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On eigenvalues of Cartan matrices

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1 Introduction

Let G be a finite group and let (O, K, F) be a p-modular system which is large enough for G. Let B be a block of FG with defect group D. We study the Cartan matrix C of B, especially the relations between eigenvalues and elementary divisors of C. Firt we recall the definition of Cartan matrix of B. Let $S_1, \ldots, S_l$ $(l = l(B))$ be the set of simple B-modules and P_i be the projective cover of S_i. The integers $c_{ij} = \dim_F \mathrm{Hom}_{FG}(P_i, P_j)$ are called Cartan invariants and the l by l matrix $C = (c_{ij})$ is the Cartan matrix of B. The following facts on the Cartan matrix C are well-known.

(Fact 1) The determinant of C, $\det C$, is a power of p.
(Fact 2) C has the unique maximal elementary divisor, which is equal to $|D|$, and the other elementary divisors are less than $|D|$.
(Fact 3) All eigenvalues of C are positive real numbers, and the maximal eigenvalue is a simple root. It is called the Frobenius eigenvalue of C, denoted by $\rho(C)$.

In [K-M-W], we posed the following two conjectures on eigenvalues of C.

(Conjecture 1) If $\rho(C) = |D|$ holds, then is it true that the eigenvalues of C coincides with the elementary divisors of C?

(Conjecture 2) If $\rho(C)$ is an integer, then is it true that $\rho(C) = |D|$?

In [K-M-W], we showed that Conjecture 1 is affirmative under one of the following three assumptions:
(a) G is p-solvable,
(b) $D \trianglelefteq G$,
(c) B is finite type or tame type, i.e. D is cyclic, dihedral, semi-dihedral or quaternion.
Conjecture 2 is slso proved under the condition (b) or (c). I can not prove it

under the condition (a).

In [W], Wada considered the following.

(Conjecture 3) Let $f_C(x)$ be the characteristic polynomial of C. Let

$$f_C(x) = f_1(x) \cdots f_t(x)$$

be the decomposition of $f_C(x)$ into monic irreducible polynomials in $\mathbf{Z}[x]$. Suppose $\rho(C)$ is a root of $f_1(x)$. Then, do we have a decomposition of the elementary divisors of C into t subsets $E_1, \cdots, E_t$ with the following properties?

(1) $deg\, f_i = |E_i| \qquad (i = 1, \ldots, t)$,

(2) $f_i(0) = \pm \prod_{e \in E_i} e \qquad (i = 1, \ldots, t)$,

(3) $|D| \in E_1$.

Note that Conjecture 3 is a generalization of Conjecture 2. Wada proved in [W] that Conjecture 3 holds when B is finite type with $l(B) \leq 5$ or tame type. If Conjecture 3 is true, then many interesting properties of the Cartan matrix are derived from it. For example, Conjecture 3 implies that if C has an integer eigenvalue λ, then λ is an elementary divisor of C. It also implies that if C has k eigenvalues which are units in the ring of algebraic integers, then first k elementary divisors of C are all 1. The last statement on unit eigenvalues is proved without Conjecture 3.

2 Results

Proposition 1 (Nomura-Kiyota) Let C be the Cartan matrix of a block B. If C has k eigenvalues which are units in the ring of algebraic integers, then first k elementary divisors of C are all 1.

For the proof, we use the following lemma.

Lemma 2 rank($\bar{C}$) = the number of multiplicity of 1 among the elementary divisors of C, where $\bar{C}$ is the matrix over GF(p) defined by C (mod p).

For p-solvable groups G , we have the following.

Proposition 3 Let C be the Cartan matrix of a block in p-solvable group. Let λ be an eigenvalue of C. If λ is a unit in the ring of algebraic integers, then we have $\lambda = 1$.

Proposition 3 comes from the following.

Proposition 4 Let C be the Cartan matrix of a block B. Suppose that every simple B-module is liftable. If λ is a unit in the ring of algebraic integers, then we have $\lambda = 1$.

3 Problems

Recall that (K, O, F) is a p-modular system. Let v be the corresponding valuation on K. We assume all eigenvalues of C are in O. We consider the following two conditions of the Cartan matrix C.

(*) There exists a 1-1 correspondence between the eigenvalues of C and the elementary divisors of C preserving the valuation v. i.e. the correspondants have the same valuations.

(**) There exists R in $\mathrm{GL}_l(O)$ such that $R^{-1}CR$ is a diagonal matrix.

We remark that (**) implies (*) and that (*) implies Conjecture 3 (except (3)). But (*) does not hold in general, as the example G = SL(2,5), p=5 shows. So we should study the following.

(Problem 1) What is the condition under which (*) holds?

We can prove the following.

Proposition 5 If G is p-solvable and $l(B) = 2$, then (**) holds.

So natural question arises.

(Problem 2) If G is p-solvable, then is it true that (**) holds?

References

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