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Blocks with defect groups which have cyclic subgroups of index p

指数pの巡回部分群を持つ群が不足群であるブロック

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This is a part of joint work with **Miles Holloway** and **Naoko Kunugi**.

In representation theory of finite groups, especially in modular representation theory of finite groups, in the last a bit more than two decades we have had well-known and important conjectures, such as, first of all, Alperin's Weight Conjecture, Dade's Conjecture (which is more precise than Alperin's Weight Conjecture), and also Broué's Abelian Defect Group Conjecture. From my point of view, all of these three conjectures originally are due to Brauer's philosophy, that is, if we are given a p-block A of a finite group G and if B is the Brauer correspondent of A in $N_G(P)$, the normalizer of a defect group P of A in G, then the p-blocks A and B should be similar, or at least they should have much in common. Here p is a prime.

I believe, more or less many people would agree with myself, hopefully. Maybe it would be better to say that I intentionally omit a kind of hypotheses for, so-called, the inertial group of the blocks A and B we are looking at.

Anyhow, in a cellebrated article of Alperin [1] he poses an interesting and important conjecture, nowadays it is called *Alperin's Weight*

Conjecture. For instance, Alperin shows in [1, Consequence 5] that, if a p-block A of G with defect group P is controlled by fusion in $N_G(P)$, then the numbers $\ell(A)$ and $\ell(B)$ of simple kG- and $kN_G(P)$-modules in A and B, respectively, are the same, provided Alperin's Weight Conjecture holds, where k is an algebraically closed field of characteristic p. Furthermore, if A is the principal p-block and controlled by fusion in $N_G(P)$, then even the numbers $k(A)$ and $k(B)$ of irreducible ordinary characters of G and $N_G(P)$ in A and B, respectively, are the same provided Alperin's Weight Conjecture holds, see [1, Consequence 7].

The purpose of the talk was to present that the above two conclusions, namely $k(A) = k(B)$ and $\ell(A) = \ell(B)$ hold where B is the principal block algebra of $\mathcal{O}N_G(P)$, $\mathcal{O}$ is a complete discrete valuation ring whose residue field is k, and that we shall give precise values of $k(A)$ and $\ell(A)$, if A is the *principal* p-block of G with defect group P which is the extra-special group $M_3(p) = p_-^{1+2}$ of order p^3 and exponent p^2. This actually answers affirmatively to a conjecture given by Hendren [2, p.490] though we look at only *principal* p-blocks.

Note that the proofs in the theorm are independent of the classification of finite simple groups.

Theorem (Holloway-Kunugi-Koshitani). *Let A be the principal block algebra of $\mathcal{O}G$ for an arbitrary finite group G with a Sylow p-subgroup $P = M_3(p) = p_-^{1+2}$ which is the extra-special group of order p^3 of exponent p^2. Then, it holds that*

$$k(A) = \frac{p^2-1}{|E|} + p|E| \qquad \text{and} \qquad \ell(A) = |E|,$$

where $|E| = |N_G(P)/P{\cdot}C_G(P)|$, the inertial index of A, and it turns out that the conjecture of Hendren [2, p.490] *is true for principal blocks.*

Remark. It is shown from [2, Theorem 5.8], or more generally from [4, Proposition 5.4] and a result of Wong [5] (see [3, IV 3.5.Satz]), that in the situation of **(1.2)** the block A is controlled by fusion in $N_G(P)$. Thus, Alperin's Weight Conjecture implies the conclusion of **(1.2)** by [1, Consequences 5 and 7].

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