.4. *For $z \in \mathbb{R}$, $q \in [0,1]$, and $g : (0,\infty) \to (0,\infty)$ with limit $c := \lim_{x\to 0+} \log g(x) \in \mathbb{R}$, we define $f(x) := [qe^{zx} + (1-q)g(x)^x]^{1/x}$ for $x > 0$. Then,*

$$\lim_{x\to 0+} \log f(x) = qz + (1-q)c.$$

For $Z \in L^\infty$ with representation (Z), we have

$$E[Z] = \sum_{t=1}^{T} z_t P(t-1 < \tau \le t) + z_{T+1} P(T < \tau).$$

We define $H_\infty(Z)$ by

$$H_\infty(Z) := \max\{z_1, \dots, z_{T+1}\}.$$

We can view $E[Z]$ (resp., $H_\infty(Z)$) as a lower (resp., upper) bound for any reasonable price of Z. From the next theorem, we see that $H_\alpha(Z)$ takes any value in $(E[Z], H_\infty(Z))$ by a suitable choice of $\alpha \in (0,\infty)^T$.

Theorem 4.5. *We assume* (EU), (R1) *and* (F1). *We also assume $0 < q_t < 1$ for all $t = 0, \dots, T-1$. Then, for $Z \in L^\infty$, the following assertions hold:*

(a) $E[Z] \le H_\alpha(Z) \le H_\infty(Z)$ *for all* $\alpha \in (0,\infty)^T$.

(b) $\lim_{\alpha\to 0+} H_\alpha(Z) = E[Z]$.

(c) $\lim_{\alpha\to\infty} H_\alpha(Z) = H_\infty(Z)$.

(d) $(E[Z], H_\infty(Z)) \subset \{H_\alpha(Z) : \alpha \in (0,\infty)^T\}$.

4.4 Numerical examples

We compare the indifference pricing method in Theorem 4.3 with traditional ones by pricing the following insurance contract:

- Type of insurance: term mortality insurance.
- Age at issue: 30 years old.
- Sex: male.
- Term of contract: from 1 year to 30 years.
- Loading of premium: excluded.
- Mortality rate: Standard Mortality Table 2007 for mortality insurance (made by the Institute of Actuaries of Japan).

- Discount rate: 2%.
- Payment method: annual payment.
- Sum assured: 1 (during the entire contract term).

By using the notation in the previous section, the aggregate risk Z of the insurance company becomes

$$Z = \sum_{t=1}^{T} \frac{1}{(1+0.02)^t} 1_{(t-1<\tau\leq t)}.$$

The traditional pricing methods that we use are as follows:

(1) Traditional method without risk loading:

$$\text{the premium } \mathrm{TP1}(T) = \sum_{t=1}^{T} \frac{1}{(1+0.02)^t} P(t-1<\tau\leq t).$$

(2) Traditional method with risk loading:

$$\text{the premium } \mathrm{TP2}(T) = \sum_{t=1}^{T} \frac{1}{(1+0.02)^t} Q'_t,$$

where

$$Q'_t := Q_t + \{Q_t(1-Q_t)\}^{1/2} \quad \text{with} \quad Q_t := P(t-1<\tau\leq t).$$

We write $\mathrm{TP1}(T)$ and $\mathrm{TP2}(T)$ for the premiums of the contract with T years of term obtained by the traditional pricing methods (1) and (2), respectively. For the values $a = 1.0$, 1.5, 2.0 and 2.5, we denote by $\mathrm{IP}_a(T)$ the premium of the same contract obtained by the indifference pricing method in Theorem 4.3 with $\alpha(t) \equiv a$. We also write $\mathrm{IP}_{\mathrm{fit}}(T)$ for the premium of the same contract calculated by the indifference pricing method with $\alpha(t) = 0.6 + 0.36\sqrt{t}$, the form of which is chosen to fit the graph of the indifference prices to that of TP2. In so doing, we used the nonlinear least-squares.

In Figures 4-1–4-3, we plot the graphs of TP1, TP2, IP_a, and $\mathrm{IP}_{\mathrm{fit}}$. We see that the fitted indifference premiums $\mathrm{IP}_{\mathrm{fit}}(T)$ simultaneously approximate the corresponding traditional prices $\mathrm{TP2}(T)$ well. We have repeated this procedure for various prices and obtained good fits in most cases.

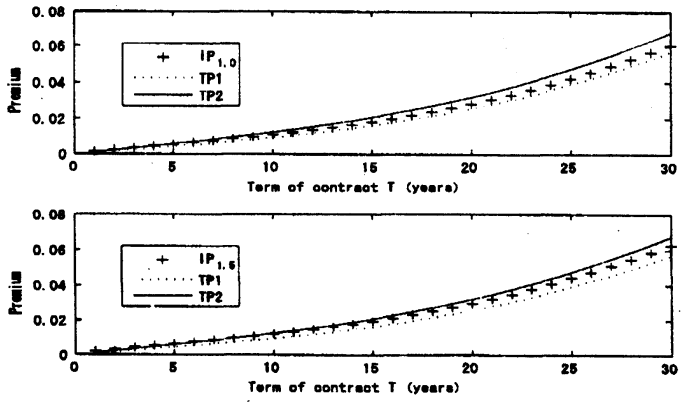

Figure 4.1: TP1 and TP2 vs. $\mathrm{IP}_{1.0}$ and $\mathrm{IP}_{1.5}$.

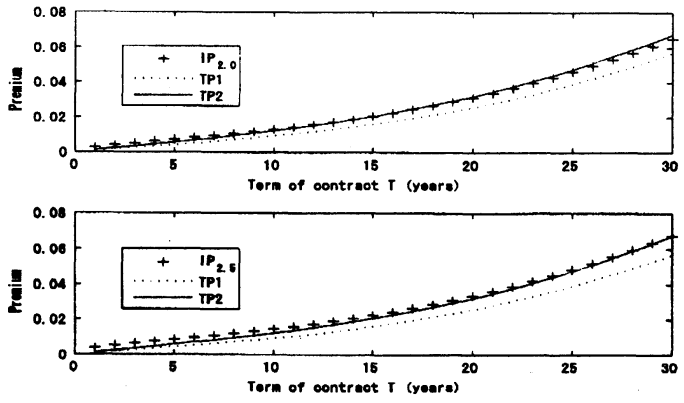

Figure 4.2: TP1 and TP2 vs. $IP_{2.0}$ and $IP_{2.5}$.

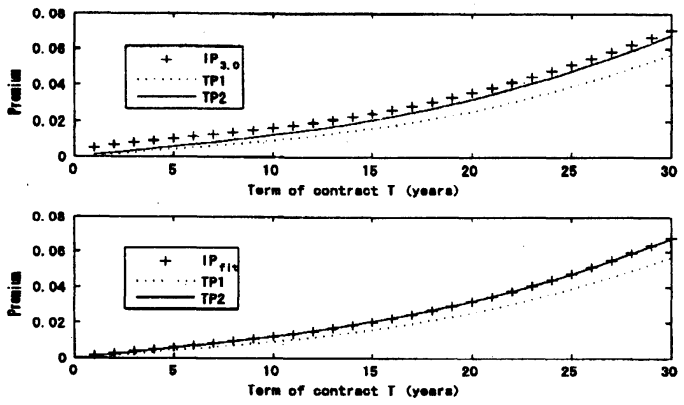

Figure 4.3: TP1 and TP2 vs. $IP_{3.0}$ and IP_{fit}.

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