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On the isomorphisms between the centers of the principapl p-block algebras induced by the Glauberman-Dade correspondence

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For a prime p, let $(\mathcal{K}, \mathcal{O}, k)$ be a p-modular system where $\mathcal{O}$ is a complete discrete valuation ring having the residue field k of characteristic p which is algebraically closed and having the quotient field $\mathcal{K}$ of characteristic zero which will be assumed to be large enough for any finite group we consider in this article. Below, by a character, we mean a $\mathcal{K}$-character.

Glauberman showed in [5] that there is a bijective correspondence between the set $\mathrm{Irr}(\mathsf{G})^{\mathsf{S}}$ of all S-invariant irreducible characters of G and the set $\mathrm{Irr}(C_{\mathsf{G}}(\mathsf{S}))$ of all irreducible characters of $C_{\mathsf{G}}(\mathsf{S})$, called the *Glauberman correspondence* of characters, where G is a finite group and S is a finite solvable group such that S acts on G via automorphism and $(|\mathsf{G}|, |\mathsf{S}|) = 1$, where $|\mathsf{G}|$ and $|\mathsf{S}|$ denote the orders of G and S, respectively. When S is cyclic, a basic relation between $\chi \in \mathrm{Irr}(\mathsf{G})^{\mathsf{S}}$ and the Glauberman corresponding character $\beta_\chi \in \mathrm{Irr}(C_{\mathsf{G}}(\mathsf{S}))$ is

$$\hat{\chi}(\mathsf{cs}) = \epsilon_\chi \beta_\chi(\mathsf{c}), \qquad (\sharp)$$

where $\hat{\chi}$ is a unique extension of χ to the semi-direct product $\mathsf{G} \rtimes \mathsf{S}$ satisfying $\mathsf{S} \subset \mathrm{Ker}(\det(\hat{\chi}))$, called the *canonical extension*, c is any element of $C_{\mathsf{G}}(\mathsf{S})$, s is any generator of S and ϵ_χ is a uniquely determined sign, see [5, Theorem 3].

Dade gives in [4] a new approach to the Glauberman correspondence without considering the canonical extension and the relation ($\sharp$) above, and partly generalizes it: he gives in [4, Theorem 6.8] a bijective correspondence between $\mathrm{Irr}(G)^E$ and $\mathrm{Irr}(G')^{E'}$, where E is a finite group, G is a normal subgroup of E such that E/G is cyclic, E' is a subgroup of E such that $E = GE'$, and $G' = G \cap E'$ (hence, G' is normal in E' and E'/G' is isomorphic to the cyclic group E/G) with the condition that E'_0, the subset of E' consisting of every element of E' whose canonical image in E'/G' is a generator of E'/G', is a trivial intersection subset of E with E' as its normalizer, that is, $E'_0 \cap {E'_0}^t$ is the empty set for any $t \in E - E'$. In fact, the correspondence of characters is given for twisted group algebras of G and G' over $\mathcal{K}$. It is shown that, with the notations above, when S is cyclic, the above conditions are satisfied by taking $\mathsf{G} \rtimes \mathsf{S}$, G, $C_{\mathsf{G}}(\mathsf{S}) \times \mathsf{S}$ and $C_{\mathsf{G}}(\mathsf{S})$ as E, G, E' and G', respectively ([4, Lemma 7.5]), and, in this case, Dade's correspondence coincides with the Glauberman correspondence ([4, Proposition 7.8]). We call the above correspondence of characters the *Glauberman-Dade correspondence.*

Watanabe began in [11] a p-block-theoretical study of the Glauberman correspondence and she showed that, when b is an S-invariant p-block of G with an S-centralized defect group, the Glauberman correspondence induces a perfect isometry (for this notion, see [2]) between $\mathbb{Z}\mathrm{Irr}(\mathsf{b})$ and $\mathbb{Z}\mathrm{Irr}(w(\mathsf{b}))$ for some uniquely determined p-block $w(\mathsf{b})$ of G^{S}.

In general, by the theorem of Broué ([2]), perfectly isometric p-blocks have isomorphic centers, and so $Z(\mathcal{O}\mathsf{Gb}) \simeq Z(\mathcal{O}\mathsf{G}^{\mathsf{S}}w(\mathsf{b}))$ as $\mathcal{O}$-algebras.

On the other hand, in [8], Okuyama gives an explicit isomorphism between the centers of $\mathcal{O}\mathsf{Gb}$ and $\mathcal{O}\mathsf{G}^{\mathsf{S}}w(\mathsf{b})$ using the relation ($\sharp$).

In this article, we note that, when $|E/G|$ is a prime, under the condition analogous to the one of Watanabe (see, Condition 3.1), the Glauberman-Dade correspondence induces a bijective correspondence between the characters belonging to the principal p-blocks of G and G' (Proposition 3.3), and give an explicit isomorphism between the centers of the principal p-block algebras of G and G' (Theorem 3.5) which coincides with the one given by Okuyama when the hypotheses are the same. For a more general situation, see [10], and for standard facts, see [7].

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Below, we always assume the following:

Condition 2.1. E is a finite group with a normal subgroup G such that the quotient group $F = E/G$ is cyclic of prime order q. E' is a subgroup of E such that $E = GE'$. G' is a normal subgroup of E' defined by $G' = G \cap E'$. Set $E_0 = E - G$ and $E'_0 = E' - G'$. E'_0 is a trivial intersection subset of E with E' as its normalizer, that is, $E'_0 \cap E'^{\tau}_0 = \emptyset$, the empty set, for any $\tau \in E - E'$.

Take $s \in E'_0$. Then $E = <G, s>$ and $E' = <G', s>$.

Denote by π the canonical epimorphism from E to E/G, and set $F = \pi(G) = \pi_{E'}(E') \simeq E'/G'$, where $\pi_{E'}$ is the restriction of π to E'.

Choose once and for all an isomorphism $F \simeq \mathrm{Hom}(F, \mathcal{K}^{\times})$ of groups. Denote by $\hat{F}$ the cyclic group $\mathrm{Hom}(F, \mathcal{K}^{\times})$, and let λ be a generator of $\hat{F}$. Then $\hat{F}$ acts on $\mathrm{Irr}(E)$ by $(\lambda\theta)(x) = \lambda(\pi(x))\theta(x)$ for $\theta \in \mathrm{Irr}(E)$ and $x \in E$, and on $\mathrm{Irr}(E')$ by $(\lambda\theta')(x') = \lambda(\pi_{E'}(x'))\theta'(x')$ for $\theta' \in \mathrm{Irr}(E')$ and $x' \in E'$. Note that a primitive q-th root of unity $\lambda(\pi(s))$ is in $\mathcal{O}^{\times}$. (For a ring R, we denote by $R^{\times}$ the multiplicative group consisting of all units of R.)

Recall from [4, Proposition 1.19] the following correspondence. Similar for E' and G'. Note that for Proposition 2.2, we only need E/G being cyclic.

Proposition 2.2. (Dade) *There is a bijective correspondence between* $\mathrm{Irr}(G)^E$ *and the set of regular* $\hat{F}$*-orbits of irreducible characters of* E. *By this correspondence,* $\phi \in \mathrm{Irr}(G)^E$ *corresponds to* $\Psi = \mathrm{Irr}(E \mid \phi) = \{\psi \in \mathrm{Irr}(E) \mid [\psi\downarrow^E_G, \phi]_G \neq 0\}$ *and* $\phi = \psi \downarrow^E_G$ *for any* $\psi \in \Psi$.

We recall the correspondence of Dade in the case of $|E/G|$ being a prime (for the statements under the condition that the order of the cyclic group E/G is divided by several primes, see [4, Theorems 6.8 and 6.9]).

Theorem 2.3. (Dade) *There is a bijection*

$$\mathrm{Irr}(G)^E \longrightarrow \mathrm{Irr}(G')^{E'}, \qquad \phi \mapsto \phi_{(G')} \tag{$*$}$$

which satisfies the following:

- *When q is odd, there are a unique sign $\epsilon_\phi \in \{\pm 1\}$ and a unique bijection*

  $$\mathrm{Irr}(E \mid \phi) \longrightarrow \mathrm{Irr}(E' \mid \phi_{(G')}), \qquad \psi \mapsto \psi_{(E')} \tag{$**$}$$

  such that

  $$(\psi - \lambda^i\psi) \downarrow^E_{E'} = \epsilon_\phi(\psi_{(E')} - \lambda^i(\psi_{(E')})) \tag{$***$}$$

  holds for any i as generalized characters.

- *When q is 2, if we choose a sign ϵ_ϕ arbitrary, there is a unique bijection $(**)$ such that $(***)$ holds.*

Moreover, $\lambda^i(\psi_{(E')}) = (\lambda^i\psi)_{(E')}$ (hence denoted by $\lambda^i\psi_{(E')}$) for any i.

We call both the correspondences $(*)$ and $(**)$ in Theorem 2.3 the *Glauberman-Dade correspondence* of characters. We denote by $\phi'_{(G)}$ the character of $\mathrm{Irr}(G)^E$ corresponding to $\phi' \in \mathrm{Irr}(G')^{E'}$.

In the remainder of this section, we rewrite the Glauberman-Dade correspondence in terms of the elements of the group algebras (Proposition 2.6).

Let $\mathcal{R} \in (\mathcal{K}, \mathcal{O}, k)$, H a finite group with a subgroup L, and $r_h \in \mathcal{R}$ for $h \in H$. Denote by Pr^H_L an $\mathcal{R}$-linear map from $\mathcal{R}H$ to $\mathcal{R}L$ defined by $\mathrm{Pr}^H_L(\sum_{h\in H} r_h h) = \sum_{l\in L} r_l l$, which induces an $\mathcal{R}$-linear map from $Z(\mathcal{R}H)$ to $Z(\mathcal{R}L)$. Denote by Tr^H_L an $\mathcal{R}$-linear map from $(\mathcal{R}H)^L(\supset Z(\mathcal{R}L))$ to $Z(\mathcal{R}H)$ defined by $\mathrm{Tr}^H_L(\tau) = \sum_{x\in[L\backslash H]} \tau^x$ for $\tau \in (\mathcal{R}H)^L$, where $(\mathcal{R}H)^L$ is the subalgebra of $\mathcal{R}H$ consisting of all the elements fixed by the conjugation action by L and $[L\backslash H]$ is a set of left coset representatives of L in H. For a conjugacy class C of H, we denote $\widehat{C} = \sum_{x\in C} x \in \mathcal{R}H$. For $\theta \in \mathrm{Irr}(H)$, we denote by e_θ the primitive idempotent of $Z(\mathcal{K}H)$ corresponding to θ.

Denote by $C(x)$ the conjugacy class of E containing $x \in E$ and by $C(x')'$ the conjugacy class of E' containing $x' \in E'$.

Since $E_0 = \sqcup_{t\in[E'\backslash E]}(E'_0)^t$ (disjoint union), see [4, Lemma 6.5], we have:

Lemma 2.4. *For $x \in E'_0$, it holds that:*

(1) $\frac{|C(x)|}{|E|} = \frac{|C(x)'|}{|E'|}$.

(2) $\mathrm{Pr}^E_{E'}(\widehat{C(x)}) = \widehat{C(x)'}$, $\mathrm{Tr}^E_{E'}(\widehat{C(x)'}) = \widehat{C(x)}$.

For $c \in \mathcal{K}$, we denote by $\bar{c}$ the complex conjugate of c (we view $\mathcal{K}$ as a subfield of the complex number field).

Lemma 2.5. *Let $\phi, \xi \in \mathrm{Irr}(G)^E$, and let $\psi, \eta \in \mathrm{Irr}(E)$ be extensions of ϕ and ξ, respectively. Then, for $x, y \in E'_0$, we have $\psi(x)\ \overline{\eta(y)} = \epsilon_\phi \epsilon_\xi\ \psi_{(E')}(x)\ \overline{\eta_{(E')}(y)}$.*

Proof. Let q be odd. Let $c, d \in \mathbb{Z}$ be such that $\pi(x) = \pi(s)^c$ and $\pi(y) = \pi(s)^d$. Let $c', d' \in \mathbb{Z}$ be such that $\underline{c'} = \underline{c}^{-1}$ and $\underline{d'} = \underline{d}^{-1}$ in $\mathbb{Z}/q\mathbb{Z}$ where the canonical image of $a \in \mathbb{Z}$ in the residue ring $\mathbb{Z}/q\mathbb{Z}$ is denoted by $\underline{a}$. By Theorem 2.3, we have

$$q\ \psi(x)\ \overline{\eta(y)} = \sum_{i=1}^{\frac{q-1}{2}} [\ (\psi - \lambda^{ic'}\psi)(x)\ \overline{(\eta - \lambda^{id'}\eta)(y)}\ ]$$

$$= \epsilon_\phi\epsilon_\xi \sum_{i=1}^{\frac{q-1}{2}} [\ (\psi_{(E')} - \lambda^{ic'}\psi_{(E')})(x)\ \overline{(\eta_{(E')} - \lambda^{id'}\eta_{(E')})(y)}\ ] = \epsilon_\phi\epsilon_\xi\ q\ \psi_{(E')}(x)\ \overline{\eta_{(E')}(y)}.$$

In the case of $q = 2$, we see that $\psi(x) = \epsilon_\phi \psi_{(E')}(x)$ since $(\psi - \lambda\psi)(x) = 2\psi(x)$. Hence, the assertions follow. □

With the notations in section 1, for cyclic $\mathsf{S} =< \mathsf{s} >$, using (♮), Okuyama showed in [8]

$$\mathrm{Pr}^{\mathsf{G}}_{\mathsf{G}^{\mathsf{S}}}(\mathsf{s}^{-1}\widehat{C(\mathsf{s})}e_\chi) = e_{\beta_\chi} \quad \text{for} \quad \chi \in \mathrm{Irr}(\mathsf{G})^{\mathsf{S}}. \qquad (\star)$$

We can show the analogous statement which implies $(\star)$ for $|\mathsf{S}| = q$ (without using (♮) since we treat the Glauberman-Dade correspondence):

Proposition 2.6. *Let $x \in E'_0$. Then*

(1) $\mathrm{Pr}^E_{E'}(\widehat{C(x)}e_\phi) = \widehat{C(x)'}e_{\phi_{(G')}}$ *for* $\phi \in \mathrm{Irr}(G)^E$.

(2) $\mathrm{Tr}^E_{E'}(\widehat{C(x)'}e_{\phi'}) = \widehat{C(x)}e_{\phi'_{(G)}}$ *for* $\phi' \in \mathrm{Irr}(G')^{E'}$.

Proof. Let $\psi \in \mathrm{Irr}(E)$ be an extension of ϕ. Then,

$$\begin{aligned}
\widehat{C(x)}e_\phi &= \widehat{C(x)}\sum_{i=0}^{q-1} e_{\lambda^i\psi} \\
&= \frac{|C(x)|}{\phi(1)}\sum_{i=0}^{q-1}\lambda^i\psi(x)e_{\lambda^i\psi} \\
&= \frac{|C(x)|}{\phi(1)}\frac{\phi(1)}{|E|}\sum_{i=1}^{q-1}\sum_{y\in E}\lambda^i\psi(x)\overline{\lambda^i\psi(y)}y \\
&= \frac{|C(x)|}{|E|}\sum_{z\in E_0,\ \pi(z)=\pi(x)} q\psi(x)\overline{\psi(z)}z.
\end{aligned}$$

From this and Lemmas 2.4 and 2.5, the assertions follow immediately. □

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In this section, we show that under Condition 3.1 below (this condition implies G' controls p-fusion in G), the Glauberman-Dade correspondence in the case $|E/G|$ being a prime induces a one-to-one correspondence between the characters belonging to the principal (p-)blocks of G and G' (Proposition 3.3) and an isomorphism of the centers of the principal (p-)block algebras of G and G' (Theorem 3.5).

Condition 3.1. There is some Sylow p-subgroup of E which is contained in G' (hence, $p \neq q$) and centralized by some element s of E'_0.

Below, we assume Condition 3.1 and take $s \in E'_0$ as in it.

A primitive idempotent of $Z(\mathcal{O}G)$ is called a block (idempotent) of G. Let b (resp. b') be the principal block (idempotent) of G (resp. G'). That is, a primitive idempotent of $Z(\mathcal{O}G)$ such that $1_G(b) \neq 0$ where 1_G is the trivial character of G. (We don't distinguish $\mathrm{Irr}(G)$ and $\mathrm{Irr}(\mathcal{K}G)$.) Denote $\mathrm{Irr}(b) = \{\phi \in \mathrm{Irr}(G) \mid \phi(b) \neq 0\}$. $\phi \in \mathrm{Irr}(b)$ is called a character belonging to the block b. Similar for the other groups and blocks.

By [3] (see also [1]), b has the following primitive idempotent decomposition in $Z(\mathcal{O}Eb)$: $b = \sum_{i=0}^{q-1} \hat{b}_i$. In this situation, $\mathrm{Irr}(b)^E = \mathrm{Irr}(b)$ and the restriction and some extension of the characters determine a one-to-one correspondence between $\mathrm{Irr}(b)$ and $\mathrm{Irr}(\hat{b}_i)$ for any i. Similar for b'.

We denote by $\hat{b}_0$ (resp. $\hat{b'}_0$) the principal block of E (resp. E'), and use ^ for the extensions of characters in $\mathrm{Irr}(b)$ (resp. $\mathrm{Irr}(b')$) belonging to the principal block, that is, $\mathrm{Irr}(\hat{b}_0) = \{\hat{\phi} | \phi \in \mathrm{Irr}(b)\}$ (resp. $\mathrm{Irr}(\hat{b'}_0) = \{\hat{\phi'} | \phi' \in \mathrm{Irr}(b')\}$).

Then, we see immediately that, for $0 \leq i \leq q-1$, a set $\{\lambda^i\hat{\phi} | \phi \in \mathrm{Irr}(b)\}$ (resp. $\{\lambda^i\hat{\phi'} | \phi' \in \mathrm{Irr}(b')\}$) forms the set of characters of E (resp. E') belonging to some block of E (resp. E') and we may denote it by $\hat{b}_i$ (resp. $\hat{b'}_i$).

Lemma 3.2. (Osima [9] or see [6, Theorem 12.4.12]) *For a finite group H and $\theta \in \mathrm{Irr}(H)$, θ belongs to the principal p-block of H if and only if $\sum_{x \in H_{p'}} \theta(x) \neq 0$, where $H_{p'}$ is the set of elements of H with the order prime to p.*

Proposition 3.3.

(1) $\mathrm{Irr}(b') = \{\phi_{(G')} | \phi \in \mathrm{Irr}(b)\}$

(2) $\mathrm{Irr}(\hat{b'}_i) = \{(\lambda^i\hat{\phi})_{(E')} | \lambda^i\hat{\phi} \in \mathrm{Irr}(\hat{b}_i)\} = \{\lambda^i\widehat{\phi_{(G')}} | \phi \in \mathrm{Irr}(b)\}$ *for* $0 \leq i \leq q-1$, *if we choose signs ϵ_ϕ appropriately in the case $q = 2$.*

Proof. By Theorem 2.3 and Lemmas 2.4(1) and 3.2, for $\phi \in \mathrm{Irr}(b)$ and $1 \leq i \leq q-1$,

$$0 \neq \sum_{x \in E_{p'}} \hat{\phi}(x) = \sum_{x \in E_{p'}} (\hat{\phi} - \lambda^i\hat{\phi})(x) = \sum_{x \in (E_0)_{p'}} (\hat{\phi} - \lambda^i\hat{\phi})(x)$$

$$= \epsilon_\phi \frac{|E|}{|E'|} \sum_{x' \in (E'_0)_{p'}} (\hat{\phi}_{(E')} - \lambda^i\hat{\phi}_{(E')})(x') = \epsilon_\phi \frac{|E|}{|E'|} \sum_{x' \in E'_{p'}} (\hat{\phi}_{(E')} - \lambda^i\hat{\phi}_{(E')})(x').$$

Hence, $\hat{\phi}_{(E')}$ or $\lambda^i\hat{\phi}_{(E')}$ must belong to $\hat{b'}_0$, and so $\phi_{(G')} \in \mathrm{Irr}(b')$. If $q = 2$, we can choose ϵ_ϕ so that $\hat{\phi}_{(E')}$ belongs to the principal block $\hat{b'}_0$. If q is odd, we see that $\hat{\phi}_{(E')} \in \mathrm{Irr}(\hat{b'}_0)$ since $i \neq 0$ is arbitrary.

Hence, in any case, the Glauberman-Dade correspondence in Theorem 2.3 induces injections from $\mathrm{Irr}(b)$ to $\mathrm{Irr}(b')$ and from $\mathrm{Irr}(\hat{b}_0)$ to $\mathrm{Irr}(\hat{b'}_0)$.

Onto part also follows from the similar consideration.

Note that $\hat{\phi}_{(E')} = \widehat{\phi_{(G')}}$. The remaining statement follows from the commutativity of the action of $\hat{F}$ and the Glauberman-Dade correspondence. □

Since $b = \sum_{\phi \in \mathrm{Irr}(b)} e_\phi$ and $b' = \sum_{\phi' \in \mathrm{Irr}(b')} e_{\phi'}$, by Propositions 2.6 and 3.3, we have:

Proposition 3.4. $\mathrm{Pr}^E_{E'}(\widehat{C(x)}b) = \widehat{C(x)'}b'$ *and* $\mathrm{Tr}^E_{E'}(\widehat{C(x)'}b') = \widehat{C(x)}b$

We see $\widehat{C(s)}b \in Z(\mathcal{O}Eb)^\times$ and $\widehat{C(s)'}b' \in Z(\mathcal{O}E'b')^\times$. In fact, $\widehat{C(s)}\hat{b}_i \in Z(\mathcal{O}E\hat{b}_i)^\times$ for any i, since $\omega_{\hat{b}_i}(\widehat{C(s)}\hat{b}_i) \neq 0$ by Condition 3.1 where $\omega_{\hat{b}_i}$ is a unique algebra homomorphism from the local algebra $Z(\mathcal{O}E\hat{b}_i)$ to k whose value at $\widehat{C(s)}\hat{b}_i$ is the canonical image in k of $\frac{|E|\lambda^i 1_E(s)}{|C_E(s)|\lambda^i 1_E(1)}$.

Note that $bb' \neq 0$. Denote $\gamma = (\widehat{C(s)'}b')^{-1}\widehat{C(s)}b \in (\mathcal{O}Ebb')^{E'}$ and let $\gamma' = \gamma^{-1} = (\widehat{C(s)}b)^{-1}\widehat{C(s)'}b'$ the inverse of γ in $(\mathcal{O}Ebb')^{E'}$. Here we mean $(\widehat{C(s)'}b')^{-1}$ the inverse of $\widehat{C(s)'}b'$ in $Z(\mathcal{O}E'b')$ and $(\widehat{C(s)}b)^{-1}$ the inverse of $\widehat{C(s)}b$ in $Z(\mathcal{O}Eb)$. Considering the grading given by π and noting $Z(\mathcal{O}Gb)$ is a local algebra, we see that $(\widehat{C(s)}b)^q \in Z(\mathcal{O}Gb)^\times$. Hence, we see $(\widehat{C(s)}b)^{-1}$ (similar for $(\widehat{C(s)'}b')^{-1}$) is a linear combination of the elements $x \in E$ such that $\pi(x) = \pi(s)^{q-1}$. Hence, $\gamma, \gamma' \in \mathcal{O}G$.

The following in the situation of the Glauberman correspondence appeared in [8].

Theorem 3.5. *With the notations above, there is an $\mathcal{O}$-algebra isomorphism from $Z(\mathcal{O}Gb)$ to $Z(\mathcal{O}G'b')$ mapping $z \in Z(\mathcal{O}Gb)$ to $\mathrm{Pr}^G_{G'}(\gamma z)$. The inverse is given by the map sending $z' \in Z(\mathcal{O}G'b')$ to $\mathrm{Tr}^G_{G'}(\gamma' z')$.*

Proof. Firstly, note that multiplying $(\widehat{C(s)'}b')^{-1}$ to

$$\widehat{C(s)'}b'e_{\phi_{(G')}} = \widehat{C(s)'}e_{\phi_{(G')}} = \mathrm{Pr}^E_{E'}(\widehat{C(s)}e_\phi)$$

where $\phi \in \mathrm{Irr}(b)$ (see Proposition 2.6(1)), we have

$$\begin{aligned} e_{\phi_{(G')}} = b'e_{\phi_{(G')}} &= (\widehat{C(s)'}b')^{-1}\mathrm{Pr}^E_{E'}(\widehat{C(s)}e_\phi) \\ &= \mathrm{Pr}^E_{E'}[(\widehat{C(s)'}b')^{-1}\widehat{C(s)}e_\phi] \\ &= \mathrm{Pr}^G_{G'}[(\widehat{C(s)'}b')^{-1}\widehat{C(s)}e_\phi] \\ &= \mathrm{Pr}^G_{G'}(\gamma e_\phi). \end{aligned}$$

Similarly, multiplying $(\widehat{C(s)}b)^{-1}$ to

$$\widehat{C(s)}be_{\phi'_{(G)}} = \widehat{C(s)}e_{\phi'_{(G)}} = \mathrm{Tr}_{E'}^{E}(\widehat{C(s)}'e_{\phi'})$$

where $\phi' \in \mathrm{Irr}(b')$ (see Proposition 2.6(2)), we have

$$\begin{aligned} e_{\phi'_{(G)}} = be_{\phi'_{(G)}} &= (\widehat{C(s)}b)^{-1}\mathrm{Tr}_{E'}^{E}(\widehat{C(s)}'e_{\phi'}) \\ &= \mathrm{Tr}_{E'}^{E}[(\widehat{C(s)}b)^{-1}\widehat{C(s)}'e_{\phi'}] \\ &= \mathrm{Tr}_{G'}^{G}[(\widehat{C(s)}b)^{-1}\widehat{C(s)}'e_{\phi'}] \\ &= \mathrm{Tr}_{G'}^{G}(\gamma' e_{\phi'}). \end{aligned}$$

Then, statements follow as in [8]. The $\mathcal{K}$-linear map sending $z_{\mathcal{K}} \in Z(\mathcal{K}Gb)$ to $\mathrm{Pr}_{G'}^{G}(\gamma z_{\mathcal{K}})$ and the $\mathcal{K}$-linear map sending $z'_{\mathcal{K}} \in Z(\mathcal{K}G'b_{(G')})$ to $\mathrm{Tr}_{G'}^{G}(\gamma' z'_{\mathcal{K}})$ are mutually inverse $\mathcal{K}$-algebra isomorphisms between $Z(\mathcal{K}Gb)$ and $Z(\mathcal{K}G'b')$, since $\{e_\phi | \phi \in \mathrm{Irr}(b)\}$ and $\{e_{\phi_{(G')}} | \phi \in \mathrm{Irr}(b)\}$ are orthogonal $\mathcal{K}$-bases of $Z(\mathcal{K}Gb)$ and $Z(\mathcal{K}G'b')$ respectively. Moreover, since these maps send elements with coefficient in $\mathcal{O}$ to elements with coefficient in $\mathcal{O}$ by definitions, these maps restrict to $\mathcal{O}$-algebra isomorphisms between $Z(\mathcal{O}Gb)$ and $Z(\mathcal{O}G'b')$. □

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