数理解析研究所講究録
第 1581 巻 2008 年 23-27

Fusion systems I

Markus Linckelmann

Kyoto, August 28, 2007

The fusion system of a finite group G encodes the p-local structure of G at a prime p in terms of a category whose objects are the p-subgroups of G and whose morphisms are the group homomorphisms between p-subgroups induced by conjugation with elements in G. Since any p-subgroup is conjugate to a subgroup of a fixed Sylow-p-subgroup P we may restrict attention to subgroups of P. This point of view leads to considering categories whose objects are subgroups of a finite p-group and whose morphisms are injective group homomorphisms, with certain properties. We start with some terminology. Let p be a prime number.

Definition 1. Let P be a finite p-group. A *category on P* is a category $\mathcal{F}$ whose objects are the subgroups of P and whose morphism sets $\mathrm{Hom}_{\mathcal{F}}(Q,R)$ consist of injective group homomorphisms, for any two subgroups Q, R of P, such that if $\varphi : Q \to R$ is a morphism in $\mathcal{F}$ then so is the induced isomorphism $Q \cong \varphi(Q)$ and its inverse, and if $Q \subseteq R$ then the inclusion morphism $Q \to R$ belongs to $\mathcal{F}$ as well; composition of morphisms in $\mathcal{F}$ is the usual composition of group homomorphisms.

In particular, categories on finite p-groups are finite *EI-categories*; that is, every endomorphism of an object is actually an automorphism of that object.

Definition 2. Let $\mathcal{F}$ be a category on a finite p-group P and let Q be a subgroup of P. We say that Q is

(a) *fully $\mathcal{F}$-normalised* if $|N_P(Q)| \geq |N_P(Q')|$ for any subgroup Q' of P such that $Q \cong Q'$ in $\mathcal{F}$;

(b) *fully $\mathcal{F}$-centralised* if $|C_P(Q)| \geq |C_P(Q')|$ for any subgroup Q' of P such that $Q \cong Q'$ in $\mathcal{F}$.

Example 3. Let G be a finite group and let P be a Sylow-p-subgroup of G. The *fusion system of G on P* is the category $\mathcal{F}_P(G)$ with object set the subgroups of P and morphism sets

$$\mathrm{Hom}_{\mathcal{F}_P(G)}(Q,R) = \mathrm{Hom}_G(Q,R) = \{\varphi : Q \to R \mid \exists x \in G : \varphi(u) = xux^{-1} (\forall u \in Q)\}$$

for any two subgroups Q, R of P. One checks that a subgroup Q of P is fully $\mathcal{F}_P(G)$-normalised if and only if $N_P(Q)$ is a Sylow-p-subgroup of $N_G(Q)$ and similarly that Q is fully $\mathcal{F}_P(G)$-centralised if and only if $C_P(Q)$ is a Sylow-p-subgroup of $C_G(Q)$.

Fusion systems of finite groups are categories on finite p-groups in the sense of the definition above with some extra properties - and the following definition, due to Puig, of fusion systems on finite p-groups captures those properties:

Definition 4 (Puig). Let P be a finite p-group. A *fusion system on* P is a category $\mathcal{F}$ on P such that $\mathcal{F}_P(P) \subseteq \mathcal{F}$ and such that the following hold:

(I) "Sylow axiom": if Q is a fully $\mathcal{F}$-normalised subgroup of P then $\mathrm{Aut}_P(Q)$ is a Sylow-p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(G)$.

(II) "Extension axiom": if $\varphi : Q \to P$ is a morphism in $\mathcal{F}$ such that $\varphi(Q)$ is fully $\mathcal{F}$-normalised then φ can be extended to a morphism $\psi : N_\varphi \to P$ in $\mathcal{F}$, where

$$N_\varphi = \{y \in N_P(Q) \mid \exists z \in N_P(\varphi(Q)) : \ \varphi(yuy^{-1}) = z\varphi(u)z^{-1} \ (\forall u \in Q)\}$$

Note that $QC_P(Q) \subseteq N_\varphi \subseteq N_P(Q)$.

Theorem 5. *Let G be a finite group and P a Sylow-p-subgroup of G. The category $\mathcal{F}_P(G)$ is a fusion system on P.*

The following combines work of Ron Solomon from the 1970's with work of Broto, Levi, Oliver (2002):

Theorem 6 (Solomon/Broto-Levi-Oliver). *Let q be an odd prime power.*

(i) There is a fusion system $\mathcal{F}_{\mathrm{Sol}(q)}$ on a Sylow-2-subgroup P of $\mathrm{Spin}_7(q)$ such that $\mathcal{F}_P(\mathrm{Spin}_7(q)) \subseteq \mathcal{F}_{\mathrm{Sol}(q)}$ and such that all subgroups of order 2 of P are isomorphic in $\mathcal{F}_{\mathrm{Sol}(q)}$.

(ii) There is no finite group G with Sylow-p-subgroup P and fusion system $\mathcal{F}_{\mathrm{Sol}(q)}$.

This shows that there are "exotic" fusion systems which do not arise as fusion systems of finite groups. Ruiz and Viruel found further exotic fusion systems on an extra-special group of order 7^3 of exponent 7. Blocks of finite groups give rise to fusion systems; more precisely, if G is a finite group, k an algebraically closed field of characteristic p and B a block of kG (that is, B is an indecomposable direct factor of kG as an algebra) then by classical work of Brauer, B determines a p-subgroup P of G, uniquely up to conjugation of G, called a *defect group of the block B*, and furthermore, as a consequence of work of Alperin and Broué, the block B determines a fusion system $\mathcal{F}_P(B)$ on P, uniquely up to conjugation by an element in $N_G(P)$. If B is the so-called *principal block* of kG then the defect group P is a Sylow-p-subgroup of G and $\mathcal{F}_P(B) = \mathcal{F}_P(G)$; thus fusion systems of finite groups are particular cases of fusion systems of blocks. It is not known whether every fusion system of a block can be realised as fusion system of a (possibly different) finite group, but there is some evidence in that direction. Using the classification of finite simple groups, Kessar showed that the Solomon fusion systems cannot arise as fusion systems of blocks:

Theorem 7 (Kessar). *$\mathcal{F}_{\mathrm{Sol}(q)}$ cannot be the fusion system of a 2-block of a finite group.*

Subsequently, Kessar and Stancu showed that the exotic fusion systems of Ruiz and Viruel are also not fusion systems of 7-blocks. It turns out, however, that fusion systems can always be realised as fusion systems of certain infinite groups: an infinite group G is said to have P as Sylow-p-subgroup if P is a finite p-subgroup of G such that any other finite p-subgroup of G is conjugate to a subgroup of P. In that case the definition of $\mathcal{F}_P(G)$ carries over verbatim.

Theorem 8 (Robinson/Leary-Stancu (2006)). *For any fusion system $\mathcal{F}$ on a finite p-group P there is a possibly infinite group G having P as Sylow-p-subgroup such that $\mathcal{F} = \mathcal{F}_G(P)$.*

Fusion systems are completely determined by automorphism groups of certain subgroups. To state this properly, we need the following terminology.

Definition 9. Let P be a finite p-group and let $\mathcal{F}$ be a fusion system on P. A subgroup Q of $\mathcal{F}$ is called *$\mathcal{F}$-centric* if $C_P(Q') = Z(Q')$ for any subgroup Q' of P such that $Q' \cong Q$ in $\mathcal{F}$. A subgroup Q of P is called *$\mathcal{F}$-radical* if $O_p(\mathrm{Aut}_{\mathcal{F}}(Q)) = \mathrm{Aut}_Q(Q)$; that is, if the largest normal p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$ consists exactly of all inner automorphisms of Q.

Theorem 10 (Alperin's Fusion Theorem). *Let $\mathcal{F}$ be a fusion system on a finite p-group P. Then $\mathcal{F}$ is completely determined by the groups $\mathrm{Aut}_{\mathcal{F}}(R)$, with R running over the $\mathcal{F}$-centric $\mathcal{F}$-radical fully $\mathcal{F}$-normalised subgroups of P. More precisely, any isomorphism in $\mathcal{F}$ can be written as composition of isomorphisms $\varphi : Q \cong Q'$ for which there exists an $\mathcal{F}$-centric $\mathcal{F}$-radical fully $\mathcal{F}$-normalised subgroup R of P containing both Q, Q', and an automorphism $\psi \in \mathrm{Aut}_{\mathcal{F}}(R)$ such that $\varphi = \psi|_Q$.*

One can define normalisers and centralisers in fusion systems, similarly to group theoretic notions:

Definition 11 (Puig). Let $\mathcal{F}$ be a category on a finite p-group P, let Q be a subgroup of P. The *normaliser of Q in $\mathcal{F}$* is the category $N_{\mathcal{F}}(Q)$ on $N_P(Q)$ with morphisms consisting of all morphisms $\varphi : R \to S$ in $\mathcal{F}$ which can be extended to a morphism $\psi : QR \to QS$ in $\mathcal{F}$ such that $\psi(Q) = Q$, where R, S are subgroups of $N_P(Q)$. Similarly, The *centraliser of Q in $\mathcal{F}$* is the category $C_{\mathcal{F}}(Q)$ on $C_P(Q)$ with morphisms consisting of all morphisms $\varphi : R \to S$ in $\mathcal{F}$ which can be extended to a morphism $\psi : QR \to QS$ in $\mathcal{F}$ such that $\psi|_Q = \mathrm{Id}_Q$, where R, S are subgroups of $C_P(Q)$.

Theorem 12 (Puig). *Let $\mathcal{F}$ be a fusion system on a finite p-group P and Q a subgroup of P.*
(i) If Q is fully $\mathcal{F}$-normalised then $N_{\mathcal{F}}(Q)$ is a fusion system on $N_P(Q)$.
(ii) If Q is fully $\mathcal{F}$-centralised then $C_{\mathcal{F}}(Q)$ is a fusion system on $C_P(Q)$.

Example 13. Let G be a finite group, P a Sylow-p-subgroup of G and set $\mathcal{F} = \mathcal{F}_P(G)$. Let Q be a subgroup of P. If Q is fully $\mathcal{F}$-normalised then $N_{\mathcal{F}}(Q) = \mathcal{F}_{N_P(Q)}(N_G(Q))$, and if Q is fully $\mathcal{F}$-centralised then $C_{\mathcal{F}}(Q) = \mathcal{F}_{C_P(Q)}(C_G(Q))$.

The folowing theorem, when specialised to fusion systems of finte groups, is a theorem of Burnside:

Theorem 14. *Let P be a finite abelian p-group and $\mathcal{F}$ a fusion system on P. We have $\mathcal{F} = N_{\mathcal{F}}(P) = \mathcal{F}_P(P \rtimes \mathrm{Aut}_{\mathcal{F}}(P))$.*

Definition 15. Let $\mathcal{F}$ be a fusion system on a finite p-group P and let $\mathcal{F}'$ be a fusion system on a subgroup P' of P. We say that *$\mathcal{F}'$ is normal in $\mathcal{F}$* for any subgroup Q of P' and any $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$ we have $\varphi(Q) \subseteq P'$ and $\varphi \circ \mathrm{Aut}_{\mathcal{F}'}(Q) \circ \varphi^{-1} = \mathrm{Aut}_{\mathcal{F}'}(\varphi(Q))$. We say that $\mathcal{F}$ is *simple* if $P \neq 1$ and $\mathcal{F}$ has no proper nontrival normal subsystem.

If $\mathcal{F} = \mathcal{F}_P(G)$, where G is a finite group and P a Sylow-p-subgroup of G, and if N is a normal subgroup of G, then $P' = N \cap P$ is a Sylow-p-subgroup of N and $\mathcal{F}_{P'}(N)$ is a normal subsystem in $\mathcal{F}$. One can show that if $\mathcal{F} = \mathcal{F}_P(G)$ is simple, then there exists a finite simple group H with P as Sylow-p-subgroup such that $\mathcal{F} = \mathcal{F}_P(H)$. Fusion systems of finite simple groups need not always be simple, though.

Theorem 16. *The fusion system $\mathcal{F}_{\mathrm{Sol}(3)}$ is simple.*

Theorem 17 (Stancu). *Let $\mathcal{F}$ be a fusion system on a finite p-group P and let Q be a normal subgroup of P. Then $\mathcal{F} = N_{\mathcal{F}}(Q)$ if and only if $\mathcal{F}_Q(Q)$ is normal in $\mathcal{F}$.*

Example 18. Let $P = D_8$ be a dihedral group of order 8. Up to isomorphism, there are exactly three fusion systems on P, and they all arise as fusion systems of finite groups with dihedral Sylow-2-subgroups, namely the fusion systems of $P = D_8$ itself, of S_4 and of $\mathrm{PSL}_2(q)$ for suitable odd prime powers q. Only the last of these three groups yields simple fusion systems.

There is a topological side to fusion systems:

Theorem 19 (Oliver, "Martino-Priddy conjecture"). *Let G be a finite group, P as Sylow-p-subgroup of G. The fusion system $\mathcal{F}_P(G)$ determines the homotopy type of the p-completed classifying space $BG^\wedge_p$.*

The proof, due to Oliver, is a tour de force using the classification of finite simple groups in conjunction with functor cohomological methods. It is not known at present whether an arbitrary fusion system gives rise to a p-complete topological space - this line of thought, first hinted at in work of Benson, led to the theory of p-local finite groups developed by Broto, Levi and Oliver. Fusion systems give also rise to analogues of orbit spaces. More precisely, let $\mathcal{F}$ be a fusion system on a finite p-group P, let $\mathcal{C}$ be a *right ideal in* $\mathcal{F}$; that is, $\mathcal{C}$ is a full subcategory of $\mathcal{F}$ with the proeprty that if Q, R are subgroups of P with Q belonging to $\mathcal{C}$ and with $\mathrm{Hom}_{\mathcal{F}}(Q, R) \neq \emptyset$ then also R belongs to $\mathcal{C}$. Non empty chains of subgroups $Q_0 < Q_1 < \cdots < Q_m$ in $\mathcal{C}$ form a poset $S(\mathcal{C})$ under taking subchains, and isomorphisms in $\mathcal{F}$ induce a notion of isomorphism classes of such chains; we denote by $[S(\mathcal{C})]$ the poset of isomorphism classes of non empty chains of subgroups in $\mathcal{C}$. Any poset can be viewed as topological space via the nerve construction.

Theorem 20. *Let $\mathcal{C}$ be a right ideal in a fusion system on a finite p-group P. The orbit space $[S(\mathcal{C})]$ is contractible, when viewed as topological space.*

"Schur multipliers" of finite EI-categories need not be finite - but in the context of right ideals in fusion systems they are:

Theorem 21. *Let $\mathcal{C}$ be a right ideal in a fusion system on a finte p-group P. Let k be an algebraically closed field of characteristic p. The group $H^2(\mathcal{C}; k^\times)$ is a finite p'-group.*

Markus Linckelmann
Department of Mathematics,
University of Aberdeen,
Aberdeen AB24 3UE,
United Kingdom