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BRAUER CORRESPONDENCE AND GREEN CORRESPONDENCE *

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1 Introduction

Let k be an algebraically closed field of prime characteristic p. Let G be a finite group of order divisible by p. We are concerned with cohomology algebras of block ideals which are in Brauer correspondence and block varieties of modules in Green correspondence.

2 Cohomology of blocks and Brauer correspondence

Let B be a block ideal of kG. Proposition 2.3 of Kessar, Linckelmann and Robinson [5] implies

$$H^{*}(G, B) \subseteq H^{*}(H, C),$$

where C is a suitably taken block ideal of a suitably chosen subgroup H of G. To understand such an inclusion via transfer map between the Hochshild cohomology algebras of the block ideals B and C we discussed in Kawai and Sasaki [4] under the following situation.

- B has D as a defect group
- H is a subgroup of G and C is a block ideal of kH
- Brauer correspondent C^{G} is defined and $C^{G}=B$ and D is also a defect group of C

We had considered the (C, B)-bimodule $M=CB$ and gave a necessary and sufficient condition for M to induce the transfer map from $HH^{*}(B)$ to $HH^{*}(C)$ which restricts to the inclusion map of $H^{*}(G, B)$ into $H^{*}(H, C)$.

Here we discuss under the following situation:

Situation (BC)

- B has D as a defect group
- H is a subgroup of G such that $DC_{G}(D) \leqslant H$ and C is a block ideal of kH
- $C^{G}=B$ and D is also a defect group of C

*The detailed version of this note will be submitted for publication elsewhere.

We shall denote by G^{op} the opposite group of the group G and consider the group algebra kG as a $k[G \times G^{\mathrm{op}}]$-module through

$$(x, y)\alpha = x\alpha y \text{ for } x, y \in G \text{ and } \alpha \in kG.$$

We have a $k[G \times G^{\mathrm{op}}]$-isomorphism

$$kG \simeq k[G \times G^{\mathrm{op}}] \otimes_{k[\Delta G]} k,$$

where $\Delta G = \{ (g, g^{-1}) \mid g \in G \}$.

Definition 2.1. Under Situation (BC), the Green correspondent of C with respect to $(G \times H^{\mathrm{op}}, \Delta D, H \times H^{\mathrm{op}})$ is defined, which turns out to be a (B, C)-bimodule; we denote it by $L(B, C)$.

The module $L(B, C)$ will play crucial role, depending on the following fact.

Theorem 2.1. *Under* Situation (BC) *let* $L = L(B, C)$.

(i) *The relatively projective elements* $\pi_L \in Z(B)$ *and* $\pi_{L^*} \in Z(C)$ *are both invertible.*
(ii) *Every* (B, A)*-bimodule is relatively* L*-projective; every* (C, A)*-bimodule is relatively* L^**-projective, where* A *is a symmetric* k*-algebra.*

Following Alperin, Linckelmann and Rouquier [1], we recall the definition of source modules of block ideals.

Definition 2.2. ([1, Definition 2]) There exists an indecomposable direct summand X of ${}_{G\times D^{\mathrm{op}}}B$ having ΔD as a vertex. The $k[G \times D^{\mathrm{op}}]$-module X is called a *source module* of the block B.

We shall write $H^*(G, B; X)$ for the block cohomology of B with respect to the defect group D and the source idempotent i such that $X = kGi$.

Green correspondence between indecomposable $k[G \times D^{\mathrm{op}}]$-modules and indecomposable $k[H \times D^{\mathrm{op}}]$-modules relates source modules of the blocks B and C in the following way.

Proposition 2.2. *Under* Situation (BC) *let* Y *be a source module of* C*. Then the Green correspondent* X *of* Y *with respect to* $(G \times D^{\mathrm{op}}, \Delta D, H \times D^{\mathrm{op}})$ *is a source module of* B.

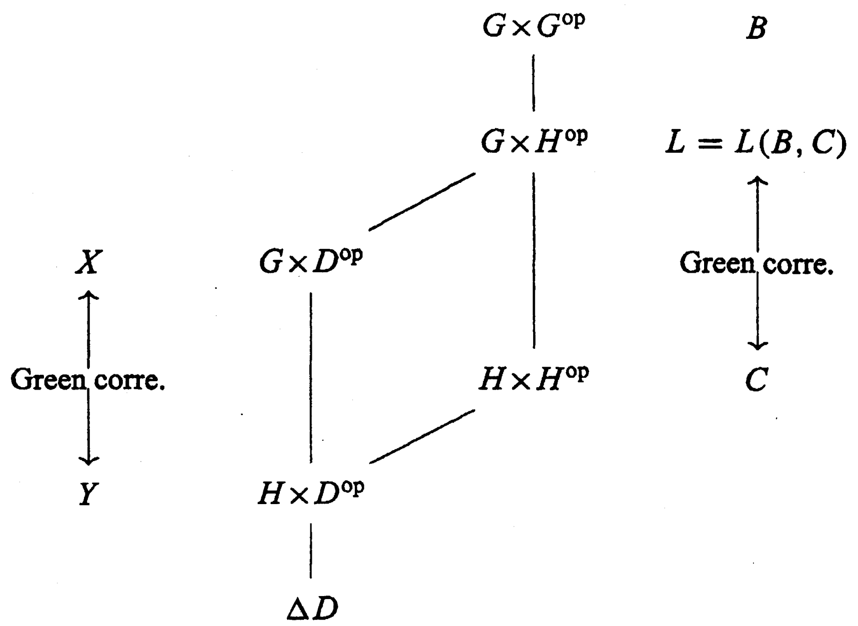
Proposition 2.3. *Under* Situation (BC) *take a source module* X *of* B *as a direct summand of* ${}_{G\times D^{\mathrm{op}}}L(B, C)$*. Then the Green correspondent* Y *of* X *with respect to* $(G \times D^{\mathrm{op}}, \Delta D, H \times D^{\mathrm{op}})$ *is a source module of* C.

Thus, under Situation (BC) we can take a source module X of the block B and a source module Y of the block C in order that X and Y are in the Green correspondence with respect to $(G \times D^{\mathrm{op}}, \Delta D, H \times D^{\mathrm{op}})$. We refer to such situation as Situation (XY).

Situation (XY)

- B has D as a defect group
- H is a subgroup of G such that $DC_G(D) \leqslant H$ and C is a block ideal of kH

- $C^G = B$ and D is also a defect group of C
- a source module X of the block B and a source module Y of the block C are in the Green correspondence with respect to $(G \times D^{\mathrm{op}}, \Delta D, H \times D^{\mathrm{op}})$



Then the (B, C)-bimodule $L = L(B, C)$ links the source modules X and Y in a similar way to induction and restriction of modules.

Theorem 2.4. *Under* Situation (XY) *the following hold.*

(i) $L^* \otimes_B X \equiv Y \oplus O(\mathscr{Y}(G \times D^{\mathrm{op}}, \Delta D, H \times D^{\mathrm{op}}))$.
(ii) $L \otimes_C Y \equiv X \oplus O(\mathscr{X}(G \times D^{\mathrm{op}}, \Delta D, H \times D^{\mathrm{op}}))$.
(iii) *If* $D \lhd H$, *then* $L \otimes_C Y \simeq X$.

The (B, C)-bimodule $L(B, C)$ has already appeared in some works. In particular, Alperin, Linckelmann and Rouquier [1] treated the case of $H = N_G(D, b_D)$, where (D, b_D) is a Sylow B-subpair. Theorem 5 in [1] corresponds to our theorem above.

Theorem 2.5. *Under* Situation (XY) *the module* $L(B, C)$ *is splendid with respect to* X *and* Y, *namely*

$$L(B, C) \mid X \otimes_{kD} Y^*.$$

The theorem above and the following, which states that the relatively projective elements associated with tensor products of the bimodules L, X and Y, including such as $X^* \otimes_B L \otimes_C Y$, are all invertible, lead us Theorem 2.8, which is one of our main theorems.

Theorem 2.6. *Under* Situation (XY) *the relatively projective elements*

(i) $\pi_{L\otimes_C Y} \in Z(B)$, $\pi_{Y^*\otimes_C L^*} \in Z(kD)$
(ii) $\pi_{X^*\otimes_B L \otimes_C Y} \in Z(kD)$, $\pi_{X^*\otimes_B L} \in Z(kD)$
(iii) $\pi_{Y^*\otimes_C L^* \otimes_B X} \in Z(kD)$, $\pi_{L^*\otimes_B X} \in Z(C)$

are all invertible.

Proposition 2.7. *Under* Situation (XY) *we have the following commutative diagram:*

$$
\begin{array}{ccccccc}
H^*(G,B;X) & \xrightarrow{\delta_D} & HH^*_{X^*}(kD) & \underset{R_{X^*}}{\overset{R_X}{\rightleftarrows}} & HH^*_X(B) & & \\
 & & \uparrow & & \uparrow & & \\
 & & HH^*_{X^*\otimes_B L\otimes_C Y}(kD) & \underset{R_{X^*}}{\overset{R_X}{\rightleftarrows}} & HH^*_{L\otimes_C Y}(B) & \hookrightarrow & HH^*_L(B) \\
 & & \| & & R_L \Uparrow\Downarrow R_{L^*} & & R_L \Uparrow\Downarrow R_{L^*} \\
 & & HH^*_{Y^*\otimes_C L^*\otimes_B X}(kD) & \underset{R_{Y^*}}{\overset{R_Y}{\rightleftarrows}} & HH^*_{L^*}(C)\cap HH^*_Y(C) & \hookrightarrow & HH^*_{L^*}(C) \\
 & & \downarrow & & \downarrow & & \\
H^*(H,C;Y) & \xrightarrow[\delta_D]{} & HH^*_{Y^*}(kD) & \underset{R_{Y^*}}{\overset{R_Y}{\rightleftarrows}} & HH^*_Y(C) & &
\end{array}
$$

Theorem 2.8. *Let B be a block ideal of kG and $D \leqslant G$ a defect group of B. Assume that a subgroup H of G containing $DC_G(D)$ normalizes a subgroup Q of D and contains $QC_G(Q)$. Let (D, b_D) be a Sylow B-subpair and let $(Q, b_Q) \leqslant (D, b_D)$. Let C be a unique block ideal of kH covering the block ideal b_Q of $kQC_G(Q)$. Then $C^G = B$ and D is a defect group of C; hence (D, b_D) is also a Sylow C-subpair.*

Let j be a source idempotent of C such that $\mathrm{Br}_D(j)e_D = \mathrm{Br}_D(j)$, where $e_D \in kC_G(D)$ is the block idempotent of the block b_D; let $Y = kHj$. Let X be a source module of B which is the Green correspondent of Y with respect to $(G\times D^{\mathrm{op}}, \Delta D, H\times D^{\mathrm{op}})$. We let $L = L(B, C)$. Then the following diagram commutes:

$$
\begin{array}{ccccc}
H^*(G,B;X) & \xrightarrow{\delta_D} & HH^*_{X^*\otimes_B L\otimes_C Y}(kD) & \underset{R_{X^*}}{\overset{R_X}{\rightleftarrows}} & HH^*_{L\otimes_C Y}(B) \\
\downarrow & & \downarrow & & \downarrow R_{L^*} \\
H^*(H,C;Y) & \xrightarrow[\delta_D]{} & HH^*_{Y^*}(kD) & \underset{R_{Y^*}}{\overset{R_Y}{\rightleftarrows}} & HH^*_Y(C)
\end{array}
$$

3 Block varieties of modules and Green correspondence

If $H^*(G, B; X) \subseteq H^*(H, C; Y)$, then Kawai and Sasaki [4, Theorem 1.3 (i)] says that the inclusion map $\iota : H^*(G, B; X) \hookrightarrow H^*(H, C; Y)$ induces a surjective map $\iota^* : V_{H,C} \to V_{G,B}$ of varieties.

Throughout this section we let $P \leqslant D$ and assume that $H \geqslant N_G(P)$. We investigate relationship between the varieties of modules in blocks B and C which are under Green correspondence.

We first note Under Situation (BC) that to tensor with $L(B, C)$ and $L(B, C)^*$ induces the Green correspondence.

Proposition 3.1. *Under* Situation (BC), *we let* $L = L(B, C)$. *If an indecomposable B-module U and an indecomposable C-module V have vertices in* $\mathscr{A}(G, P, H)$ *and are in the Green correspondence with respect to* (G, P, H), *then*

$$L \otimes_C V \equiv U \oplus O(\mathscr{X}(G, P, H))$$
$$L^* \otimes_B U \equiv V \oplus O(\mathscr{Y}(G, P, H)).$$

The block variety of an indecomposable module is determined by particular vertex and a particular source by Benson and Linckelmann [2].

Definition 3.1. (Benson and Linckelmann [2, Proposition 2.5]) Let X be a source module of a block ideal B. Let U be an indecomposable B-module. There exists a vertex Q of U such that

$$Q \leqslant D, \ U \mid X \otimes_{kQ} X^* \otimes_B U.$$

We would like to call such a vertex Q of U an *X-vertex*. For an X-vertex Q of U we can take a Q-source S of U such that

$$S \mid {}_{kQ}X^* \otimes_B U, \ U \mid X \otimes_{kQ} S$$

We would like to call such a source a *(Q, X)-source*.

[2, Theorem 1.1] says that the block variety $V_{G,B}(U)$ in the block cohomology $H^*(G, B; X)$ is the pull back of the variety $V_Q(S)$ of S, where Q is an X-vertex and S is a (Q, X)-source of U.

Proposition 3.2. *Under* Situation (XY), *let U and V be as in Proposition 3.1. Then the following hold.*

(i) *If* $Q \in \mathscr{A}(G, P, H)$ *is a Y-vertex of V and S is a* (Q, Y)*-source of V, then Q is an X-vertex of U and S is a* (Q, X)*-source of U.*
(ii) *If* $Q \in \mathscr{A}(G, P, H)$ *is an X-vertex of U and S is a* (Q, X)*-source of U, then Q is a Y-vertex of V and S is a* (Q, Y)*-source of V.*

It is well known that the Green correspondent of an indecomposable module lies in a block ideal of a subgroup of G lies in its Brauer correspondent. The following is a partial converse to this fact.

Proposition 3.3. *Under* Situation (XY), *assume that an indecomposable B-module U has an X-vertex belonging to* $\mathscr{A}(G, P, H)$. *Then the Green correspondent V of U with respect to* (G, P, H) *lies in the block C.*

The following is our main theorem.

Theorem 3.4. *Under* Situation (XY) *assume that* $H^*(G, B; X) \subseteq H^*(H, C; Y)$.

(i) *Assume that an indecomposable B-module U has an X-vertex belonging to* $\mathscr{A}(G, P, H)$. *Then the Green correspondent V of U with respect to* (G, P, H) *lies in the block C and*

$$V_{G,B}(U) = \iota^* V_{H,C}(V).$$

(ii) *Assume that an indecomposable C-module V has a Y-vertex belonging to $\mathscr{A}(G, P, H)$. Then the Green correspondent U of V with respect to (G, P, H) lies in the block B and*

$$V_{G,B}(U) = \iota^* V_{H,C}(V).$$

Example. (cf [2, Corollary 1.4]) Let B be a block ideal of kG and $D \leqslant G$ a defect group of B. Let X be a source module of B. Let U be an indecomposable B-module and Q an X-vertex of U and S a (Q, X)-source of U. Assume that the X-vertex Q of U in normal in D and let $H = N_G(Q)$. Let $P \leqslant D$ and assume that $H \geqslant N_G(P)$ and that $Q \in \mathscr{A}(G, P, H)$.

Let (D, b_D) be a Sylow B-subpair such that $b_D X(D) = X(D)$ and let $(Q, b_Q) \leqslant (D, b_D)$. Let C be a unique block of kH covering the block b_Q.

Then we have

(i) $H^*(G, B) \subseteq H^*(H, C)$;
(ii) Q is a Y-vertex of V and S is a (Q, Y)-source of V;
(iii) V lies in C and $V_{G,B}(U) = \iota^* V_{H,C}(V)$.

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