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Three Phased Switching of Operations under Uncertainty*

東京大学・大学院工学系研究科　髙嶋 隆太 (Ryuta Takashima)†
Graduate School of Engineering,
The University of Tokyo
早稲田大学・創造理工学部　後藤 允 (Makoto Goto)
School of Creative Science and Engineering,
Waseda University

龍谷大学・経済学部　辻村 元男 (Motoh Tsujimura)
Faculty of Economics,
Ryukoku University

1 Introduction

Real options theory, pioneered by Brennan and Schwartz (1985), and McDonald and Siegel (1986), and summarized in Dixit and Pindyck (1994), and Trigeorgis (1996), is a useful methodology for the economic evaluation and analysis of various investment projects under uncertainty. In particular, the analysis of entry and exit problem is an important task in real options research since the entry and exit model enables us to analyze the actual investment problems as the start-up and shut-down of a power plant, the switching of production capacity, and the suspension and restarting of an investment project.

The pioneering literature on entry and exit decision under uncertainty includes Brennan and Schwartz (1985) and Dixit (1989). Many research groups have studied the extension of model by using the Dixit model as a baseline model. Bar-Ilan and Strange (1996) introduce a model of investment lags which is the time it takes to build the project. Boyarchenko and Levendorskiĭ (2000) develop a discrete-time model taking into account non-Gaussian stochastic processes. Zervos (2003) considers the operation problem with the abandonment of project. Sødal (2006) discusses the entry and exit problems by use of a discount factor approach. Additionally, there also exist the following application problems. Näsäkkälä and Fleten (2004) analyze the investment and operating flexibility of gas-fired power plants. Wirl (2006) investigates a switching problem of greenhouse gas emissions. Zhdanov (2007) examines the interactions between entry-exit and financing decisions of firms in a competitive industry. Furthermore, Dixit (1988) extends the entry-exit model by incorporating construction and scrapping decisions, and then Dixit and Pindyck (1994) examine lay-up decisions for an oil tanker by using this entry-exit-scrapping model. In these studies, two operating states of the mothballing and the operation have been

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†7-3-1 Hongo, Bunkyo-ku, Tokyo 113-8656, Japan, E-mail: takashima@n.t.u-tokyo.ac.jp

analyzed. To our knowledge, more than three operating states and a operating state between the mothballing and the operation, such as a partial operation, have not been investigated.

In this paper, we extend the entry-exit-scrapping model to allow for three operating states. The dependence of switching thresholds on uncertainty and operating cost is investigated. Specifically, we demonstrate the value of the investment option by comparing our proposed model with the Dixit model. We also show the effect of the capacity in the partial operation on switching thresholds. In addition, in order to examine the interactions between entry-exit-scrapping and financing decisions, we propose a model for analyzing the investment policy of the firm which is financed by issuing equity and debt.

The remainder of this paper is organized as follows. The next section describes the model and derives the equations satisfied by the option value. Section 3 provides some numerical results with respect to the effect of uncertainty and costs on each threshold value. In section 4, we present a model in which the firm is partially financed with debt. Section 5 concludes.

2 The Model

Following Dixit (1988), we propose the entry-exit-scrapping model taking into account three operating states such as a full operation, a partial operation, and a mothballing. The evolution of the output price is assumed to follow a geometric Brownian motion:

$$dP_t = \mu P_t dt + \sigma P_t dW_t, \quad P_0 = p, \tag{2.1}$$

where μ is the instantaneous risk-adjusted expected growth rate of P_t, σ is the associated volatility, and W_t is a standard Brownian motion. We assume that the firm is risk-neutral, and thus the appropriate discount rate is a risk-free rate $r > \mu$.

Suppose that the firm has two states of the idle and operation which contains three states of the full operation, the partial operation, and the mothballing. We also assume that the firm can change the idle state and three operating states $Z_t \in \{0, 1, 2, 3\}$ with a fixed cost. $Z_t = 0, 1, 2, 3$ represent the states of the idle, the full operation, the partial operation, and the mothballing, respectively. The firm's operation strategy is defined by the sequence of the pair of i-th switching times T_i and i-th switching operations. Therefore, the firm's investment strategy v is defined as the following:

$$v = (T_i, \zeta_i)_{i \geq 0}, \tag{2.2}$$

where $\zeta_i = Z_{T_i}$ is the control of the state. The profit functions in each state $\pi(P_t; Z_t)$ are given by,

$$\pi(P_t; Z_t) = \begin{cases} 0, & Z_t = 0, \\ P_t - C, & Z_t = 1, \\ \alpha(P_t - C), & Z_t = 2, \\ -M, & Z_t = 3, \end{cases} \tag{2.3}$$

where C is the operating cost, $0 < \alpha < 1$ is the rate of partial operation, M is the maintenance cost.

The firm's problem is to maximize the expected discounted value by selecting an optimal control rule. The value function is expressed by the following equation:

$$V(p;v)=\sup_{v\in\mathcal{V}}\mathbb{E}\left[\int_0^{\infty}e^{-rt}\pi\left(P_t;Z_t\right)dt-\sum_{i=0}^{\infty}e^{-rT_i}K(Z_{T_i-},Z_{T_i})\mathbf{1}_{\{T_i<\infty\}}\right], \tag{2.4}$$

where $\mathcal{V}$ is the collection of admissible controls, and

$$\begin{aligned}
K(0,0) &= K(1,1)=K(2,2)=0, &(2.5)\\
K(0,1) &= K_I, &(2.6)\\
K(1,2) &= K_{C2}, &(2.7)\\
K(2,3) &= K_{C1}, &(2.8)\\
K(3,2) &= K_{E1}, &(2.9)\\
K(2,1) &= K_{E2}, &(2.10)\\
K(3,0) &= K_A, &(2.11)
\end{aligned}$$

are fixed costs to change each state.

Using standard method in the literature as Dixit and Pindyck (1994), it can be shown that the value of idle state V_0, the full operation V_1, the partial operation V_2, and the mothballing V_3 must satisfy the following ordinary differential equations:

$$\frac{1}{2}\sigma^2p^2V_0''+\mu pV_0'-rV_0=0, \tag{2.12}$$

$$\frac{1}{2}\sigma^2p^2V_1''+\mu pV_1'-rV_1+p-C=0. \tag{2.13}$$

$$\frac{1}{2}\sigma^2p^2V_2''+\mu pV_2'-rV_2+\alpha(p-C)=0, \tag{2.14}$$

$$\frac{1}{2}\sigma^2p^2V_3''+\mu pV_3'-rV_3-M=0, \tag{2.15}$$

The general solutions of Eqs. (2.12)–(2.15) are given by the following equations:

$$\begin{aligned}
V_0(p) &= a_1p^{\beta_1}, &(2.16)\\
V_1(p) &= a_2p^{\beta_2}+\frac{p}{r-\mu}-\frac{C}{r}, &(2.17)\\
V_2(p) &= a_3p^{\beta_1}+a_4p^{\beta_2}+\frac{\alpha p}{r-\mu}-\frac{\alpha C}{r}, &(2.18)\\
V_3(p) &= a_5p^{\beta_1}+a_6p^{\beta_2}-\frac{M}{r}, &(2.19)
\end{aligned}$$

where a_1,a_2,a_3,a_4,a_5, and a_6 are unknown constants, and

$$\beta_1=\frac{1}{2}-\frac{\mu}{\sigma^2}+\sqrt{\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2r}{\sigma^2}}>1,\quad \beta_2=\frac{1}{2}-\frac{\mu}{\sigma^2}-\sqrt{\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2r}{\sigma^2}}<0. \tag{2.20}$$

Unlike the entry-exit-scrapping model in Dixit (1988), there exists a partial operating state as Eq. (2.18) in this model. The first term in Eq. (2.18) represents the value of the option to switch from the partial operating to the full operating state. The second term in Eq. (2.18) is the value of the option to mothball.

For the threshold prices of investment, each switching, and scrapping, the value of firm must satisfy the following value-matching and smooth-pasting conditions:

$$V_0(P_I) = V_1(P_I) - K_I, \qquad V_0'(P_I) = V_1'(P_I), \tag{2.21}$$

$$V_1(P_{C2}) = V_2(P_{C2}) - K_{C2}, \qquad V_1'(P_{C2}) = V_2'(P_{C2}), \tag{2.22}$$

$$V_2(P_{C1}) = V_3(P_{C1}) - K_{C1}, \qquad V_2'(P_{C1}) = V_3'(P_{C1}), \tag{2.23}$$

$$V_3(P_{E1}) = V_2(P_{E1}) - K_{E1}, \qquad V_3'(P_{E1}) = V_2'(P_{E1}), \tag{2.24}$$

$$V_2(P_{E2}) = V_1(P_{E2}) - K_{E2}, \qquad V_2'(P_{E2}) = V_1'(P_{E2}), \tag{2.25}$$

$$V_3(P_A) = V_0(P_A) - K_A, \qquad V_3'(P_A) = V_0'(P_A). \tag{2.26}$$

As can be seen from these equations, an embedding of a operating state leads to an increase of four equations compared with the Dixit model. These twelve equations provide a simultaneous nonlinear equation system, which can determine the six thresholds $P_I, P_{C1}, P_{C2}, P_{E1}, P_{E2}, P_A$ and the six option value coefficients $a_1, a_2, a_3, a_4, a_5, a_6$ by means of a numerical calculation method such as the Newton-Raphson method.

3 Numerical Analysis

In the previous section, we proposed the entry-exit-scrapping model which is extended to allow for three operating states. In this section, we present the calculation results of each state value and the difference of the value between the Dixit model and our proposed model. The base case parameters used in this section are as follows: $\mu = 0$, $\sigma = 0.2$, $r = 0.04$, $C = 2$, $M = 0.2$, $\alpha = 0.5$, $K_I = 10$, $K_{C1} = 0.1$, $K_{C2} = 0.5$, $K_{E1} = 2$, $K_{E2} = 1$, and $K_A = 3.5$. Figure 1 shows the value of the investment option and each operation, and the six thresholds. The difference of the value at the threshold corresponds to fixed costs to change each state. We make a comparison between the Dixit model and our proposed model by means of these values.

The dependence of the thresholds for each switching on the volatility of output price are presented in Fig. 2. It is clear from this figure that thresholds P_I, P_{E2}, and P_{E1} increase and the thresholds P_{C2}, P_{C1}, and P_A decrease with volatility. As in Dixit (1988), a large uncertainty increases the inaction state. In order to investigate the effect of the switching times on the option value, the difference of the value between the Dixit model and our proposed model is calculated. Figure 3 illustrates the difference of the idle state value at $p = 3$ between the Dixit model and our proposed model, that is, $V_0^{\text{This}}(p) - V_0^D(p)$ as a function of the volatility. The difference of the value decreases as the volatility increases. This is because an increase in volatility induces a increases in the option value, and the difference of the value with respect to the number of the switching state, which is the degree of the operating flexibility, becomes small.

In Fig. 4, the dependence of the thresholds for each switching on the operating cost are shown. As shown in this figure, when the operating cost C is less than 1.06, the mothballing option becomes worthless. This boundary value is larger than 0.73 calculated by use of the

Dixit model. Although an increase in the switching state leads to an increases in the value, the existence region of each option decreases with the number of state. We show the difference of the idle state value at $x = 3$ between two models as a function of the operating cost in Fig. 5. This difference of the value indicates the flexibility value in which the operation state increases. It turns out that this flexibility value decreases as the operating cost C increases.

Figure 6 presents the dependence of the thresholds for each switching on the rate of partial operation. Although the switching costs are constants so far, in this calculation, we assume that the switching costs except K_I and K_A is a proportional to α as follows:

$$K_{E1} = k_{E1}\alpha, \quad K_{E2} = k_{E2}(1-\alpha), \quad K_{C2} = k_{C2}, \alpha \quad K_{C2} = k_{C2}(1-\alpha),$$

where k_{E1}, k_{E2}, k_{C1}, and $k_{C2} > 0$ are constants. It can be seen from this figure that P_{E1} and P_{C1} are relatively sensitive to α. Thus, the influence of α on the switching between the partial operation and the mothballing states is larger than that of other switching.

4 Equity and Debt Financing

In previous section, we considered an unlevered firm which is financed entirely with equity. This case is a standard framework in literature on real options. However, many firms are financed by issuing not only equity but also debt. Recently many researchers have studied the interaction among firm's investment and financing decisions under uncertainty by means of real option framework (Mauer and Ott, 2000; Mauer and Sarkar, 2005; Lyandres and Zhdanov, 2006; Sundaresan and Wang, 2006). We therefore expand the model illustrated above to allow for debt financing in this section.

Since equityholders choose the switching times $T_{i\geq 0}$ and the default time T_d optimally, the value function of equity, after the investment option is exercised and then the firm starts operating, is given by the following equation:

$$E(p) = \sup_{T_{i\geq 0}, T_d > 0} \mathbb{E}\left[\int_t^{T_d} e^{-r(s-t)}(1-\tau)\left(\pi\left(P_t; Z_t\right) - c\right) ds - \sum_{i=0}^{\infty} e^{-r(T_i - t)} K(Z_{T_i -}, Z_{T_i}) \mathbf{1}_{\{T_i < T_d\}}\right], \tag{4.1}$$

where τ is a constant corporate tax rate, and c is a continuous coupon payment. Unlike the model presented above, suppose that the firm can abandon the project forever at the scrapping time, that is, has no idle state because there exists the default in this model.

Since the holders of debt receive the continuous coupon payment, the value of debt can be expressed as

$$D(p) = \mathbb{E}\left[\int_t^{T_d} e^{-r(s-t)} c\, ds + e^{-r(T_d - t)}(1-\theta)\frac{(1-\tau)(P_{T_d} - C)}{r-\mu}\right], \tag{4.2}$$

where θ is the proportional bankruptcy cost, $0 \leq \theta \leq 1$. The holders of debt receive the value of the unlevered firm net of bankruptcy costs.

When the output price rises above the investment threshold P_I, the firm can start operating by incurring the investment cost K_I. The firm is partially financed by debt at the investment time T_I. The value of the investment option $E_0(p)$ is given by

$$E_0(p) = \sup_{T_I > 0} \mathbb{E}\left[e^{-rT_I}\left\{E_1(P_t) - (K_I - D(P_t))\right\}\right], \tag{4.3}$$

where $E_1(P_t)$ is the equity value of full operation. For the threshold price of investment P_I^L, the value of the investment option must satisfy the following value-matching and smooth-pasting conditions:

$$E_0(P_I^L) = E_1(P_I^L) - (K_I - D(P_I^L)) = V_L(P_I^L) - K_I, \tag{4.4}$$

$$E_0'(P_I^L) = V_L'(P_I^L), \tag{4.5}$$

where $V_L(p) = E_1(p) + D(p)$. From these calculations, the value of levered firm with three operating states, and the threshold for investment, switching, and default can be obtained.

5 Concluding Remarks

We have proposed the entry-exit-scrapping model with three operating states. The values of the idle state, the full operation, the partial operation and the mothballing, and the threshold values of investment, each switching, and abandonment were presented. In particular, we showed the dependence of switching thresholds on uncertainty and operating cost. We also showed the difference of the value between the Dixit model and our proposed model in order to examine the value of operation flexibility. It was found that the embedding of a operating state leads to increase in the value of the firm and the investment opportunity. Moreover, we presented the entry-exit-scrapping model of the levered firm which is financed with equity and debt.

For future work, we will analyze the entry-exit-scrapping strategy of the levered firm by use of the model presented above, and show the optimal capital structure and the agency cost between the firm-value-maximizing and the equity-value-maximizing policies. Additionally, we will investigate the entry-exit-scrapping problems with n operating states such as more than four states.

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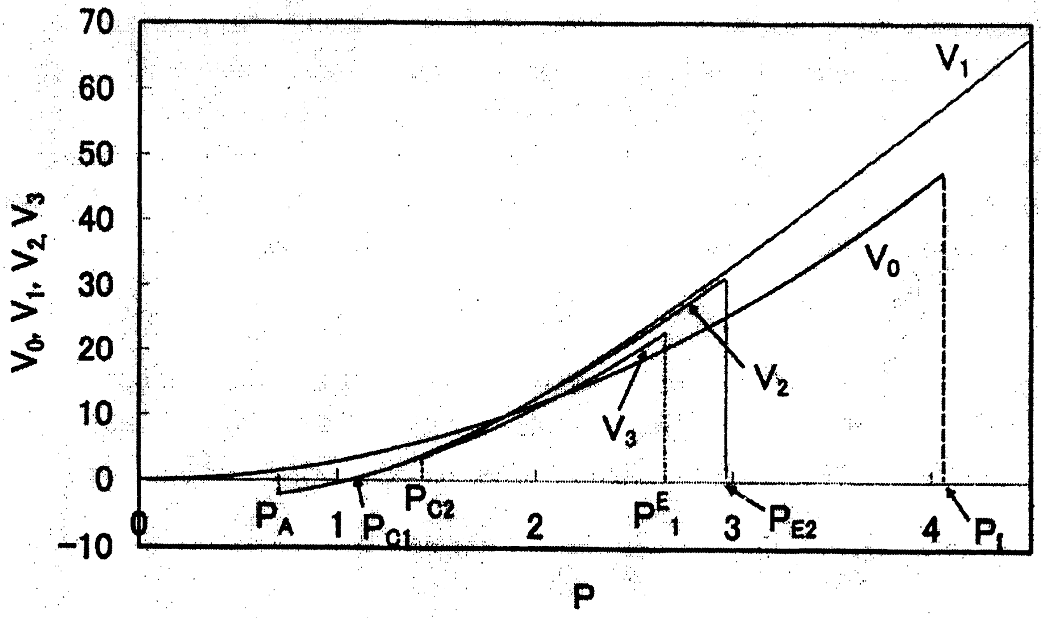

Figure 1: Values of idle state, each operation, and mothballing.

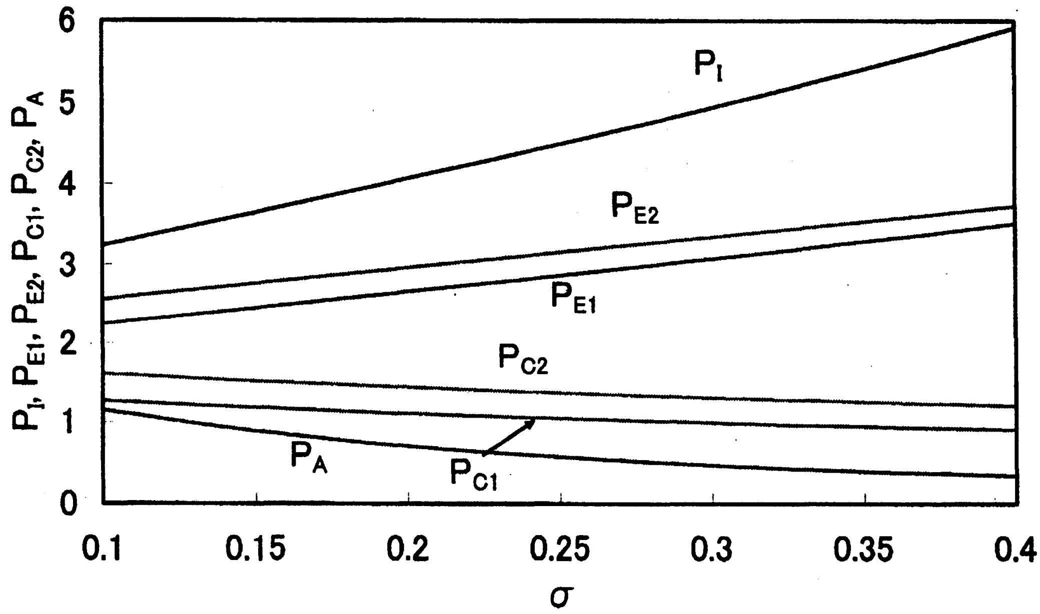

Figure 2: Thresholds of the switching as a function of the volatility.

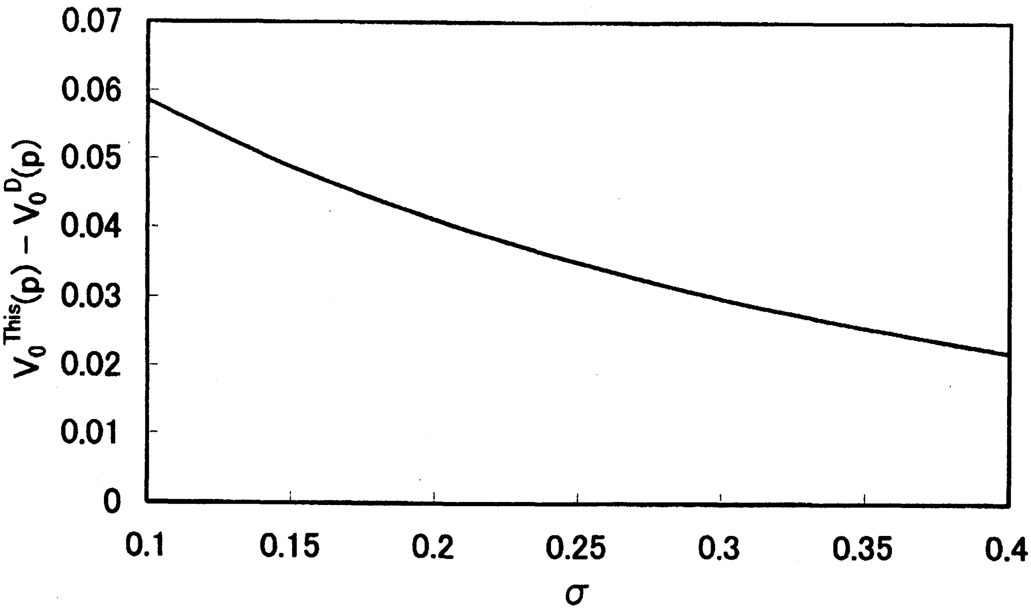

Figure 3: Difference of the value of the investment option as a function of the volatility.

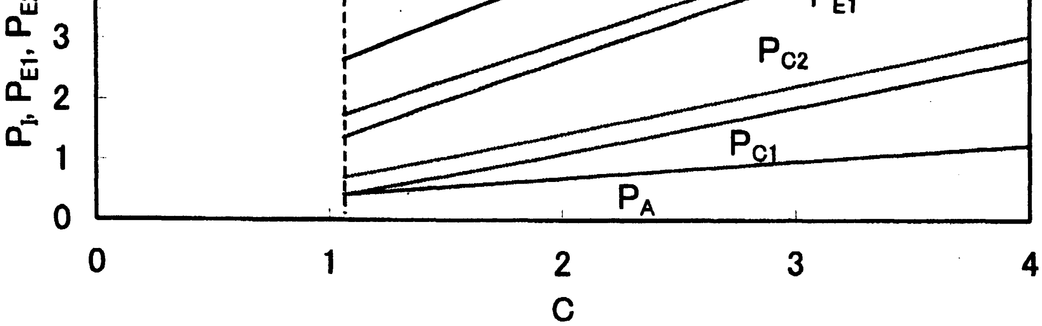

Figure 4: Thresholds of the switching as a function of the oparating cost.

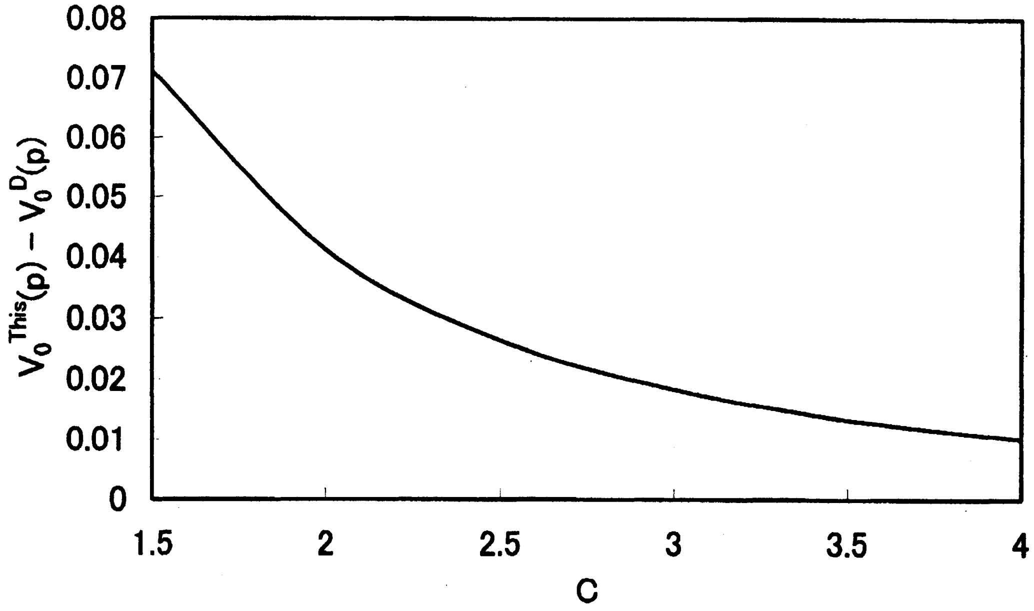

Figure 5: Difference of the value of the investment option as a function of the operating cost.

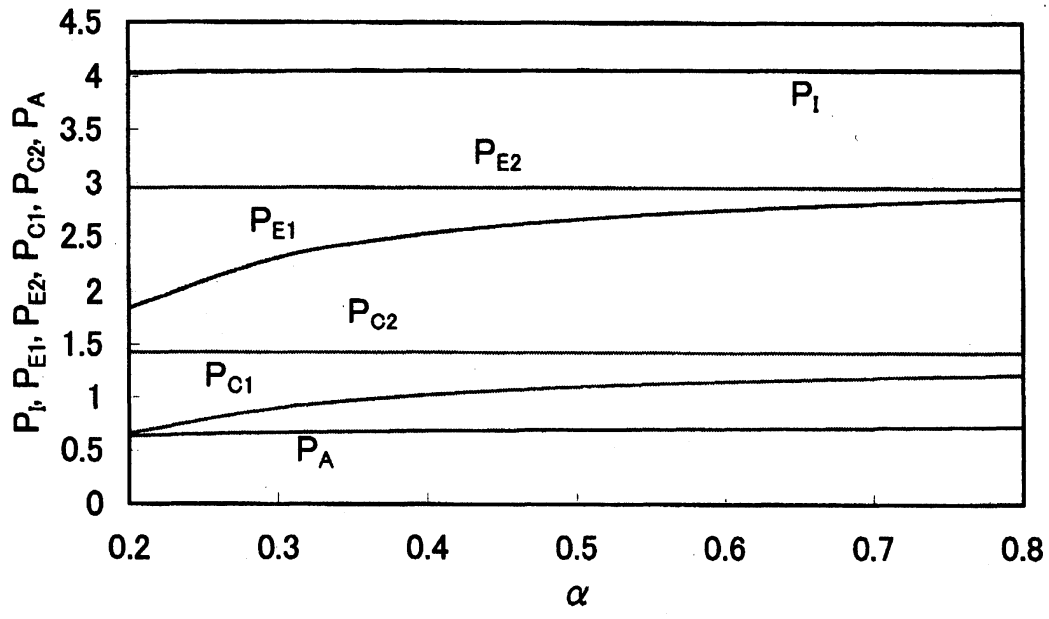

Figure 6: Thresholds of the switching as a function of the rate of partial operation.