数理解析研究所講究録
第 1581 巻 2008 年 55-57

FUSION SYSTEMS II

RADHA KESSAR

Kyoto, August 29, 2007

Let p denote a prime number. The question of when fusion of p-elements in a finite group G is controlled by a single proper subgroup of G is an important aspect of finite group theory. Here, we present various results which generalize classical "control of fusion " results in finite group theory to arbitrary fusion systems.

Throughout we will use the notation of the article "Fusion Systems I" of this volume.

Definition 1.1. Let P be a finite p-group. The Thompson subgroup of P is the subgroup $J(P)$ of P generated by the set of abelian subgroups of P of maximal order.

The p-nilpotency theorem of Glauberman and Thompson [5, Ch. 8, Theorem 3.1] states that a finite group G is p-nilpotent if and only if $N_G(Z(J(P)))$ is p-nilpotent. By a theorem of Frobenius [4, 8.6], G is p-nilpotent if and only if P controls G-fusion in P, or equivalently, if and only if $\mathcal{F}_P(G)=\mathcal{F}_P(P)$. The following theorem proves an analogue for arbitrary fusion systems [7, Theorem A].

Theorem 1.2. *Let p be an odd prime and let $\mathcal{F}$ be a fusion system on a finite p-group P. We have $\mathcal{F}=\mathcal{F}_P(P)$ if and only if $N_{\mathcal{F}}(Z(J(P)))=\mathcal{F}_P(P)$.*

A finite group A is said to be *involved* in another finite group G if there are subgroups H, K of G such that $K\trianglelefteq H$ and $A\cong H/K$. Glauberman's ZJ-Theorem in [3] asserts that if p is odd and $Qd(p)$ is not involved in G then $N_G(Z(J(P)))$ controls strong p-fusion in G, where P is Sylow-p-subgroup of G and where $Qd(p)$ is the semi-direct product $(C_p\times C_p)\rtimes SL_2(p)$ with $SL_2(p)$ acting naturally on the elementary abelian group of rank 2. By [4, 14.8] the conclusion holds in fact with ZJ replaced by any Glauberman functor (see Definition 1.5 below). In order to extend this to arbitrary fusion systems, we introduce the following notation and terminology.

Definition 1.3. Let $\mathcal{F}$ be a fusion system on a finite p-group P, and Qbe a subgroup of P. We say that

(i) Q is $\mathcal{F}$ normalized if $|N_P(Q)| \geq |N_P(Q')|$ for any $Q' \leq P$ such that $Q \cong Q'$ in $\mathcal{F}$.

(ii) Q is $\mathcal{F}$-centric if $C_P(Q') = Z(Q')$ for any $Q' \leq P$ such that $Q \cong Q'$ in $\mathcal{F}$.

(iii) Q is $\mathcal{F}$-radical if $O_p(\mathrm{Aut}_{\mathcal{F}}(\mathrm{Q})/\mathrm{Inn}(\mathrm{Q})) = 1$, where Inn(Q) denotes the group of inner automorphisms of Q.

By [1, 4.3], if $\mathcal{F}$ is a fusion system on a finite p-group P and Q an $\mathcal{F}$-centric fully normalized subgroup of P, there is, up to isomorphism, a unique finite group $L = L_Q^{\mathcal{F}}$ having $N_P(Q)$ as Sylow-p-subgroup such that $C_L(Q) = Z(Q)$ and $N_{\mathcal{F}}(Q) = \mathcal{F}_{N_P(Q)}(L)$.

Definition 1.4. A fusion system $\mathcal{F}$ on a finite p-group P is called *$Qd(p)$-free*, if $Qd(p)$ is not involved in any of the groups $L_Q^{\mathcal{F}}$, with Q running over the set of $\mathcal{F}$-centric radical fully normalized subgroups of P.

Definition 1.5. ([8, 1.3]) A *positive characteristic p-functor* is a map sending any finite p-group P to a characteristic subgroup $W(P)$ of P such that $W(P) \neq 1$ if $P \neq 1$ and such that any isomorphism of finite p-groups $P \cong Q$ maps $W(P)$ onto $W(Q)$. A *Glauberman functor* is a positive characteristic p-functor with the following additional property: whenever P is a Sylow-p-subgroup of a finite group L which satisfies $C_L(O_p(L)) = Z(O_p(L))$ and which does not involve $Qd(p)$, then $W(P)$ is normal in L.

Any of the maps sending a finite p-group P to $Z(J(P))$ or $K_\infty(P)$ or $K^\infty(P)$ are Glauberman functors, where $J(P)$ is the Thompson subgroup of P, and where K_∞, K^∞ are as defined in [4, Section 12].

Theorem 1.6. *Let p be an odd prime, let W be a Glauberman functor and let $\mathcal{F}$ be a fusion system on a finite p-group P. If $\mathcal{F}$ is $Qd(p)$-free then $\mathcal{F} = N_{\mathcal{F}}(W(P))$. In particular, if $\mathcal{F}$ is $Qd(p)$-free then $\mathcal{F} = N_{\mathcal{F}}(ZJ(P))$.*

For fusion systems of finite groups, and when $W(P) = ZJ(P)$, this is Glauberman's ZJ-theorem; for fusion systems of p-blocks of finite groups this has also been noted by G. R. Robinson, generalizing [8, 1.4] where it was shown that the conclusion of Theorem B holds under the slightly stronger assumption that $SL_2(p)$ is not involved in any of the automorphism groups $\mathrm{Aut}_{\mathcal{F}}(\mathrm{Q})$, with Q running over the set of $\mathcal{F}$-centric radical subgroups of P.

Since there exist Glauberman functors mapping P to a subgroup $W(P)$ satisfying $C_P(W(P)) = Z(W(P))$ (for example, K_∞, K^∞ have this property), the above Theorem in conjunction with [1, 4.3] implies that a $Qd(p)$-free fusion system on a finite p-group P is in fact equal to the fusion system of a finite group L having P as Sylow-p-subgroup and satisfying $C_L(O_p(L)) \subseteq O_p(L)$. In particular, a $Qd(p)$-free fusion system is the underlying fusion system of a unique p-local finite group in the sense of [2].

References

[1] C. Broto, N. Castellana, J. Grodal, R. Levi, B. Oliver, Subgroup families controlling p-local finite groups *Proc. London Math. Soc. (3)* **91** (2005), no.2, 325–354.

[2] C. Broto, R. Levi, B. Oliver, The homotopy theory of fusion systems *J. Amer. Math. Soc.* **16** (2003), 779–856.

[3] G. Glauberman, A characteristic subgroup of a p-stable group *Canadian J. Math.* **20** (1968) 1101–1135

[4] G. Glauberman, Global and local properties of finite groups *Finite simple groups* (eds. Powell-Higman), Academic Press, London (1971) 1–64

[5] D. Gorenstein, Finite Groups, Chelsea Publishing Company, New York, (1980)

[6] R. Kessar, M. Linckelmann, A block theoretic analogue of a theorem of Glauberman and Thompson *Proc. Amer. Math. Soc.***131** (2003), 35–40.

[7] R. Kessar, M. Linckelmann, ZJ-theorems for fusion systems, *Transactions of the American Mathematical Society*, in press, ISSN/ISBN: 0002-9947

[8] R. Kessar, M. Linckelmann, G. R. Robinson, Local control in fusion systems of p-blocks of finite groups *J. Algebra***257** (2002) 393–413.

Department of Mathematics, University of Aberdeen, King's College, Aberdeen AB24 3UE, Scotland

E-mail address: `/\/k\e/s\s/a\r/\/ (no slashes) at maths dot abdn dot ac dot uk`