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Swaption pricing: A Binomial Approach

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1 Introduction

The Libor market model and Swap market model are inconsistent with each other in that they can not be simultaneously described by log-normal processes.

The market quots the at-the-money caps in term of their Black implied volatilities. From these, one can infer caplet volatilites. Caplet implied volatilities give information about the distribution of forward Libor. The market seems to assume that it is log-normal with volatility. At-the-money European swaptions are also quoted in term of their Black implied volatilities which give information about distribution of swap rate. The Black model pricing is assumed that forward swap rate follows log-normal distribution.

The purpose of this paper is to build an arbitrage-free lattice model for swaption, which is consistent with Libor market model and which provides an implementation method for the theoretical closed-form formula which is difficult to get numerical solution. We furthermore compare the approximations of swaption pricing between swap market model and binomial lattice in theoretical and numerical aspect.

There are several papers on solving inconsistency of two market models. We can see the swap volatility approximation by Libor volatility in Rebonate[5] or Brigo[1]. These approaches are mainly to adjest the swap volatility by using Libor volatilty. Recently, however, Davis and Mataix-Pastor[2] have shown the possibility of negative forward Libor rate from coexistence of Libor market model and Swap market model. This negative forward Libor could give us arbitrage opportunity. Our approximation by lattice would make it possible to get arbitrage opportunity.

The rest of this paper is organized as follows. In Section 2 we provide notation and introduce Libor market model which is based on HJM model. In Section 3, we derive European payer swaption price formula for Gaussian volatility. The formula is a weighted average of discount bonds with Gaissian distribution weight. However, it is not easy task to find numerical solution of function which satisfy positivity of swaption. In section 4, we propose the numerical method to get a solution of this function by binomial lattice, which uses the change of measure techique based on Jamishidan [3]. In Section 5 is devoted to numerical example of flat term structure of Libor and volatility. After providing Swap market model with European payer swaption formula, we compare numerical values of coefficeient for discount bonds in the portfolio of bonds replicating the swap. Finally, we discuss the replication strategy for arbitrage and closing remarks.

2 Libor model

We assume HJM-model for discouted bond prices of maturity T_i, $\{B_i(t)\}_{0\leq t\leq T_i}$ under risk neutral measure Q,

$$dB_i(t)/B_i(t) = r(t)dt + \sigma^i(t)dW(t)$$

where the time span is $\delta = t_{i+1} - t_i, for \quad i = 0, \cdots, N-1$. Let $r(t)$ be spot rate and $\sigma^i(t)$ be the volatility of discount bond. The bond price of maturity of T_i at time T_m is, for $t \leq T_m \leq T_i$

$$B_i(T_m) = B_i(t) \exp\left(\int_t^{T_m} r(s)ds + \int_t^{T_m} \sigma^i(s)dW(s) - \frac{1}{2}\int_t^{T_m} |\sigma^i(s)|^2 ds\right), \qquad (2.1)$$

and for the bond price of maturity T_{i+1} is

$$B_{i+1}(T_m) = B_{i+1}(t) \exp\left(\int_t^{T_m} r(s)ds + \int_t^{T_m} \sigma^{i+1}(s)dW(s) - \frac{1}{2}\int_t^{T_m} |\sigma^{i+1}(s)|^2 ds\right). \qquad (2.2)$$

Let $L_i(t)$ be a forward Libor from T_i to T_{i+1}, then the Libor process is defined as

$$L_i(t) = \delta^{-1}(\frac{B_i(t)}{B_{i+1}(t)} - 1).$$

Dividing (2.1) by (2.2) and from the definition of Libor we get

$$\frac{1+\delta L_i(T_m)}{1+\delta L_i(t)} = \exp\left(\int_t^{T_m} [\sigma^i(s) - \sigma^{i+1}(s)]dW(s) - \frac{1}{2}\int_t^{T_m} [|\sigma^i(s)|^2 - |\sigma^{i+1}(s)|^2]ds\right) \qquad (2.3)$$

In HJM-model the forward process of settlement time T is modeled as

$$df_T(t) = \mu_T(t)dt + \sigma_T(t)dW(t)$$

where σ_T is the volatility of forward process $\{f_T(t)\}$. For the settlement time T_i we write the volatility $\sigma_i(t)$ instead of $\sigma_{T_i}(t)$. The bond price at t of maturity T in (2.1) divided by (2.2) and let $B_t(t) = 1$, then

$$B_T(t) = \frac{B_T(0)}{B_t(0)} \exp\left(\int_0^t (\sigma^T(s) - \sigma^t(s))dW(s) - \frac{1}{2}\int_0^t \left(|\sigma^T(s)|^2 - |\sigma^t(s)|^2\right) ds\right)$$

Forward rate is defined as $f_T(t) = -\frac{\partial}{\partial T} \log B_T(t)$ and then

$$df_T(t) = \sigma^T(t)\frac{\partial}{\partial T}\sigma^T(t)dt - \frac{\partial}{\partial T}\sigma^T(t)dW(t)$$

By Itô's division rule

$$\frac{d(B_i(t)/B_{i+1}(t))}{B_i(t)/B_{i+1}(t)} = (\sigma^i(t) - \sigma^{i+1}(t))(dW(t) - \sigma^{i+1}(t)dt)$$

Under the risk adjusted measure Q^{i+1}

$$\frac{dL_i(t)\delta}{1+\delta L_i(t)} = (\sigma^i(t) - \sigma^{i+1}(t))dW^{i+1}(t),$$

when we define the risk adjested Measure Q^{i+1} by $dQ^{i+1}/dQ = \mathcal{E}(\int_0^{T_{i+1}} \sigma^{i+1}(t)dW(t))$, then $W^{i+1}(t) = W(t) - \int_0^t \sigma^{i+1}(s)ds$ is Brownian motion under Q^{i+1} where $\mathcal{E}(\cdot)$ is stochastic exponential.

Therefor Libor $L_i(t)$ is Q^{i+1}-martingale as,

$$E_t^{i+1}[L_i(T_m)] = L_i(t).$$

The Bond Volatility $\sigma^T(t) = -\int_t^T \sigma_s(t)ds$ and let $v_i(t)$ be the volatility of Libor $L_i(t)$;

$$v_i(t) = \sigma^i(t) - \sigma^{i+1}(t) = -\int_t^{T_i} \sigma_u(t)du + \int_t^{T_{i+1}} \sigma_u(t)du = \int_{T_i}^{T_{i+1}} \sigma_u(t)du$$

In section 4 of binomial lattice model we assume v_i is constant for (T_i, T_{i+1}) and in numerical experiment section 5 assume a constant $v = v_i, \ \forall i$. Libor process is expressed under Q^{i+1} from (2.3) as follows,

$$L_i(T_m) = \delta^{-1}(1 + \delta L_i(t)) \exp\{\int_t^{T_m} v_i(s)dW^{i+1}(s) - \frac{1}{2}\int_t^{T_m} |v^i(s)|^2 ds\} - 1$$

3 Swaption price of Gaussian volatility

The payer swaption is the option with strike swap rate k and the maturity T_n, where the underlying swap contract starts from T_n to T_N and payment period $\delta = T_i - T_{i-1}, \quad i = n+1, \cdots, N$. The payment at the maturity is

$$A(T_n) = max(B_n(T_n) - B_N(T_n) - k\delta \sum_{i=n+1}^{N} B_i(T_n), 0)$$

where, $B_i(T_j)$ denotes the price at T_j of bond of maturity time T_i.

The bond price of maturity T_n is 1 at time T_n then $A(T_n) = \max(1 - V(T_n), 0)$ is a put option on bonds portfolio, where

$$V(T_n) = B_N(T_n) + k\delta \sum_{i=n+1}^{N} B_i(T_n)$$

Under risk neutral measure Q, the price of swaption at time 0 is

$$S(0) = E^Q[\exp\{-\int_0^{T_n} r(s)ds\}A(T_n)]$$

Theorem 1 The swaption price of Gaussian volatility HJM model is given as follows,

$$S(0) = B_n(0)N(d_n) - B_N(0)N(d_N) - k\delta \sum_{i=n+1}^{N} B_i(0)N(d_i) \tag{3.1}$$

where $d_i = d_n - \int_0^{T_n}(\sigma^i(s) - \sigma^n(s))ds, \quad i = n+1, \cdots, N$ and d_n is the solution of equation;

$$\begin{aligned} f(x) &= \frac{B_N(0)}{B_n(0)} \exp\{v(0, T_n, T_N)\sqrt{T_n}x - \frac{1}{2}v(0, T_n, T_N)^2 T_n \\ &+ k\delta \sum_{i=n+1}^{N} \frac{B_i(0)}{B_n(0)} \exp\{v(0, T_n, T_i)\sqrt{T_n}x - \frac{1}{2}v(0, T_n, T_i)^2 T_n\} - 1 = 0 \end{aligned} \tag{3.2}$$

where let the variance process $v(t, T_n, T_i)^2 = \frac{1}{T_n - t}\int_t^{T_n} |\sigma^i(t) - \sigma^n(t)|^2 dt$.

Proof. Taking $B_n(t)$ as the numeraire for the payoff at time T_n;

$$\begin{aligned}
\frac{V(T_n)-1}{B_n(T_n)} &= \frac{B_N(0)}{B_n(0)}\exp\{\int_0^{T_n}(\sigma^N(t)-\sigma^n(t))dW^n(t)-\frac{1}{2}\int_0^{T_n}|\sigma^N(t)-\sigma^n(t)|^2dt \\
&+ k\delta\sum_{i=n+1}^{N}\frac{B_i(0)}{B_n(0)}\exp\{\int_0^{T_n}(\sigma^i(t)-\sigma^n(t))dW^n(t)-\frac{1}{2}\int_0^{T_n}|\sigma^i(t)-\sigma^n(t)|^2dt\}-1
\end{aligned}$$

Let U_n be a standarad normal distributed variate, i.e. $U_n \sim N(0,1)$ and define the function;

$$\begin{aligned}
f(U_n) &= \frac{B_N(0)}{B_n(0)}\exp\{v(0,T_n,T_N)\sqrt{T_n}U_n-\frac{1}{2}v(0,T_n,T_N)^2T_n \\
&+ k\delta\sum_{i=n+1}^{N}\frac{B_i(0)}{B_n(0)}\exp\{v(0,T_n,T_i)\sqrt{T_n}U_n-\frac{1}{2}v(0,T_n,T_i)^2T_n\}-1
\end{aligned}$$

where the normal variate $\int_0^{T_n}(\sigma^i(t)-\sigma^n(t))dW^n(t) \sim N(0,v(0,T_i,T_N)^2T_n)$.

The swaption price under risk neutral becomes as follows, with using the change of numeraire technique as $dQ^i/dQ = B_i(T_n)/B_i(0)\exp\{-\int_0^{T_n}r(s)ds\}, \quad i=n,\cdots,N$;

$$\begin{aligned}
S(0) &= E^Q[\exp\{-\int_0^{T_n}r(s)ds\}\max(1-V(T_n),0)] \\
&= E^Q[\exp\{-\int_0^{T_n}r(s)ds\}(1-V(T_n))\mathbf{1}_{\{1\geq V(T_n)\}}] \\
&= E^Q[\exp\{-\int_0^{T_n}r(s)ds\}(B_n(T_n)-B_N(T_n)-k\delta\sum_{i=1}^{n}B_i(T_n))\mathbf{1}_{\{1\geq V(T_n)\}}] \\
&= B_n(0)Q^n(V(T_n)\leq 1)-B_N(0)Q^N(V(T_n)\leq 1)-k\delta\sum_{i=n+1}^{N}B_i(0)Q^i(V(T_n)\leq 1)
\end{aligned}$$

To compute $Q^i(V(T_n)\leq 1)$ we use the function $f(x)$,

$$Q^n(V(T_n)\leq 1) = Q^n(f(U_n)\leq f(d_n))$$

Since $f(d_n)=0$ and $f(\cdot)$ is a montone increasing function and U_n is a standard normal variate,

$$Q^n(V(T_n)\leq 1) = N(d_n).$$

On the other hand, $Q^i(V(T_n)\leq 1) = Q^i(f(U_n)\leq f(d_n))$,

$$\begin{aligned}
\frac{dQ^i}{dQ^n}|_{\mathcal{F}_t} &= \frac{B_i(t)}{B_i(0)}\exp\{-\int_0^t r(s)ds\}/\left(\frac{B_n(t)}{B_n(0)}\exp\{-\int_0^t r(s)ds\}\right) \\
&= \frac{B_i(t)}{B_n(t)}\frac{B_n(0)}{B_i(0)} \\
&= \exp\{\int_0^t(\sigma^i(s)-\sigma^n(s))dW^n(s)-\frac{1}{2}\int_0^t|\sigma^i(s)-\sigma^n(s)|^2ds\}
\end{aligned}$$

By Girsanov theorem, $W^i(t) = W^n(t)-\int_0^t(\sigma^i(s)-\sigma^n(s))ds$ is Brownian motion under Q^i.

$$\begin{aligned}
Q^i(V(T_n)\leq 1) &= Q^i(f(U_n-\int_0^{T_n}(\sigma^i(s)-\sigma^0(s))ds)\leq f(d_n-\int_0^{T_n}(\sigma^i(s)-\sigma^0(s))ds)) \\
&= N(d_n-\int_0^{T_n}(\sigma^i(s)-\sigma^0(s))ds) = N(d_i)
\end{aligned}$$

□

4 Lattice model

The above described one factor swaption model has difficulty to find the solution of equation (3.2) but we can easily get the numerical solution by Binomial approximation of Libor model. First see the main theorem of Libor model.

Theorem 2 The following equations are satisfied in transition probability in Libor binomial model between Q^i and Q^{i+1} which are respectly martingale measures for $L_i(t)$ and $L_{i+1}(t)$, where q_i is upward transitinal probality in binomial tree in the measure Q^i, and q_{i+1} is that in Q^{i+1}.

$$
\begin{aligned}
q_i &= q_{i+1}\frac{1+\delta L_i^u(t)}{1+\delta L_i(t)} && (4.1)\\
1-q_i &= (1-q_{i+1})\frac{1+\delta L_i^d(t)}{1+\delta L_i(t)}, && (4.2)
\end{aligned}
$$

where the binomial states are $L_i^u(t)$ and $L_i^d(t)$.

Proof. From Jamshidian's theorem*

$$E_t^i[L_i(t+\Delta t)] = E_t^{i+1}[L_i(t+\Delta t)\frac{1+\delta L^i(t+\Delta t)}{1+\delta L^i(t)}]$$

Since $L_i(t)$ is Q^{i+1}-martingale,

$$E_t^i[L_i(t+\Delta t)] = \frac{L_i(t)+\delta E^{i+1}[L_i^2(t+\Delta t)]}{1+\delta L_i(t)}$$

By Binomial modeling assumption, the Libor moves in one step for measures Q^{i+1} and Q^i;

$$L_i(t+\Delta t) = \begin{cases} L^u & Q_t^{i+1}(\omega_u) = q_{i+1} \quad Q_t^i(\omega_u) = q_i \\ L^d & Q_t^{i+1}(\omega_d) = 1-q_{i+1} \quad Q_t^i(\omega_d) = 1-q_i \end{cases}$$

To simplify notations, we use L^i instead of $L_i(t)$.

$$
\begin{aligned}
q^iL^u + (1-q^i)L^d &= \frac{L_i}{1+\delta L_i} + \frac{\delta}{1+\delta L_i}\left((L^u)^2 q_{i+1} + (L^d)^2(1-q_{i+1})\right)\\
q_i(L^u - L^d) &= \frac{L_i - L^d(1+\delta L_i) + \delta(L^d)^2}{1+\delta L_i} + \delta q_{i+1}\frac{(L^u)^2-(L^d)^2}{1+\delta L_i}
\end{aligned}
$$

Using the martingale measure $q_{i+1} = (L_i - L^d)/(L^u - L^d)$,

$$q_i = q_{i+1}\left(\frac{1-\delta L^d}{1+\delta L_i} + \frac{\delta L^u + \delta L^d}{1+\delta L_i}\right).$$

Then we get (4.1). The equation (4.2) is also obtained by using the martingale measure,

$$1-q_i = 1 - q_{i+1}\frac{1+\delta L^u}{1+\delta L_i} = (1-q_{i+1})\frac{1+\delta L^d}{1+\delta L_i}$$

□

*see Jamshidian[3]pp.25-26

The forward bond price from T_n to T_N a t is

$$B(t;T_n,T_N)=\frac{B_N(t)}{B_n(t)}=\frac{1}{\prod_{i=n}^{N-1}(1+\delta L_i(t))},\qquad t\le T_n$$

In the binomial lattice, the forward bond price at $t+\Delta t$ has two states,

$$\begin{aligned}B_N^u &= \frac{1}{\prod_{i=n}^{N-1}(1+\delta L_i^u)}\\ B_N^d &= \frac{1}{\prod_{i=n}^{N-1}(1+\delta L_i^d)}\end{aligned}$$

Q^N is called the terminal measure and the transition prbability Q^n of Libor L_n is changed to Q^N,

$$q_n/q_N=\prod_{i=n}^{N-1}\frac{1+\delta L_i^u}{1+\delta L_i}$$

for upward state and

$$(1-q_n)/(1-q_N)=\prod_{i=n}^{N-1}\frac{1+\delta L_i^d}{1+\delta L_i}$$

for the downward state. Swaption payoff at T_n is $\max(1-V(T_n),0)$ and the price at time 0 is

$$\begin{aligned}S(0)/B_n(0) &= E^n[\frac{(1-V(T_n))^+}{B_n(T_n)}]\\ &= E^n[\mathbf{1}_{\{1\ge V(T_n)\}}]-E^n[\frac{B_N(T_n)}{B_n(T_n)}\mathbf{1}_{\{1\ge V(T_n)\}}]-k\delta\sum_{i=n+1}^{N}E^n[\frac{B_i(T_n)}{B_n(T_n)}\mathbf{1}_{\{1\ge V(T_n)\}}]\end{aligned}\qquad(4.3)$$

Using change of measure as (4.1),

$$\begin{aligned}E^n[B(t+\Delta t;T_n,T_N)\mathbf{1}_{\{1\ge V(T_n)\}}|\mathcal{F}_t] &= (q_nB_N^u+(1-q_n)B_N^d)\mathbf{1}_{\{1\ge V(T_n)\}}\\ &= \frac{q_N\mathbf{1}^u_{\{1\ge V(T_n)\}}+(1-q_N)\mathbf{1}^d_{\{1\ge V(T_n)\}}}{\prod_{i=n}^{N-1}1+\delta L_i(t)}\\ &= B(t;T_n,T_N)Q^N(\mathbf{1}_{\{1\ge V(T_n)\}})\end{aligned}$$

Then the unconditional expectation becomes

$$E^n[\frac{B_N(T_n)}{B_n(T_n)}\mathbf{1}_{\{1\ge V(T_n)\}}]=B(0,T_n,T_N)Q^N(\{1\ge V(T_n)\})$$

In general, by change of measure to Q^i from Q^n,

$$E^n[\frac{B_i(T_n)}{B_n(T_n)}\mathbf{1}_{\{1\ge V(T_n)\}}]=B(0,T_n,T_i)Q^i(\{1\ge V(T_n)\})$$

Therefore (4.3) becomes

$$S(0)/B_n(0)=Q^n(\{1\ge V(T_n)\})-B(0,T_n,T_N)Q^N(\{1\ge V(T_n)\})-k\delta\sum_{i=n+1}^{N}B(0,T_n,T_i)Q^i(\{1\ge V(T_n)\})$$

Then we get swaption pricing formula like (3.1),

$$S(0)=B_n(0)Q^n(\{1\ge V(T_n)\})-B_N(0)Q^N(\{1\ge V(T_n)\})-k\delta\sum_{i=n+1}^{N}B_i(0)Q^i(\{1\ge V(T_n)\})\qquad(4.4)$$

Theorem 3 The payer swaption price is in binomial model as foolows,

$$S(0) = B_n(0)F_n(l^*) - B_N(0)F_N(l^*) - k\delta \sum_{i=n}^{N} B_i(0)F_i(l^*) \tag{4.5}$$

where l^* is the smallest integer which satisfies

$$\begin{aligned} 1 - \frac{1}{\prod_{i=n}^{N-1} 1 + \delta L_i(T_n)} &- k\delta \sum_{i=n}^{N} \frac{1}{\prod_{i=n}^{j-1} 1 + \delta L_i(T_n)} \geq 0 \\ 1 + \delta Li(T_n) &= 1 + \delta L_i(0) u_i^{l^*} d_i^{n-l^*} \end{aligned} \tag{4.6}$$

where $L_i^u(T_{k+1}) = L_i(T_k)u_i$ and $L_i^d(T_{k+1}) = L_i(T_k)d_i$ are for $k \leq i$. The binomial distribution function is defined as

$$F_i(l) = 1 - \sum_{j=0}^{l} \begin{pmatrix} n \\ j \end{pmatrix} q_i^i (1 - q_i)^{n-i}.$$

Proof. For binomial lattice the probability in (4.4) is binomial distribution $F_i(l)$. The positive payoff condtion $1 \geq V(T_n)$ satisfies

$$1 - B(0, T_n, T_N) - k\delta \sum_{i=n+1}^{N} B(0, T_n, T_i) \geq 0$$

and it is (4.6). □

5 Example of flat term structure and volatilty

The simplest case is of flat term strucure and flat volatility structure so as $Li(t) = L(t)$ and $u_i = u, \quad d_i = d$. The bond price at time 0 is for the maturity T_i due to flat term structure,

$$B_i(0) = \frac{1}{(1 + \delta L(0))^i}.$$

The Libor is at T_n is

$$L_i(T_n) = L(0)u^l d^{n-l}, \quad l = 0, \cdots, n$$

Because of assumption of flat volatility structure, the forward bond price at T_n is

$$B(T_n, T_n, T_j) = \frac{1}{\prod_{i=n}^{j} 1 + \delta L_i(T_n)} = \frac{1}{(1 + \delta L(0) u^l d^{(n-l)})^{j-n}}, \quad j = n+1, \cdots, N$$

The minimal integer to satisfy (4.6) is

$$\begin{aligned} & 1 - \frac{1}{(1 + \delta L(0) u^l d^{(n-l)})^{N-n}} - k\delta \sum_{j=n+1}^{N} \frac{1}{(1 + \delta L(0) u^l d^{(n-l)})^{j-n}} \\ = & \left(1 + \delta L(0) u^l d^{(n-l)}\right)^{N-n} - k\delta \sum_{j=n+1}^{N} \left(1 + \delta L(0) u^l d^{(n-l)}\right)^{N-j} - 1 \geq 0 \end{aligned}$$

Let $a_0 = -(1 + k\delta)$, $a_{N-n} = 1$, $a_i = -k\delta$, and $x = (1 + \delta L(0) u^l d^{(n-l)})$, then

$$\sum_{i=0}^{N-n} a_i x^i = 0$$

There exist a positive solution x^* because only $a_{N-n} > 0$ and others $a_i < 0$, by Decartes' rule of signs. The number of upward moves becomes

$$l^* = \min\{l \geq \log(x^* - 1)/\delta L(0)) - n \log d/(\log u - \log d)\}$$

From (4.5) for flat term and volatility structure, the positive payment condition l^* is same for all binominal distributions. Thus

$$S(0) = B_n(0)F_n(l*) - B_N(0)F_N(l*) - k\delta \sum_{i=n}^{N} B_i(0)F_i(l^*) \tag{5.1}$$

where $F_i(l^*) = 1 - \sum_{j=0}^{l^*} \binom{n}{j} q_i^j (1-q_i)^{n-j}$ and $q_n = (1-d)/(u-d), q_i = q_{i+1}(1+\delta Lu)/(1+\delta L)$

5.1 Swap market model

Swap market model is utilized for calibration of implied volatility term structure. Let $B_{nN}(t)$ the portfolio value of discount bonds whose maturities are from T_{n+1} to T_N.

$$B_{nN}(t) = \sum_{i=n+1}^{N} B_i(t)$$

There exists the martingale measure Q^{nN} whose numeraire is this portfolio. For any attainable portfolio process $\{C(t)\}$

$$E^{nN}[\frac{C(T)}{B_{nN}(T)}|\mathcal{F}_t] = \frac{C(t)}{B_{nN}(t)}$$

Payer Swaption payoff of swap rate k at Maturity T_n is $\max(B_n(T_n) - B_N(T_n) - k\delta B_{nN}(T_n), 0)$, by taking the portfolio $B_{nN}(t)$ as numeraire, the swaption premium at 0 is

$$\begin{aligned} \frac{C(0)}{B_{nN}(0)} &= E^{nN}[\frac{\max(B_n(T_n) - B_N(T_n) - k\delta B_{nN}(T_n), 0)}{B_{nN}(T_n)}] \\ &= \delta E^{nN}[\max(S_{nN}(T_n) - k, 0)] \end{aligned}$$

where $S_{nN}(t) = \frac{B_n(t) - B_N(t)}{\delta B_{nN}(t)}$ is swap rate at t $(0 \leq t \leq T_n)$. The swap rate is also Q^{nN}-martingale and in the swap market model the swap rate is assumed to be the log normal process;

$$dS_{nN}(t) = \theta(t) S_{nN}(t) dW_{nN}(t)$$

where $W_{nN}(t)$ is Brownian process under Q^{nN}. The swap rate at T_n is

$$S_{nN}(T_n) = S_{nN}(0) \exp\{-\frac{1}{2}\int_0^{T_n} \theta^2(s)ds + \int_0^{T_n} \theta(s) dW_{nN}(s)\}.$$

From this simplified assumption the swaption price is given by Black formula,

$$C(0) = \delta B_{nN}(0)(S_{nN}(0)N(d_1) - kN(d_2))$$

where $d_1 = \log(S_{nN}(0)/k)/v_{nN}(T_n) + v_{nN}(T_n)/2$, $d_2 = d_1 - v_{nN}(T_n)$. The volatility is $v_{nN}(T_n) = \int_0^{T_n} \theta^2(s)ds$. We compare the swaption premium (3.1)

$$\begin{aligned} C(0) &= \frac{\delta B_{nN}(0)}{\delta B_{nN}(0)}(B_n(0) - B_N(0))N(d_1) - k\delta B_{nN}(0)N(d_2) \\ &= B_n(0)N(d_1) - B_N(0)N(d_1) - \delta k \sum_{i=n+1}^{N} B_i(0)N(d_2) \end{aligned} \tag{5.2}$$

The difference is coeffients of bond prices $B_i(0)$.

$$\begin{aligned} d_1 &= \left(\log(B_n(0) - B_N(0)) - \log(k\delta B_{nN})\right)/v_{nN}(T_n) + \frac{1}{2}v_{nN}(T_n) \\ d_2 &= d_1 - v_{nN}(T_n) \end{aligned}$$

The payer swaption price could take the general equation form;

$$C(0) = B_n(0)c_n - B_N c_N - k\delta \sum_{i=n+1}^{N} B_i c_i \tag{5.3}$$

We juxtapose coeffients of discount bonds in equations of (3.1),(4.5) and(5.2) in Table 1.

Bond maturity	T_n	T_N	T_i	
coeffient	c_n	c_N	c_i	equation
Gaussian closed-form	$N(d_n)$	$N(d_N)$	$N(d_i)$	(3.1)
Binomial approx.	$F_n(l^*)$	$F_N(l^*)$	$F_i(l^*)$	(4.5)
Swap market model	$N(d_1)$	$N(d_1)$	$N(d_2)$	(5.2)

Table 1: Payer swaption coeffients

5.2 The hedging strategy and numerical example

We calculate the payer swaption of 3×7 and 5×5 cases of flat Libor and volatilities strucure, where Libor are (i) 2% (ii) 5% and the volatilities are (a)0.4 (b)0.2. These maturities are 3×7 swaption for strike swap-rates for case of (i) are 1%, 2% and 3%. For the case of (ii) strike swap-rates are 4%, 5% and 6%. We compare the binomial lattice, Monte Carlo method and Black formula which assumption is Swap market model.

Table 2: Payer Swaption prices

Vol=40%		3 × 7 Swaption	Strike rate
Initial Libor=2%	0.01	0.02	0.03
Lattice	672.23	335.85	173.34
M.C.	660.20	331.33	173.03
Black	665.76	335.63	176.31
Vol=20%		3 × 7 Swaption	Strike rate
Initial Libor=5%	0.04	0.05	0.06
Lattice	647.42	360.99	188.38
M.C.	625.22	346.17	179.56
Black	648.30	363.39	191.06
Vol=40%		5 × 5 Swaption	Strike rate
Initial Libor=2%	0.01	0.02	0.03
Lattice	491.18	289.99	182.16
M.C.	495.04	297.95	191.95
Black	496.43	298.72	192.08
Vol=20%		5 × 5 Swaption	Strike rate
Initial Libor=5%	0.04	0.05	0.06
Lattice	485.65	308.81	191.81
M.C.	475.94	302.70	188.79
Black	489.86	313.75	196.87

all swaption prices are Basis point(1/100%) unit and M.C.are Monte Carlo Method which are provided by Dr. Yasuoka, Mizuho Information & Reseach Insitute, (100,000 runs))
Lattice method prices are 1000 node for a year and total steps are $1000 \times x$ for swaption maturity x years.

From inconsistency of Libor market model and Swap market model, we could have arbtrage opportunity if we had constructed a hedging strategy. Davis [2] has shown the existence of negative libor rate in the case of coexistense of Libor and swap market models.

For the swaption if we take Gaussian model, the hedging strategy is as follows,

$$dC(t) = N(d_n)dB_n(t) - N(d_N)dB_N(t) - k\delta \sum_{i=n+1}^{N} N(d_i)dB_i(t)$$

which is easily shown. Delta hedging is change of the portfolio which is $N(d_i)$ unit of bond of maturity T_i.

The hedging strategy of swap market model is obtained from (5.2)

$$dC(t) = N(d_1)dB_n(t) - N(d_1)dB_N(t) - k\delta \sum_{i=n+1}^{N} N(d_2)dB_i(t)$$

Provided the longer term interest is changed, the delta hedging of swap market model is not senstive due to the same delta $N(d_2)$ for all $dB_i(t)$. The price differences are caused by coeffients c_i as Table 1. We calculate coefficient for 3x7 swaption (volatity =40%, interest=2%, strike=2%).

Bond maturity	T_3	T_{10}	$T_i(i = 3.5, \cdots, 10)$
coeffient	c_n	c_N	c_i
Binomial approx.	0.57170	0.53369	0.56919,···,0.53643
Swap market model	0.63819	0.63819	0.36723

Table 3: Payer swaption coeffients

In this data case, we can see in Table 3 the $B_n(t)$ trading amount is excessive and other maturity bonds trading is insufficient in swap market model. For this case we could take arbitrage opportunity if change of longer term interest shift upward and we trade swaption and the hedging strategy of bonds.

References

[1] Brigo, D Interest Rate Models Theory and Practice, Springer 2001

[2] Davis, M.H.A., and Mataix-Pastor, V. Negative Libor Rates in the Swap Market Model. Finance and Stochastics 11(2), pp 181-193, (2007)

[3] Jamshidian, F. Libor and Swap Market Models and Measures, Sakura Global Capital, 1996

[4] Jamshidian, F. Bivariate support of forward Libor and swap rates. Univ. of Twente working paper, (2005)

[5] Rebonato, R Modern pricing of interest-rate derivatives, Princeton Univesity Press, 2002