ON A CERTAIN SIMPLE MODULE AND COHOMOLOGY OF THE SYMMETRIC GROUP OVER $GF(2)$

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INTRODUCTION

My talk is concerned with some simple module of the symmetric group over a field of characteritic 2. The simple module concerned is called the spin module and we shall discuss its cohomological properties, especially, want to discuss a problem raised by Uno at the 2001 conference of this meeting [8].

Let Σ_n be the symmetric group of degree n and k be an algebraically closed field of characteritic 2. Simple modules of Σ_n is parametrized by the set of 2-regular partitions of n. For $n = 2(m+1)$, the simple module $D^{(m+2,m)}$ corresponding to the partition $(m + 2, m)$ is called the spin module of $\Sigma_{2(m+1)}$. $D^{(m+2,m)} \downarrow_{\Sigma_{2m+1}}$ remains to be simple and isomorphic to $D^{(m+1,m)}$, the simple $k\Sigma_{2m+1}$-module corresponding to the partition $(m + 1, m)$ of $2m + 1$. $D^{(m+1,m)}$ is also called the spin module of Σ_{2m+1}.

Uno's conjecture on the spin module $D^{(m+1,m)}$ proposed in [8] concerns with the cohomological variety of it and he remarked that the comlexity of $D^{(m+1,m)}$ is $[m/2]$ if the conjecture was true. We do not know whether the conjecture is true or not, but we could caluculate its complexity. We shall give a proof of the following.

Theorem *The complexity of the spin module $D^{(m+1,m)}$ of Σ_{2m+1} is equal to $[m/2]$.*

We first summarize some results from the cohomology theory of finite groups in section 1 and then give a definition of the spin module of the symmetric group in section 2. Our proof of the theorem heavily depends on the observations of Nagai and Uno [8], [6] on which we shall discuss in section 3. A proof of the proposition will be given in the last of section 3. In my lecture, we could not discuss Uno's conjecture in detail. In section 4, we include here my idea to answer the conjecture.

1. FROM COHOMOLOGY THEORY OF FINITE GROUPS

Let kG be the group algebra of a finite group G over an algebraically closed field k of characteristic $p > 0$. The cohomology algebra $H^*(G, k)$ of G over k is a finitely generated, graded commutative algebra over k.

Let $V_G(k)$ be the variety, the maximal ideal spectrum corresponding to $H^*(G, k)$. For a finitely generated kG-module M, the cohomology algebra $Ext^*_{kG}(M, M)$ is an $H^*(G, k)$-module by cup products and is finitely generated. Let $I_G(M)$ be the annihilator ideal of $Ext^*_{kG}(M, M)$ in $H^*(G, k)$. The variety of M, denoted by $V_G(M)$,

is a subvariety of $V_G(k)$ defined by the ideal $I_G(M)$. The complexity of M, denoted by $c(M) = c_G(M)$, is defined to be the Krull dimension of $H^*(G,k)/I_G(M)$.

For a subgroup H of G, we have the restriction map $\mathrm{res}_{G,H} : H^*(G,k) \to H^*(H,k)$ and the corestriction map (or the transfer map) $\mathrm{cor}_{H,G} : H^*(H,k) \to H^*(G,k)$. The restriction map is a ring homomorphism so that it induces a map $\mathrm{res}^*_{G,H} :$ $V_H(k) \to V_G(k)$, $\mathfrak{l} \mapsto \mathrm{res}^{-1}_{G,H}(\mathfrak{l})$.

Quillen's dimension theorem and stratification theorem say the following.

Theorem 1.1 (Quillen). *The following assertions hold.*

(1) $V_G(M) = \cup_{E \subset G, elem.abelian}\, \mathrm{res}^*_{G,E}(V_E(M))$

(2) $c_G(M) = max._{E \subset G, elem.abelian} c_E(M)$

For these results and a general theory of cohomology of finite group, see the book of Benson [2].

2. The Spin Module of the Symmetric Group

In the rest of the note, let $p = 2$.

Let Σ_n be the symmetric group of degree n. Simple modules of Σ_n over k is parametrized by the set of 2-regular partitions of n.

Assume that $n = 2(m+1)$ is even. Then the simple module $D^{(m+2,m)}$ of $\Sigma_{2(m+1)}$ corresponding to the partition $(m+2, m)$ is called the spin module of $\Sigma_{2(m+1)}$. It is known that $\dim_k D^{(m+2,m)} = 2^m$.

$D^{(m+2,m)} \downarrow_{\Sigma_{2m+1}}$ remains to be simple and corresponding to the partition $(m + 1, m)$ of $2m + 1$.

$$D^{(m+2,m)} \downarrow_{\Sigma_{2m+1}} = D^{(m+1,m)}$$

The restriction of $D^{(m+1,m)}$ to Σ_{2m} is no longer simple, and is a (unique) non-split self extension of the spin module $D^{(m+1,m-1)}$ of Σ_{2m}.

$$D^{(m+1,m)} \downarrow_{\Sigma_{2m}} = \begin{array}{c} D^{(m+1,m-1)} \\ D^{(m+1,m-1)} \end{array}$$

We denote this $k\Sigma_{2m}$-module by $M^{(m+1,m-1)}$. Cohomological properties $D^{(m+1,m)}$ are covered by those of $M^{(m+1,m-1)}$ because $[\Sigma_{2m+1} : \Sigma_{2m}]$ is odd.

For representations of symmetric groups, see the book of James [5]. Several properties of the spin module are shown by Gow and Kleshchev [4]. See also the study of Sheth [7].

3. Observations by Nagai and Uno

In this section, we summarize results by Nagai and Uno [8] and give a proof of the theorem mentioned in the introduction.

3.1. Restriction to Young Subgroups.

The module $M^{(m+1,m-1)}$ behaves well when restricted to Young subsgroups of Σ_n. The following two results are obtained by Theorem 2 [8] and proved by Nagai and Uno. See also Uno's discussions there.

Proposition 3.1 (Nagai-Uno). *The following assertions hold.*

(1) $D^{(m+2,m)} \downarrow_{\Sigma_{2(i-1)+2} \times \Sigma_{2(m-i)+2}}$ *is a self extension* ; $\begin{array}{c} D^{(i+1,i-1)} \otimes D^{(m-i+2,m-i)} \\ D^{(i+1,i-1)} \otimes D^{(m-i+2,m-i)} \end{array}$

of the $k[\Sigma_{2(i-1)+2} \times \Sigma_{2(m-i)+2}]$-*module* $D^{(i+1,i-1)} \otimes D^{(m-i+2,m-i)}$.

(2) $D^{(m+2,m)} \downarrow_{\Sigma_{2i+1} \times \Sigma_{2(m-i)+1}} \cong D^{(i+1,i)} \otimes D^{(m-i+1,m-i)}$

(3) $M^{(m+1,m-1)} \downarrow_{\Sigma_{2i} \times \Sigma_{2(m-i)}} \cong M^{(i+1,i-1)} \otimes M^{(m-i+1,m-i-1)}$

Lemma 3.2. *Let* $\sigma = (1\ 2)(3\ 4) \cdots (2m - 1\ 2m) \in \Sigma_{2m}$. *Then* $M^{(m+1,m-1)} \downarrow_{\langle \sigma \rangle}$ *is projective.*

3.2. Elementary Abelian Subgroups of Σ_n.

Let
$$n = 2^{s_1} + 2^{s_2} + \cdots + 2^{s_k}, \quad 0 \leq s_1 < s_2 < \cdots < 2^k$$

be the 2-adic expansion of n. Then
$$\Sigma_{2^{s_1}} \times \Sigma_{2^{s_2}} \times \cdots \times \Sigma_{2^{s_k}} \subset \Sigma_n$$

is of odd index. So, in order to determine the variety $V_{\Sigma_{2m}}(M^{(m+1,m-1)})$ and the complexity $c_{\Sigma_{2m}}(M^{(m+1,m-1)})$, we may assume that $n = 2^s$ for some s by Proposition 1 and Quillen's Theorem.

Let $E(s)$ be a (maximal) regular elementary abelian subgroup of Σ_{2^s}. Elementary abelian subgroups of Σ_n are well understood. The following fact is known. See [1].

Lemma 3.3. *Each elementary abelian 2-group of* Σ_{2^s} *is conjugate to a subgroup* $E(s)$ *or* $\Sigma_{2^{s-1}} \times \Sigma_{2^{s-1}}$.

3.3. The Dickson Invariants in Polynomial Ring.

We easily see that
$$N_{\Sigma_{2^s}}(E(s)) = GL(E(s)) \ltimes E(s)$$

where $GL(E(s)) \cong GL(s, 2)$ is a general linear group on the vector space $E(s)$ over $GF(2)$.

The cohomology ring $H^*(E(s), k)$ is a polynomial ring of s variables over k on which $GL(E(s))$ acts in obvious way. And the invariant subring $H^*(E(s), k)^{GL(E(s))}$ is generated by so called Dickson invariants denoted by
$$c_t = c_t(E(s)), \quad 0 \leq t \leq s - 1, \quad \deg c_t = 2^s - 2^t = 2^t(2^{s-t} - 1)$$

c_t satisfies that
$$for\ F \subset E(s),\ \mathrm{res}_{E(s),F}(c_t) \neq 0 \iff |E(s) : F| \leq 2^t$$

For precise definition and properties of Dickson invariants, see [3].

3.4. Uno's Conjecture.

Denote by $M(s)$ the $k\Sigma_{2^s}$-module $M^{(2^{s-1}+1,2^{s-1}-1)}$. The assertion 1 in the following lemma follows from Lemma 3.2 and a proof of the assertion 2 is given in [8].

Lemma 3.4. *The following assertions hold.*

(1) $c_{E(s)}(M(s)) \leqq s - 1$.

(2) $c_{E(1)}(M(1)) = 0$, $\quad c_{E(2)}(M(2)) = 1$ and $c_{E(3)}(M(3)) = 2$.

Uno conjectured the following.

Conjecture 3.5 (Uno [8]). $\sqrt{I_{E(s)}(M(s))}$ *is a principal ideal generated by the Dickson invariants* $c_{s-1}(E(s))$.

And he remarked that the following fact is true.

Theorem 3.6. *If the conjecture is true, then* $c_{\Sigma_{2m}}(M^{(m+1,m-1)}) = [m/2]$.

3.5. Proof of Theorem.

We do not know whether the conjecture is true or not. But the conclusion of Theorem 3.6 can be proved without answering to the conjecture.

Theorem 3.7. *The following assertions hold.*

(1) $c_{\Sigma_{2^s}}(M(s)) = 2^{s-2}$

(2) $c_{\Sigma_{2m}}(M^{(m+1,m-1)}) = [m/2]$

Proof. We have an inequality $2^{s-2} \geqq s - 1$ for $s \geqq 2$. By Lemma 3.4, $c_{E(s)}(M(s)) \leqq s - 1$. By Propositin 3.1,(3) and induction,

$$c_{\Sigma_{2^{s-1}} \times \Sigma_{2^{s-1}}}(M(s)) = c_{\Sigma_{2^{s-1}} \times \Sigma_{2^{s-1}}}(M(s-1) \otimes M(s-1)) = 2^{s-3} + 2^{s-3} = 2^{s-2}$$

Lemma 3.3 and Quillen's theorem imply the assertion 1. The assertion 2 follows from 1. $\qquad\square$

4. Toward the Uno's Conjecture

Although we could determined the complexity of the $k\Sigma_{2m}$-module $M^{(m+1,m-1)}$, our result does not give any information concerning the variety $V_{\Sigma_{2m}}(M^{(m+1,m-1)})$ and the annihlator ideal $I_{\Sigma_{2m}}(M^{(m+1,m-1)})$. In this section, we shall explain our idea toward the Uno's conjecture. For a cohomology element ζ in $H^*(\Sigma_n, k)$, let $V_{\Sigma_n}(\zeta)$ be the subvariety in $V_{\Sigma_n}(k)$ determined by the ideal $\zeta H^*(\Sigma_n, k)$.

Problem 4.1. *Let s be an positive integer and $M(s) = M^{(2^{s-1}+1,2^{s-1}-1)}$ be the $k\Sigma_{2^s}$-module defined in subsection 3.4.*

(1) *Determine the annihilator ideal* $I_{\Sigma_{2^s}}(M(s))$.

(2) *Find cohomology elements* $\zeta_1, \cdots, \zeta_{2^s-2}$ *in* $H^*(\Sigma_{2^s}, k)$ *which cover the variety* $V_{\Sigma_{2^s}}(M(s))$, *that is, which satisfy*

$$V_{\Sigma_{2^s}}(\zeta_1) \cap \cdots \cap V_{\Sigma_{2^s}}(\zeta_{2^s-2}) \cap V_{\Sigma_{2^s}}(M(s)) = \{0\}$$

In the rest of the section, let $G = \Sigma_{2^s}$ and $m = 2^{s-1}$. For the cohomology of the symmetric group, we refer to the book of Adem and Milgram [1].

4.1. A Cohomlogy Element in $H^*(\Sigma_{2^s}, k)$ Related to $c_{s-1}(E(s))$.

Let $E(s)$ be a regular elementary abelian subgroup of G. Let

$$\sigma_i = (2i-1\ 2i),\ 1 \leq i \leq m, \quad \sigma = \sigma_1 \cdots \sigma_m = (1\ 2)(3\ 4)\cdots(2m-1\ 2m) \in \Sigma_{2^s}.$$

$A = \langle \sigma_1, \cdots, \sigma_m \rangle$ is a subgroup of G isomorphic to $\mathbb{Z}_2^m$.

Set $H = C_G(\sigma)$. H is a stabilazer of the following set of 2-points sets

$$\{\ \{1,2\}, \{3,4\}, \cdots, \{2m-1, 2m\}\ \}$$

Let

$$\tau_i = (2i-1\ 2i+1)(2i\ 2i+2),\ 1 \leq i \leq m-1, \quad S = \langle \tau_1, \cdots, \tau_{m-1} \rangle \subset G$$

Then S is a subgroup of H isomorphic to the symmetric group of degree $m = 2^{s-1}$ and $H = S \ltimes A$, a semidirect product of S by A.

For an integer i with $0 \leq i \leq m-1$, set

$$S_i \times S_{m-i} = \langle \tau_1, \cdots, \tau_{i-1} \rangle \times \langle \tau_{i+1}, \cdots, \tau_{m-1} \rangle \subset S$$

$$A_i \times A_{m-i} = \langle \sigma_1, \cdots, \sigma_i \rangle \times \langle \sigma_{i+1}, \cdots, \sigma_m \rangle \subset A$$

$$L_i = (S_i \times S_{m-i}) \ltimes (1 \times A_{m-i}) = S_i \times (S_{m-i} \ltimes A_{m-i}) \subset H$$

Consider the following subgroups of H,
$$\begin{array}{c} H \\ | \\ K = (S_1 \times S_{m-1}) \ltimes A \\ \triangledown \\ L = (S_1 \times S_{m-1}) \ltimes (1 \times A_{m-1}) \end{array}$$

, $[H:K] = m$, $[K:L] = 2$ ($H = L_0$, $L = L_1$).

Let $\alpha \in H^1(K, \mathbb{Z}_2) \subset H^1(K, k)$ be the cohomology elements corresponding to the extension of k_K by k_K ;

$$\alpha \quad : 0 \to k_K \to k_L \uparrow^K \to k_K \to 0$$

Set $\beta = \mathrm{nrm}_{K,H}(\alpha) \in H^m(H, k)$, where $\mathrm{nrm}_{K,H}$ is the norm map of Evens. And set $\rho = \mathrm{cor}_{H,G}(\beta) \in H^m(G, k)$. Using Mackey formula and properties of Dickson invariants, we can prove the following lemma.

Lemma 4.2.
$$\mathrm{res}_{G,E(s)}(\rho) = c_{s-1}(E(s))$$

Uno's conjecture asserts that the fact that $\rho \in \sqrt{\mathrm{Ker}\,\mathrm{res}_{G,E(s)} + I_G(M(s))}$ is true. We propose the folowing problem.

Problem 4.3. Is $\rho \in \sqrt{I_G(M(s))}$?

We have no idea to solve the problem, but we think the following fact may help to have a solution.

Lemma 4.4. An m-fold self-extension of k_H corresponding to $\beta \in H^m(H, k)$ has the following form ;

$$\beta \quad : 0 \to k_H \to k_{L_m} \uparrow^H \to \cdots \to k_{L_i} \uparrow^H \to \cdots \to k_{L_1} \uparrow^H \to k_H \to 0$$

4.2. A Cohomlogy Element in $H^*(\Sigma_{2^s}, k)$ Related to $c_0(E(s))$.

Let $n = 2^s = 2m$ and set $\Omega = \{ 1, 2, \cdots, n \}$. For an integer t with $0 \le t \le n$, set $\Gamma_t = \{ I \subset \Omega \; ; \; |I| = t \}$. $G = \Sigma_n$ acts on Γ_t as a permutation group and the stabilizer of $I \in \Gamma_t$ is a Young subgroup isomorphic to $\Sigma_t \times \Sigma_{n-t}$.

Let X_t be the permutation module of $G = \Sigma_n$.on the set Γ_t. $X_0 \cong X_n \cong k_G$. We use the same symbols $I \in \Gamma_t$ to denote k-basis of X_t.

Define a k-map $f_t : X_t \to X_{t+1}$, $0 \le t \le n - 1$ by

$$f_t \; : X_t \to X_{t+1}, \quad f_t(I) = \sum_{j \in \Omega \backslash I} I \cup \{ j \}, \quad for \; I \in \Gamma_t$$

It is easy to see that f_t is a $k\Sigma_n$-homomorphism and $f_{t+1} \circ f_t = 0$. And a computation shows that Ker $f_{t+1} = $ Im f_t for $0 \le t \le n - 1$. Thus we have the following lemma.

Lemma 4.5. *We have an* $(n - 1)$*-fold self-extension of* k_G *of the form ;*

$$0 \to k_G \to X_1 \to \cdots \to X_t \to \cdots \to X_{n-1} \to k_G \to 0$$

Let $\zeta \in H^{n-1}(G, k)$ be the cohomology element corresponding to the above extension. Then we have the following.

Lemma 4.6.

$$\mathrm{res}_{G,E(s)}(\zeta) = c_0(E(s))$$

Let $L_\zeta = $ Ker $\tilde{\zeta}$ be the Carlson module of ζ so that we have a short exact sequence of kG-modules ;

$$0 \to L_\zeta \to \Omega^{n-1}(k_G) \xrightarrow{\tilde{\zeta}} k_G \to 0$$

Problem 4.7. *Is* $c_G(L_\zeta \otimes M(s)) = c_G(M(s)) - 1$ *?*

If the above problem is answered affirmatively, then the exact sequence in the lemma will help to find cohomology elements which covers the variety $V_G(M(s))$ and will give us some information to attack Uno's conjecture.

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